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HOW BIG ARE THE INCREMENTS OF A WIENER PROCESS?

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Let $\beta_T = (2a_T[\log(T/a_T) + \log\log T])^{-\frac{1}{2}}$, $0 < a_T \leqslant T < \infty$ and $\{W(t); 0 \leqslant t < \infty\}$ be a standard Wiener process. This paper studies the almost sure limiting behaviour of $\sup_{0\leqslant t\leqslant T-a_T} \beta_T|W(t + a_T) - W(t)|$ as $T \to \infty$ under varying conditions on a_T and T/a_T. With $a_T = T$ we get the law of iterated logarithm and with $a_T = c \log T$, $c > 0$, the Erdős-Rényi law of large numbers for the Wiener process. A number of other results for the Wiener process also follow via choosing a_T appropriately. Connections with strong invariance principles and the P. Lévy modulus of continuity for $W(t)$ are also established.

1. Introduction. The Erdős-Rényi law of large numbers ([2]) has the following form when applied to the Wiener process:

THEOREM A. *Let $W(t)$ $(0 \leqslant t < \infty)$ be a standard Wiener process. Then, for any $c > 0$, we have*

$$\lim_{T\to\infty} \sup_{0\leqslant t\leqslant T-c\log T} \frac{|W(t + c\log T) - W(t)|}{(2c)^{\frac{1}{2}} \log T} = 1 \qquad \text{w.p. } 1.$$

Strassen's law of iterated logarithm [10] implies:

THEOREM B. *Let $0 < c < 1$ and $W(t)$ $(0 \leqslant t < \infty)$ be a standard Wiener process. Then*

$$\limsup_{T\to\infty} \sup_{0\leqslant t\leqslant T-cT} \frac{|W(t + cT) - W(t)|}{(2cT \log\log T)^{\frac{1}{2}}} = 1 \qquad \text{w.p. } 1.$$

A common property of these two theorems is that they study the increments of a Wiener process on an interval $[0, T]$. The first one considers increments on subintervals of length $c \log T$ of $[0, T]$, the second one does the same on subintervals of length cT. In this paper we intend to investigate the increments of a Wiener process on subintervals of length $a_T \leqslant T$. Our main result is

THEOREM 1. *Let a_T $(T \geqslant 0)$ be a nondecreasing function of T for which*
(i) $0 < a_T \leqslant T$ $(T \geqslant 0)$,
(ii) *a_T/T is nonincreasing.*
Then

$$\limsup_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \beta_T|W(t + a_T) - W(t)| = 1 \qquad \text{w.p. } 1 \tag{1}$$

and

$$\limsup_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} \beta_T|W(t + s) - W(t)| = 1 \qquad \text{w.p. } 1, \tag{2}$$

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where $\beta_T = (2a_T[\log(T/a_T) + \log\log T])^{-\frac{1}{2}}$.

If we also have

(iii)

$$\lim_{T\to\infty} \frac{\log T/a_T}{\log\log T} = \infty,$$

then

$$\lim_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \beta_T |W(t+a_T) - W(t)| = 1 \qquad \text{w.p. 1}, \tag{3}$$

and

$$\lim_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} \beta_T |W(t+s) - W(t)| = 1 \qquad \text{w.p. 1}. \tag{4}$$

This theorem clearly implies Theorems A and B and it also implies the following well-known result:

THEOREM C.

$$\lim_{T\to\infty} \sup_{0\leqslant t\leqslant T-1} \frac{|W(t+1) - W(t)|}{(2\log T)^{\frac{1}{2}}} = 1 \qquad \text{w.p. 1}.$$

Our method of proof can also be used to prove

THEOREM 2. *Suppose that* a_T *satisfies conditions* (i), (ii) *of Theorem* 1. *Then*

$$\limsup_{T\to\infty} \beta_T |W(T+a_T) - W(T)| = 1 \qquad \text{w.p. 1}, \tag{5}$$

$$\limsup_{T\to\infty} \sup_{0\leqslant s\leqslant a_T} \beta_T |W(T+s) - W(T)| = 1 \qquad \text{w.p. 1}. \tag{6}$$

This theorem is a simple generalization of a theorem of Lai ([5], see also [6]) who proved this result under somewhat stronger conditions on a_T.

