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THE ALMOST SURE STABILITY OF QUADRATIC FORMS[1]

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Let w_{jk} be a doubly indexed sequence of weights, let $\{X_k\}$ be a sequence of independent random variables and let $Q_n = \sum_{j,k=1}^n w_{jk}X_jX_k$. Sufficient conditions for the almost sure stability of Q_n are given and the "tightness" of these conditions is investigated. These quadratic forms are weighted sums of dependent variables; however, their stability properties are very much like those established in the literature for weighted sums of independent variables.

1. Introduction and statement of results. Let $X_1, X_2, \cdots$ be independent random variables, at least two of which are not degenerate, let w_{jk} be a doubly indexed sequence of real numbers and let $Q_n = \sum_{j,k=1}^n w_{jk}X_jX_k$. If $\{Y_n\}$ is any sequence of random variables for which $B_n^{-1}Y_n - a_n \to 0$ a.s. (in probability) with $\{a_n\}$ a sequence of real numbers and $\{B_n\}$ a sequence of positive numbers which diverge monotonically, then Y_n is said to be stable almost surely (in probability) with respect to $\{a_n\}$ and $\{B_n\}$. If it is clear which sequences $\{a_n\}$ and $\{B_n\}$ are under consideration we will simply say that Y_n is stable almost surely (in probability). Our main purpose is to consider sequences which stabilize Q_n almost surely. Varberg (1966, 1968) has considered the almost sure and quadratic mean convergence of Q_n and Whittle (1964) has investigated the asymptotic normality of certain quadratic forms. Griffiths, Platt and Wright (1973) studied sequences which stabilize Q_n in probability.

A large portion of classical probability theory consists of the study of the stability properties of $S_n = \sum_{k=1}^n X_k$. More recently, the almost sure convergence of the weighted sums $\sum_k a_{nk}X_k$ has been considered as well as dependent sequences $\{X_k\}$ such as martingales, Markov sequences and stationary random variables. (Chapters 3 and 4 of Stout (1974) contain a discussion of many of these results.) In the dependent case, the treatment of the weighted sums has been less thorough than that of S_n. The quadratic forms Q_n can be viewed as weighted sums of dependent variables and, under certain assumptions, are also martingales. We shall see that the almost sure stability properties of Q_n are very much like those of the weighted sums of independent variables $T_n = \sum_{k=1}^n v_kX_k$.

For comparison we state some of the stability results for T_n provided in the literature. Let $v_k > 0$ for $k = 1, 2, \cdots$, let $V_n = \sum_{k=1}^n v_k \to \infty$, and let $M(x) =$ card. $\{k: V_k/v_k \leqslant x\}$ for $x > 0$. Assuming the X_k are independent and identically

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distributed as X, Jamison, Orey and Pruitt (1965) proved the following results:

THEOREM 1. *If* $E|X| < \infty$ *and*

$$\int x^2 \int_{y \geq |x|} y^{-3} M(y)\, dy\, dP[X \leq x] < \infty \tag{1}$$

then $V_n^{-1} T_n - E(X) \to 0$ a.s.

THEOREM 2. *Fix* $r \in [1, 2)$. *The conclusion of Theorem* 1 *holds for a given sequence of weights* $\{v_k\}$ *and all* X *with* $E|X|^r < \infty$ *if and only if* $\lim \sup_{x \to \infty} M(x)/x^r < \infty$.

For Theorem 2 they only gave the proof of the case $r = 1$, however, the proof for $1 < r < 2$ is essentially the same. Heyde (1968) considered the more general norming constants B_n, with B_n a sequence of positive numbers which diverge monotonically. (Throughout our investigation B_n shall always denote such a sequence.) With $M_1(x)$ defined as $M(x)$, except V_n is replaced by B_n and with $\{X_k\}$ independent and identically distributed as X, he proved the following:

THEOREM 3. *If* $M_1(x)$ *satisfies* (1), *then* T_n *is stable with respect to* $\{B_n\}$ *and* $a_n = B_n^{-1} \sum_{k=1}^n v_k E(X_k I_{[|X_k| < B_k / v_k]})$. *Furthermore*, $a_n \to 0$ *if*

$$E(X) = 0 \quad \textit{and} \quad \int |x| \int_0^{|x|} y^{-2} M_1(y)\, dy\, dP[X \leq x] < \infty. \tag{2}$$

or

$$\int |x| \int_{|x|}^{\infty} y^{-2} M_1(y)\, dy\, dP[X \leq x] < \infty. \tag{3}$$

