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ASYMPTOTIC INDEPENDENCE IN THE MULTIVARIATE CENTRAL LIMIT THEOREM[1]

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Necessary and sufficient conditions are given for asymptotic independence in the multivariate central limit theorem. If $\{X_n\}$ is a sequence of independent, identically distributed random variables whose common distribution is symmetric, and if the distribution of X_1^2 is in the domain of attraction of a stable distribution of characteristic exponent α, then $\bar{X}$ and s^2 are asymptotically independent if and only if $1 \leqslant \alpha \leqslant 2$. If the components of a multivariate infinitely divisible distribution are pairwise independent, then they are independent.

1. Introduction and summary. One particular question related to the multivariate central limit theorem is that of asymptotic independence of the component random variables. In [10] and in [4] the special case was asymptotic independence of the sums of the positive parts and of the negative parts for an infinitesimal system of random variables satisfying the univariate central limit theorem. A use for such asymptotic independence is given in [1]. In what follows a brief look will be taken at the general problem of asymptotic independence with solutions provided for two particular problems.

The results obtained here are as follows. First a general theorem is obtained giving necessary and sufficient conditions for the components in the multivariate central limit theorem to be asymptotically independent. This is applied toward obtaining the main result: if $\{X_n\}$ are independent, identically distributed random variables with symmetric common distribution function F, and if the distribution function G of X_1^2 is in the domain of attraction of some stable distribution, then

$$\bar{X}_n = (X_1 + \cdots + X_n)/n \tag{1}$$

and

$$s_n^2 = \frac{1}{n-1}\Sigma_{i=1}^{n}(X_i - \bar{X}_n)^2 \tag{2}$$

are asymptotically independent if and only if G belongs to the domain of attraction of a stable distribution of characteristic exponent not less than 1. Finally, a theorem due to Pierre [7] is proved without any assumption of finite moments: if $\{X_\theta, \theta \in \Theta\}$ is a family of random variables such that every finite dimensional

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marginal distribution is infinitely divisible, then pair-wise independence implies independence.

It is of course recognized that the theorem on asymptotic independence of $\bar{X}_n$ and s_n^2 extends in a way the classical theorem due to Geary, Lukacs, Kawata, Sakamoto and Zinger that states: if $X_1, \cdots, X_n$ are independent and identically distributed, then $\bar{X}_n$ and s_n^2 are independent if and only if their common distribution is normal. (See [6], page 80 for details and references.) The basic assumption on our extension must contain the hypothesis that G is in the domain of attraction of some stable distribution, for otherwise there would be no joint convergence even of the partial sums and the partial sums of squares of $\{X_n\}$. The underlying assumption of symmetry of F might be removable. The weakening of the hypothesis in Pierre's theorem shows that in the multivariate central limit theorem, asymptotic pairwise independence of the components implies asymptotic independence.

The starting point is the general form of the multivariate central limit theorem, possibly given for the first time by Rvacheva [8] and quoted by us in [4]. We quote it here for quick reference:

THEOREM 1. *Let $\{\{\mathbf{X}_{nj}\}\}$ be an infinitesimal system of row-wise independent p-dimensional random vectors, and define the probability measure H_{nj} over the measurable space $(\mathbb{R}^p, \mathfrak{P}^{(p)})$ by $H_{nj}(A) = P[\mathbf{X}_{nj} \in A]$ for all $A \in \mathfrak{P}^{(p)}$. Then there exists a sequence $\{\mathbf{c}_n\} \subset \mathbb{R}^p$ such that the distribution of $\Sigma_{j=1}^{k_n}\mathbf{X}_{nj} + \mathbf{c}_n$ converges completely to a (necessarily infinitely divisible) distribution function F if and only if there exist a Lévy spectral measure N over the Borel subsets $\mathfrak{P}^{(p)}$ of $\mathbb{R}^p$ and a nonnegative definite quadratic form $Q(\mathbf{u})$ defined over $\mathbb{R}^p$ which satisfy the following:*