In their paper [2] Erdős and Rényi pointed out that their Theorem A is related to the following theorem of Lévy ([7], see also [9]):

THEOREM D. *We have*

$$\lim_{h\to 0} \sup_{0\leqslant t\leqslant 1} \frac{|W(t+h) - W(t)|}{(2h\log 1/h)^{\frac{1}{2}}}$$

$$= \lim_{h\to 0} \sup_{0\leqslant t\leqslant 1} \sup_{0\leqslant s\leqslant h} \frac{|W(t+s) - W(t)|}{(2h\log 1/h)^{\frac{1}{2}}} = 1 \qquad \text{w.p. 1}.$$

The exact relationship between Theorems A and D and also that of Theorems 1 and D is not completely clear yet. All three of them can, however, be proved from our Lemma 1 of Section 2, but it is probably not true that they should directly imply each other.

Chan ([1]) also proved a theorem which is closely related to our Theorem 1, and which deals with the multi-time parameter Wiener process. His condition concerning a_T is more restrictive than those in this paper.

Our Theorem 1 and the strong invariance principle of Komlós-Major-Tusnády ([3]) easily imply:

THEOREM 3. *Let* $X_1, X_2, \cdots$ *be a sequence of* i.i.d. rv's *satisfying the conditions*
(i) $EX_1 = 0$, $EX_1^2 = 1$,
(ii) *there exists a* $t_0 > 0$ *such that* $Ee^{tX_1} < \infty$ *if* $|t| < t_0$.
Then for the sums $S_n = X_1 + X_2 + \cdots + X_n$ *we have*

$$\limsup_{n\to\infty} \sup_{1 \leqslant k \leqslant n-a_n} \beta_n |S_{k+a_n} - S_k| = 1 \qquad \text{w.p. } 1$$

provided that a_n *satisfies conditions* (i)–(ii) *of Theorem* 1, *and* $a_n/\log n \to \infty$. *The analogous statements for* S_n *fashioned after* (2), (3), (4), (5) *and* (6) *are similarly true.*

The case when $a_n = c \log n$, $c > 0$, was first treated by Erdős and Rényi ([2]) and further developed by Komlós and Tusnády ([4]). The case when $a_n = o(\log n)$ seems to be unknown. These two cases cannot be treated by invariance-principle-like methods.

Assuming some stronger restrictions on a_n and applying some further results of [3] (see also [8]), condition (ii) of Theorem 3 can be replaced by weaker moment conditions and results like those in [6] can be similarly proved by invariance considerations.

2. An inequality. In this section we prove our

LEMMA 1. *For any* $\varepsilon > 0$ *there exists a constant* $C = C(\varepsilon) > 0$ *such that the inequality*

$$(7) \quad P\left\{\sup_{0 \leqslant s,\, s+t \leqslant 1} \sup_{0 \leqslant t \leqslant h} |W(s+t) - W(s)| \geqslant v h^{\frac{1}{2}}\right\} \leqslant C h^{-1} e^{-v^2/(2+\varepsilon)}$$

holds for every positive v *and* $0 < h < 1$.

PROOF. Let R be the smallest integer for which $1/R \leqslant \varepsilon^2 h/4$. Then for each $\omega \in \Omega$ we have

$$\begin{aligned}\sup_{0 \leqslant s,\, s+t \leqslant 1} & \sup_{0 \leqslant t \leqslant h} |W(s+t) - W(s)| \\ & \leqslant \max_{0 \leqslant i \leqslant R-1} \sup_{0 \leqslant t \leqslant h} \left|W\left(\frac{i}{R} + t\right) - W\left(\frac{i}{R}\right)\right| \\ & \quad + 2 \max_{0 \leqslant i \leqslant R-1} \sup_{0 \leqslant \tau \leqslant (1/R)} \left|W\left(\frac{i}{R} + \tau\right) - W\left(\frac{i}{R}\right)\right|\end{aligned}$$