Heyde has also developed the results necessary to obtain an analogue to Theorem 2. Fix $r \in (1, 2)$ and consider fixed sequences $\{v_n\}$ and $\{B_n\}$. If $\lim \sup_{x \to \infty} M_1(x)/x^r < \infty$, $E(X) = 0$ and $E|X|^r < \infty$ then the hypotheses of Theorem 3 hold (notice that $M_1(x) = 0$ for some $x > 0$) and so $B_n^{-1} T_n \to 0$ a.s. Conversely, if $\lim \sup_{x \to \infty} M_1(x)/x^r = \infty$ then there exists a symmetric random variable X with $E(X) = 0$, $E|X|^r < \infty$ and $EM_1(|X|) = \infty$. So appealing to Theorem 5 of Heyde's work $B_n^{-1} T_n$ cannot converge to zero almost surely. We state this result as

THEOREM 4. *Let* $1 < r < 2$, *let* $\{X_k\}$ *be independent and identically distributed as* X *with* $E(X) = 0$, *and let* $\{v_n\}$ *and* $\{B_n\}$ *be fixed sequences. Then* $B_n^{-1} T_n \to 0$ a.s. *for all* X *with* $E|X|^r < \infty$ *if and only if* $\lim \sup_{x \to \infty} M_1(x)/x^r < \infty$.

(Heyde (1968), Chow and Teicher (1971) and Wright, Platt and Robertson (1977) have investigated the stability properties of T_n when $E|X| = \infty$.)

In investigating the quadratic forms Q_n we wish to allow for the possibility that the X_k's are not identically distributed and so, for $y > 0$, we define

$$F(y) = \sup_k P[|X_k| \geq y] \quad \text{and} \quad G(y) = \sup_{j \neq k} P[|X_j X_k| \geq y].$$

Typically, the assumptions on the random variables are expressed in terms of moment or moment-like conditions. However, the diagonal terms $w_{jj} X_j^2$ and the off-diagonal terms $w_{jk} X_j X_k$ $(j \neq k)$ are different in this respect; for instance, $E|X_j|^r < \infty$ and $E|X_k|^r < \infty$ imply that $E|X_j X_k|^r < \infty$ but only that $E[(X_j^2)^{r/2}] <$

∞. (A more detailed discussion of the relationships between the tail probabilities F and G is found in Section 2 of Griffiths et al. (1973).) For this reason we first assume $w_{jj} \equiv 0$ and then consider the linear sum $\Sigma_{j=1}^n w_{jj} X_j^2$ later. Furthermore, we will assume $w_{jk} = w_{kj}$ for $j, k = 1, 2, \cdots$; this is no restriction since one can always obtain such symmetric weights without changing the value of Q_n (i.e., $(w_{jk} + w_{kj})/2$). While we will state the results for these quadratic forms in terms of $B_n^{-1} Q_n$, the sequence motivating the discussion is $B_n = W_n$ where $W_n = \Sigma_{j,k=1}^n |w_{jk}|$. Define $N(x) =$ card. $\{(j, k) : j < k, B_k/|w_{jk}| \leqslant x\}$ for $x > 0$. If Q_n is stable in probability with respect to $\{B_n\}$ and some sequence $\{a_n\}$, then Theorem 3.1 of Griffiths et al. shows that, in the identically distributed case, $\max_{1 \leqslant j, k \leqslant n} |w_{jk}|/B_n \to 0$ and so, for sequences $\{w_{jk}\}$ and $\{B_n\}$ to be considered, $N(x)$ will be finite for each x.

In the results that follow an integral of the form $\int_A f(x)|dG(x)|(\int_A f(x)|dF(x)|)$, with $G(F)$ a nonincreasing function, should be interpreted as a Lebesgue-Stieltjes integral with respect to the measure determined by $-G(-F)$. The proofs of the following results are given in Section 2.

THEOREM 5. *Let* $w_{jj} = 0$ *and* $E(X_j) = 0$ *for* $j = 1, 2, \cdots$. *If*

$$G(x) \to 0 \quad \text{as} \quad x \to \infty \quad \text{and} \quad \int_0^\infty x^2 \int_x^\infty y^{-3} N(y)\, dy |dG(x)| < \infty \tag{4}$$

and

$$\int_0^\infty x \int_0^x y^{-2} N(y)\, dy |dG(x)| < \infty, \tag{5}$$

then $Q_n / B_n \to 0$ a.s.