(i) *for Borel sets of the form $S = \{\mathbf{x} \in \mathbb{R}^p : |\mathbf{x}| > R,\ \omega_{\mathbf{x}} \in A\}$, where A is a Borel subset of the surface of the unit sphere, $\omega_{\mathbf{x}}$ denotes the point of intersection of the vector $\mathbf{x}$ with the surface of the unit sphere, and such that $N(\text{bdry } S) = 0$,*

$$\Sigma_{j=1}^{k_n} H_{nj}(S) \to N(S) \qquad as \quad n \to \infty,$$

and

(ii) $\lim_{\varepsilon\downarrow 0} \overline{\lim}_{n\to\infty} \Sigma_{j=1}^{k_n}\{\int_{\|\mathbf{x}\|<\varepsilon}(\mathbf{u}^t\mathbf{x})^2 H_{nj}(d\mathbf{x}) - (\int_{\|\mathbf{x}\|<\varepsilon}\mathbf{u}^t\mathbf{x} H_{nj}(d\mathbf{x}))^2\} = Q(\mathbf{u})$, *where* $\mathbf{u}^t\mathbf{x} = \Sigma_{i=1}^p u_i x_i$. *The characteristic function $\hat{F}(\mathbf{u})$ of $F(\mathbf{x})$ is given by*

$$\hat{F}(\mathbf{u}) = \exp\left\{ i\gamma^t\mathbf{u} - Q(\mathbf{u})/2 + \int_{\|\mathbf{x}\|>0}\left(e^{i\mathbf{u}^t\mathbf{x}} - 1 - \frac{i\mathbf{u}^t\mathbf{x}}{1 + \|\mathbf{x}\|^2}\right) N(d\mathbf{x}) \right\},$$

where $\gamma \in \mathbb{R}^p$ is constant.

2. Asymptotic independence of components. The general theorem for asymptotic independence is as follows. Notation is provided in Section 1.

THEOREM 2. *Let $\{\{\mathbf{X}_{nj}, 1 \leqslant j \leqslant k_n\}\}$ be an infinitesimal system of row-wise independent p-dimensional random vectors for which there exists a sequence $\{\mathbf{c}_n\}$ of p-dimensional constant vectors such that the distribution of $\Sigma_{j=1}^{k_n}\mathbf{X}_{nj} + \mathbf{c}_n$ converges to that of some random vector $\mathbf{X}$. Necessary and sufficient conditions for the components* *of* $\mathbf{X}$ *to be independent are*:

(i) *for every closed ball* $S \subset \mathbb{R}^p$ *that has an empty intersection with each axis in* $\mathbb{R}^p$, $\Sigma_{j=1}^{k_n} H_{nj}(S) \to 0$ *as* $n \to \infty$, *and*

(ii) *for* $1 \leqslant i < j \leqslant p$,

$$\lim_{\varepsilon \downarrow 0} \overline{\lim}_{n \to \infty} \Sigma_{l=1}^{k_n} \Big\{ \textstyle\int_{|x_i| < \varepsilon;\, |x_j| < \varepsilon} x_i x_j H_{nl}(d\mathbf{x}) - \int_{|x_i| < \varepsilon} x_i H_{nl}(d\mathbf{x}) \int_{|x_j| < \varepsilon} x_j H_{nl}(d\mathbf{x}) \Big\} = 0.$$

PROOF. For $1 \leqslant r \leqslant p$, let $X_{nj;\,r}$, X_r and $c_{n;\,r}$ denote the rth components of $\mathbf{X}_{nj}$, $\mathbf{X}$ and $\mathbf{c}_n$ respectively. We first prove that conditions (i) and (ii) are sufficient. From the basic hypothesis it follows that, for $1 \leqslant r \leqslant p$, the distribution of $\Sigma_{j=1}^{k_n} X_{nj;\,r} + c_{n;\,r}$ converges completely to that of X_r (since marginals of a convergent sequence converge to the marginal of the limit). Thus by the converse of the univariate central limit theorem, it follows that

$$\lim_{\varepsilon \downarrow 0} \overline{\lim}_{n \to \infty} \Sigma_{j=1}^{k_n} \Big\{ \textstyle\int_{|x_r| < \varepsilon} x_r^2 H_{nj}(d\mathbf{x}) - \big(\int_{|x_r| < \varepsilon} x_r H_{nj}(d\mathbf{x}) \big)^2 \Big\} = \text{(some)} \quad \sigma_{rr} \geqslant 0. \tag{3}$$