and for any $x > 0$

$$P\left\{\max_{0 \leqslant i \leqslant R-1} \sup_{0 \leqslant t \leqslant h} \left|W\left(\frac{i}{R} + t\right) - W\left(\frac{i}{R}\right)\right| \geqslant x h^{\frac{1}{2}}\right\} \leqslant 4Re^{-x^2/2} \leqslant Ch^{-1}e^{-x^2/2},$$

$$\begin{aligned} & P\left\{2 \max_{0 \leqslant i \leqslant R-1} \sup_{0 \leqslant \tau \leqslant (1/R)} \left|W\left(\frac{i}{R} + \tau\right) - W\left(\frac{i}{R}\right)\right| \geqslant \varepsilon x h^{\frac{1}{2}}\right\} \\ & \leqslant P\left\{\max_{0 \leqslant i \leqslant R-1} \sup_{0 \leqslant \tau \leqslant (1/R)} \left|W\left(\frac{i}{R} + \tau\right) - W\left(\frac{i}{R}\right)\right| \geqslant x R^{-\frac{1}{2}}\right\} \\ & \leqslant 4Re^{-x^2/2} \leqslant Ch^{-1}e^{-x^2/2}.\end{aligned}$$

Combining the above three inequalities we get

(8) $P\{\sup_{0\le s,\, s+t\le 1}\sup_{0\le t\le h}|W(s+t)-W(s)| \geq xh^{\frac{1}{2}}+\varepsilon xh^{\frac{1}{2}}\} \leq 2Ch^{-1}e^{-x^2/2}$.

Choosing $v = x(1+\varepsilon)$ in (8) we have our inequality (7) with a different choice of ε and C.

A simple analogue of this lemma is

LEMMA 1*. *For any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that the inequality*

$$P\{\sup_{0\le s,\, s+t\le T}\sup_{0\le t\le h}|W(s+t)-W(s)| \geq vh^{\frac{1}{2}}\} \leq CTh^{-1}e^{-v^2/(2+\varepsilon)}$$

holds for every positive v, h and T.

This Lemma 1* follows from Lemma 1 and from the following observation:

LEMMA A. *For any fixed $T > 0$ we have*

$$\{W(s);\ 0 \leq s \leq T\} =_{\mathfrak{D}} \{T^{\frac{1}{2}}W(sT^{-1});\ 0 \leq s \leq T\},$$

that is, for any $0 \leq s_1 < s_2 < \cdots < s_n \leq T$ $(n = 1, 2, \cdots)$ the joint distributions of $\{W(s_1), W(s_2), \cdots, W(s_n)\}$ and that of $\{T^{\frac{1}{2}}W(s_1T^{-1}), T^{\frac{1}{2}}W(s_2T^{-1}), \cdots, T^{\frac{1}{2}}W(s_nT^{-1})\}$ are equal to each other.

3. Proof of Theorem 1.

STEP 1. *Let*

$$A(T) = \sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\beta_T|W(t+s)-W(t)|.$$

Suppose that conditions (i), (ii) *of Theorem* 1 *are fulfilled. Then*

$$\limsup_{T\to\infty} A(T) \leq 1 \quad \text{w.p. } 1. \tag{9}$$

PROOF. By Lemma 1* we have

$$P(A(T) \geq 1+\varepsilon) \leq C\frac{T}{a_T}\exp\left\{-(1+\varepsilon)\left[\log\frac{T}{a_T}+\log\log T\right]\right\}$$
$$= C\left(\frac{a_T}{T}\right)^{\varepsilon}\frac{1}{(\log T)^{1+\varepsilon}}.$$

Let $T_k = \theta^k$ $(\theta > 1)$. Then

$$\Sigma_{k=1}^{\infty}P(A(T_k) \geq 1+\varepsilon) < \infty$$

for every $\varepsilon > 0$, $\theta > 1$, hence by the Borel-Cantelli lemma

$$\limsup_{k\to\infty} A(T_k) \leq 1 \quad \text{w.p. } 1. \tag{10}$$

We also have

$$1 \leq \frac{\beta_{T_k}}{\beta_{T_{k+1}}} \leq \theta \tag{11}$$

if k is big enough.