To clarify the meaning of conditions (4) and (5), it should be noted that the sum of expressions (4) and (5) is

$$\begin{aligned}\int_0^\infty x \int_0^\infty ((x/y) \wedge 1) y^{-2} N(y)\, dy |dG(x)| \\ &= \Sigma_{j<k} \int_0^\infty x \int_{B_k/|w_{jk}|}^\infty ((x/y) \wedge 1) y^{-2}\, dy |dG(x)| \\ &= \Sigma_{j<k} \int_0^\infty \int_0^{|w_{jk}|x/B_k} (u \wedge 1)\, du |dG(x)| \\ &= \Sigma_{j<k} \int_0^\infty \phi(w_{jk} x / B_k) |dG(x)|,\end{aligned}$$

where ϕ is the symmetric function defined by $\phi(v) = \int_0^v (u \wedge 1)\, du$ for $v \geqslant 0$. Consequently, $\Sigma_{j<k} E(\phi(w_{jk} B_k^{-1} X_j X_k))$ is finite if conditions (4) and (5) hold.

The question arises, could stability results like those in the first part of Theorem 3 be obtained for Q_n if the X_k's are not necessarily centered at their means? Consider the following example: let $w_{1k} = w_{k1} = v_{k-1}$ for $k = 2, 3, \cdots$, let $w_{jk} = 0$ otherwise and X_1, X_2 be i.i.d. as X with $E(X) = \mu \neq 0$. Then Theorem 1 gives conditions under which $Q_n / W_n \to \mu X_1$ a.s. So with the "natural" choice for B_n the answer in general is no. However, one could write

$$\begin{aligned} B_n^{-1} Q_n - B_n^{-1} \Sigma_{j,k=1}^n w_{jk} \mu_j \mu_k &= B_n^{-1} \Sigma_{j,k=1}^n w_{jk} (X_j - \mu_j)(X_k - \mu_k) \\ &\quad + 2 B_n^{-1} \Sigma_{j,k=1}^n w_{jk} \mu_j (X_k - \mu_k) \end{aligned}$$

and apply Theorem 5 to obtain the almost sure convergence of the first expression on the right-hand side. The second expression is a linear sum with weights $v_k = 2\Sigma_{j=1}^{n} w_{jk}\mu_j$ and so conditions obtained from Theorem 3 could be stated which would guarantee its almost sure stability. We will not state such results, but we do combine the linear results with Theorem 5 to provide a result for quadratic forms with possibly nonzero diagonal elements. Let $N_D(x) = \text{card.}\ \{k\colon w_{kk} \neq 0, B_k/|w_{kk}| \leqslant x\}$ for $x > 0$.

THEOREM 6. *Let* $E(X_j) = 0$ *for* $j = 1, 2, \cdots$. *If* G *satisfies* (4) *and* (5) *and if* F *satisfies*

$$\int_0^\infty x \int_x^\infty y^{-2} N_D(y)\, dy |dF(x^{\frac{1}{2}})| < \infty, \tag{6}$$

then $Q_n/B_n \to 0$ a.s.

The next result is an analogue of Theorems 2 and 4.

THEOREM 7. *Let* $1 \leqslant r < 2$, *let* $E(X_j) = 0$ *for* $j = 1, 2, \cdots$ *and let* $\{w_{jk}\}$ *and* $\{B_n\}$ *be fixed sequences. Suppose that*

$$\int_1^\infty x \log x |dG(x)| < \infty, \tag{7}$$

$$G(x) \to 0 \quad \text{as} \quad x \to \infty \quad \text{and} \quad \int_0^\infty x^r |dG(x)| < \infty, \tag{8}$$

and $\limsup_{x\to\infty} N(x)/x^r < \infty$. *In addition, if the* w_{jj} *are not all zero suppose that* $\limsup_{x\to\infty} N_D(x)/x^{r/2} < \infty$, *then* $Q_n/B_n \to 0$ a.s.

It should be noted that for $r > 1$, (8) implies (7) and for $r = 1$, (7) is a stronger assumption than (8). It would be of interest to know if (7) can be omitted in Theorem 7. A comparison of Theorems 2 and 7 suggests that this might be the case for $B_n = W_n$. For the case $1 < r < 2$ the hypotheses of the theorem are tight in the following sense.

REMARK 8. (a) If w_{jk} is a sequence of weights with $w_{jk} = 0$ for $j \neq k$ and $\limsup_{x\to\infty} N_D(x)/x^{r/2} = \infty$, then there is a sequence of independent and identically distributed random variables $\{X_k\}$ which satisfy (8) but Q_n is not stable with respect to $\{B_n\}$ and $\{a_n\}$ for any $\{a_n\}$.

(b) If $w_{2k-1,2k} = w_{2k,2k-1} = v_k$ for $k = 1, 2, \cdots$, $w_{jk} = 0$ otherwise and $\limsup_{x\to\infty} N(x)/x^r = \infty$, then there exists a sequence of independent and identically distributed random variables $\{X_k\}$ which satisfy (8) but Q_n is not stable with respect to $\{B_n\}$ and $\{a_n\}$ for any $\{a_n\}$.

To show that the more general quadratic forms $Q_n^* = \Sigma_{j,k} a_{jk}^{(n)} X_j X_k \to 0$ a.s. one could show that $\Sigma_n P[|Q_n^*| > \varepsilon] < \infty$ for each $\varepsilon > 0$. Hanson and Wright (1971) and Wright (1973) have provided bounds for these probabilities which could be used to show that this series is finite in certain cases.

2. Proofs.

We state without proof the following result: if $H_i(x)$ is a nonincreasing function with $H_i(x) \to 0$ as $x \to \infty$ for $i = 1, 2$, if $H_1(x) \geqslant H_2(x)$ for all $x \geqslant 0$ and if $f(x)$ is a nondecreasing function of x for $x \geqslant 0$, then $\int_a^\infty f(x)|dH_1(x)| \geqslant \int_a^\infty f(x)|dH_2(x)|$ for any $a \geqslant 0$. So that this result could be used in the proofs of Theorems 5 and 6 it has been assumed that $G(y) \to 0$ as $y \to \infty$. Of course, $F(y) \to 0$ and $G(y) \to 0$ are equivalent since $P[|X_j| \geq y^{\frac{1}{2}}] + P[|X_k| \geq y^{\frac{1}{2}}] \geq P[|X_jX_k| \geq y]$ and $P[|X_jX_k| \geq \varepsilon y] \geq P[|X_j| \geq \varepsilon]P[|X_k| \geq y]$ for $j \neq k$.

PROOF OF THEOREM 5. By the Kronecker lemma $Q_n/B_n \to 0$ a.s. if $Y_n = \Sigma_{1 \leq j < k \leq n} B_k^{-1} w_{jk} X_j X_k$ converges almost surely to a finite limit. Since Y_n is a martingale, we need only show that $\sup_n E|Y_n| < \infty$. But $E|Y_n| \leq 1 + 2E(\phi(Y_n))$ with $\phi(u) = u^2/2$ for $|u| \leq 1$ and $\phi(u) = |u| - \frac{1}{2}$ for $|u| > 1$. By the argument given by Kurtz (1972) for part (a) of Lemma 2.2, $E(\phi(Y_n)) \leq \Sigma_{k=1}^{n-1} E(\phi(Y_{k+1} - Y_k))$. Since $Y_{k+1} - Y_k = \Sigma_{j=1}^{k} w_{jk+1} B_{k+1}^{-1} X_j X_{k+1}$, conditioning on X_{k+1} and applying the same argument shows that

$$\sup_n E(\phi(Y_n)) \leq \Sigma_{j<k} E\big(\phi\big(w_{jk} B_k^{-1} X_j X_k\big)\big),$$

which was seen to be finite in the discussion following Theorem 5.

PROOF OF THEOREM 6. We show that $B_n^{-1}\Sigma_{k=1}^n w_{kk} X_k^2 \to 0$ a.s. follows from a slight modification of Theorem 3. Examining the proofs given by Heyde, we see that by taking the absolute value of the weights in the definition of M_1 and in the truncation value and defining $F(x)$ as here the following result holds: if (4) holds for F then $B_n^{-1}T_n - a_n \to 0$ a.s. with $a_n = B_n^{-1}\Sigma_{k=1}^n v_k E(X_k I_{[|X_k| < B_k/|v_k|]})$ and $a_n \to 0$ if $\int_0^\infty x \int_x^\infty y^{-2} M_1(y)\, dy\, dF(x) < \infty$. In modifying these proofs, it should be noted that for $P = 1, 2$

$$\Sigma_k(|v_k|/B_k)^P E|X_k|^P I_{[|X_k| < B_k/|v_k|]} \leq \Sigma_k F(B_k/|v_k|) + \Sigma_k(|v_k|/B_k)^P \int_{[0, B_k/|v_k|)} x^P |dF(x)|.$$

(Integrate by parts, bound the tail probability by F and integrate by parts again.)