Now choosing θ near enough to 1, (9) follows from (10) and (11), because $\beta_T^{-1} A(T)$ is nondecreasing and β_T is nonincreasing in T.

STEP 2. *Let*

$$B(T) = \beta_T |W(T) - W(T - a_T)|.$$

Suppose that conditions (i), (ii) *of Theorem* 1 *are fulfilled. Then*

$$\limsup_{T\to\infty} B(T) \geqslant 1 \quad \text{w.p. } 1. \tag{12}$$

PROOF. We have

$$P(B(T) \geqslant 1 - \varepsilon) \geqslant \frac{\exp\left\{-(1-\varepsilon)^2\left[\log\frac{T}{a_T} + \log\log T\right]\right\}}{(2\pi)^{\frac{1}{2}}\left(2\left(\log\frac{T}{a_T} + \log\log T\right)\right)^{\frac{1}{2}}} \geqslant \left(\frac{a_T}{T\log T}\right)^{1-\varepsilon} \tag{13}$$

for T big enough.

Let $T_1 = 1$ and define T_{k+1} by

$$T_{k+1} - a_{T_{k+1}} = T_k \qquad \text{if} \quad \rho < 1$$

and

$$T_{k+1} = \theta^{k+1} \qquad \text{if} \quad \rho = 1,$$

where $\theta > 1$ and $\lim_{T\to\infty} a_T/T = \rho$ (we note that our conditions (i) and (ii) imply that a_T is a continuous function of T and $T - a_T$ is a strictly monotone increasing function of T).

In case $\rho < 1$, (12) follows from the simple fact that

$$\sum_{k=2}^{\infty}\left(\frac{a_{T_k}}{T_k \log T_k}\right)^{1-\varepsilon} = \infty$$

and that the rv's $B(T_k)$ $(k = 1, 2, \cdots)$ are independent.

In case $\rho = 1$, $a_{T_{k+1}} \geqslant T_{k+1} - T_k$ (if k is big enough), hence

$$B(T_{k+1}) \geqslant \beta_{T_{k+1}} |W(T_{k+1}) - W(T_k)| - \beta_{T_{k+1}} \sup_{0\leqslant u\leqslant v\leqslant T_k} |W(v) - W(u)|.$$

By Step 1,

$$\limsup_{k\to\infty} \beta_{T_{k+1}} \sup_{0\leqslant u\leqslant v\leqslant T_k} |W(v) - W(u)| \leqslant 2\theta^{-\frac{1}{2}}.$$

We also have

$$P\{\beta_{T_{k+1}} |W(T_{k+1}) - W(T_k)| \geqslant 1 - \varepsilon\} = O(k^{-(1-\varepsilon)^2\theta/(\theta-1)}).$$

The latter two formulas imply (12), since θ can be arbitrarily large.

STEP 3. *Let*

$$C(T) = \sup_{0\leqslant t\leqslant T-a_T} \beta_T |W(t + a_T) - W(t)|.$$

Suppose the conditions (i)–(iii) *of Theorem* 1 *are fulfilled. Then*

(14) $$\liminf_{T\to\infty} C(T) \geqslant 1 \qquad \text{w.p. 1.}$$

PROOF. Since the rv's

$$\beta_T|W((k+1)a_T) - W(ka_T)| \qquad (k = 0, 1, 2, \cdots, [T/a_T] - 1)$$

are independent, by (13) we have

$$P\{\max_{0\leqslant k\leqslant [T/a_T]-1}\beta_T|W((k+1)a_T) - W(ka_T)| \leqslant 1-\varepsilon\}$$
$$\leqslant \left(1 - \left(\frac{a_T}{T\log T}\right)^{1-\varepsilon}\right)^{[T/a_T]} \leqslant 2\exp\left\{-\left(\frac{T}{a_T}\right)^{\varepsilon}\left(\frac{1}{\log T}\right)^{1-\varepsilon}\right\}.$$