To apply this result to the sum of diagonal elements we consider $F_D(x) = \sup_k P[X_k^2 \geq x] = F(x^{\frac{1}{2}})$. The hypotheses of the modification of Heyde's result must be checked. Recall that $G(y) \to 0$ as $y \to \infty$ if and only if $F(y) \to 0$ as $y \to \infty$ and so $F_D(y) \to 0$ as $y \to \infty$. Also

$$\int_0^\infty x^2 \int_x^\infty y^{-3} N_D(y)\, dy |dF_D(x)| \leq \int_0^\infty x \int_x^\infty y^{-2} N_D(y)\, dy |dF_D(x)|$$

and so we need only to show that the latter is finite, but this is assumption (6).

PROOF OF THEOREM 7. To apply Theorem 6 we must show that (4), (5) and (6) hold. Since $N(x) \leq Cx^r$ for all $x \geq 0$ and some $C > 0$ and since $N(x) = 0$ for some $x > 0$, (4) and (5) are easily established. Since $N_D(x) \leq C_1 x^{r/2}$ for all $x \geq 0$ and some $C_1 > 0$, expression (6) is bounded above by $C_1(1 - r/2)^{-1}\int_0^\infty x^{r/2}|dF(x^{\frac{1}{2}})|$. Griffiths et al. (1973) have shown that there exist positive constants ε, δ for which $F(x) \leq \delta^{-1}G(\varepsilon x)$ for all $x \geq 0$ and so it suffices to show that $\int_1^\infty x^{r/2}|dG(\varepsilon x^{\frac{1}{2}})| < \infty$. However, changing the variable of integration this integral becomes $\varepsilon^{-r}\int_\varepsilon^\infty x^r |dG(x)|$, which is finite by assumption.

PROOF OF REMARK 8. We establish part (a) first. Let $X_1, X_1', X_2, X_2', \cdots$ be a sequence of independent and identically distributed variables. If Q_n is stable with respect to $\{B_n\}$ and some $\{a_n\}$ then $B_n^{-1}\Sigma_{k=1}^n w_{kk}(X_k^2 - (X_k')^2) \to 0$ a.s. and by Theorem 5 of Heyde $EN_D(|X_1^2 - (X_1')^2|) < \infty$. We now construct a discrete distribution for the X_k's which satisfies (8) but $EN_D(|X_1^2 - (X_1')^2|) = \infty$. Since $\limsup_{x\to\infty} N_D(x)/x^{r/2} = \infty$, select a sequence of numbers $x_k > 1$, which diverge monotonically and which satisfy $N_D(x_k^2)/x_k^r \geqslant k^2$ for $k = 1, 2, \cdots$. Consider the symmetric distribution which satisfies $P[X_1 = 0] = \frac{1}{4}$, $P[|X_1| = 1] = \frac{1}{4}$ and $P[|X_1| = x_k] = c/(k^2 x_k^r)$ for $k = 1, 2, \cdots$ where c is chosen so that $\Sigma_{k=1}^{\infty} P[|X_1| = x_k] = \frac{1}{2}$. The integral in (8), in the identically distributed case, is $E|X_1X_2|^r$ which is finite for this distribution since $E|X_1|^r = \frac{1}{4} + \Sigma_{k=1}^{\infty} ck^{-2}$. However, $EN_D(|X_1^2 - X_1'^2|) \geqslant \Sigma_{k=1}^{\infty} N_D(x_k^2) P[|X_1| = x_k] P[X_1' = 0] = 4^{-1} c \Sigma_{k=1}^{\infty} N_D(x_k^2) k^{-2} x_k^{-r} = \infty$.

In part (b), Q_n is again a sum of independent identically distributed variables if the X_k are independent and identically distributed. If Q_n is almost surely stable with respect to $\{B_n\}$ and some $\{a_n\}$, then by symmetrizing and applying Theorem 5 of Heyde we see that $EN(|X_1X_2 - X_1'X_2'|) < \infty$. Choose x_k as in the proof of part (a) except $N(x_k)/x_k^r \geqslant k^2$ and consider the same distribution given there. Again (8) holds but

$$EN(|X_1X_2 - X_1'X_2'|) \geqslant \Sigma_{k=1}^{\infty} N(x_k) P[|X_1| = x_k] P[|X_2| = 1] P[X_1' = 0] = 16^{-1} c \Sigma_{k=1}^{\infty} N(x_k) k^{-2} x_k^{-r} = \infty.$$

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