By condition (iii) we have

$$\Sigma_{j=1}^{\infty} \exp\left\{-\left(\frac{j}{a_j}\right)^{\varepsilon}\left(\frac{1}{\log j}\right)^{1-\varepsilon}\right\} < \infty,$$

and whence, so far, we have proved

(15) $$\liminf_{j\to\infty} C(j) \geqslant \liminf_{j\to\infty} \max_{0\leqslant k\leqslant [j/a_j]-1} \beta_j|W((k+1)a_j) - W(ka_j)| \geqslant 1 \qquad \text{w.p. 1.}$$

Considering now the case of in-between-times $j \leqslant T < j+1$, we first observe that $0 \leqslant a_T - a_j$ by condition (i), and that, by condition (ii), $0 \leqslant a_T - a_j \leqslant a_j/j \leqslant \delta a_j$ for any $\delta > 0$, if $j \leqslant T < j+1$ and j is big enough. (The latter inequality is immediate, since $a_T/T \leqslant a_j/j$ by (ii), and so, via $a_T \leqslant a_j(T/j)$, we have $a_T - a_j \leqslant a_j((T/j) - 1) \leqslant a_j/j$). Whence, for $j \leqslant T < j+1$ and j large, we have

(16) $$\begin{aligned} C(T) \geqslant\ & \max_{0\leqslant k\leqslant [j/a_j]-1} \beta_{j+1}|W((k+1)a_j) - W(ka_j)| \\ & - \sup_{0\leqslant t\leqslant T-\delta a_T} \sup_{0\leqslant s\leqslant \delta a_T} \beta_T|W(t+s) - W(t)|. \end{aligned}$$

On the other hand, by Step 1 we have

$$\limsup_{T\to\infty} \sup_{0\leqslant t\leqslant T-\delta a_T} \sup_{0\leqslant s\leqslant \delta a_T} \beta_T|W(t+s) - W(s)|$$
$$\leqslant \limsup_{T\to\infty} \frac{\left(2\delta a_T\left(\log\dfrac{T}{\delta a_T} + \log\log T\right)\right)^{\frac{1}{2}}}{\left(2a_T\left(\log\dfrac{T}{a_T} + \log\log T\right)\right)^{\frac{1}{2}}} = \delta^{\frac{1}{2}}.$$

This, by (15) and (16), also completes the proof of (14).

REMARK. In the submitted version of our paper we have asked the question whether the statements of (3) and (4) can be true if (iii) fails. In his report the referee gave a negative answer to this question, outlining a proof of the following statement:

$$\begin{aligned}(17)\quad &\liminf_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \beta_T|W(t+a_T)-W(t)| \\ &= \liminf_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} \beta_T|W(t+s)-W(t)| = \left(\frac{r}{r+1}\right)^{\frac{1}{2}},\end{aligned}$$

w.p. 1, where

(iii)*

$$r = \liminf_{T\to\infty} \frac{\log T/a_T}{\log\log T}, \qquad 0 \leqslant r \leqslant \infty.$$

Earlier S. A. Book and T. R. Shore (personal communication) gave the same negative answer. C. M. Deo (personal communication) also gave a negative answer to our question. In the light of (17) we formulate the

Problem. Find the normalizing factors $\gamma_T^{(1)}$, $\gamma_T^{(2)}$ for which we have

$$\begin{aligned}(18)\quad &\liminf_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \gamma_T^{(1)}|W(t+a_T)-W(t)| \\ &= \liminf_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} \sup_{0\leqslant s\leqslant a_T} \gamma_T^{(2)}|W(t+s)-W(t)| = 1 \qquad \text{w.p. } 1,\end{aligned}$$

when r of (iii)* is equal to zero.

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