Thus by (3) and condition (ii) we have

$$\lim_{\varepsilon \downarrow 0} \overline{\lim}_{n \to \infty} \Sigma_{j=1}^{k_n} \Big\{ \textstyle\int_{\|\mathbf{x}\| < \varepsilon} (\mathbf{u}'\mathbf{x})^2 H_{nj}(d\mathbf{x}) - \big(\int_{\|\mathbf{x}\| < \varepsilon} (\mathbf{u}'\mathbf{x}) H_{nj}(d\mathbf{x}) \big)^2 \Big\} = \Sigma_{r=1}^{p} \sigma_{rr} u_r^2. \tag{4}$$

If, for $A \in \mathfrak{B}^{(1)}$, we define $M_r(A) = N\{\mathbf{x} : x_r \in A\}$, then by condition (i) and by (4) we determine the characteristic function of $\mathbf{X}$ to be

$$f_{\mathbf{X}}(\mathbf{u}) = \Pi_{r=1}^{p} \exp\left\{ i\gamma_r u_r - \sigma_{rr} u_r^2/2 + \int_{x_r \neq 0} \left(e^{iu_r x_r} - 1 - \frac{iu_r x_r}{1 + x_r^2} \right) M_r(dx_r) \right\},$$

i.e., the components of $\mathbf{X}$ are independent. Conversely, let us assume that the components of $\mathbf{X}$ are independent. Since each component, X_r, of $\mathbf{X}$ is infinitely divisible, then its characteristic function may be written in the form

$$f_{X_r}(u_r) = \exp\left\{ i\gamma_r u_r - \sigma_{rr} u_r^2/2 + \int_{x_r \neq 0} \left(e^{iu_r x_r} - 1 - \frac{iu_r x_r}{1 + x_r^2} \right) M_r(dx_r) \right\}, \tag{5}$$

and thus we obtain as the characteristic function of $\mathbf{X}$ the following:

$$f_{\mathbf{X}}(\mathbf{u}) = \exp\left\{ i\gamma'\mathbf{u} - \tfrac{1}{2}\Sigma_{r=1}^{p} \sigma_{rr} u_r^2 + \Sigma_{r=1}^{p} \int_{x_r \neq 0} \left(e^{iu_r x_r} - 1 - \frac{iu_r x_r}{1 + x_r^2} \right) M_r(dx_r) \right\}.$$

However, by the hypothesis and Theorem 1 we know there exist a vector $\mathbf{a} \in \mathbb{R}^p$, a nonnegative definite quadratic form $Q(\mathbf{u})$ over $\mathbb{R}^p$ and a Lévy spectral measure N over $\mathbb{R}^p \setminus \{0\}$ such that

$$f_{\mathbf{X}}(\mathbf{u}) = \exp\left\{ i\mathbf{a}'\mathbf{u} - Q(\mathbf{u})/2 + \int_{x\neq 0}\left(e^{i\mathbf{u}'\mathbf{x}} - 1 - \frac{i\mathbf{u}'\mathbf{x}}{1+\mathbf{x}'\mathbf{x}}\right)N(d\mathbf{x})\right\}.$$

By the uniqueness of the canonical representation of $\mathbf{a}$, $Q(\mathbf{u})$, $N(\cdot)$, it follows that N is the measure over $(\mathbb{R}^p, \mathfrak{B}^{(p)})$ concentrated along the *axes only* of $\mathbb{R}^p$ and defined by (for $A \in \mathfrak{B}^{(p)}$)

$$N(A) = \Sigma_{r=1}^{p} M_r\{x_r : (0, \cdots, 0, x_r, 0, \cdots, 0) \in A\},$$

$\mathbf{a} = \gamma$, and $Q(\mathbf{u}) = \Sigma_{r=1}^{p}\sigma_{rr}u_r^2$. Therefore, for every closed ball $S \subset \mathbb{R}^p$ that has an empty intersection with each axis, $N(S) = 0$. By Theorem 1, condition (i) follows. We now show that condition (ii) is true. Since the rth components obey the (univariate) central limit theorem, it follows that (3) holds for $1 \leqslant r \leqslant p$. In condition (ii) in the statement of Theorem 1, let $u_i = u_j = 1$ for $1 \leqslant i < j \leqslant p$, and let $u_k = 0$, $k \neq i$, $k \neq j$, $1 \leqslant k \leqslant p$. Combining this with (3) yields condition (ii) of our theorem.

3. Asymptotic independence of $\bar{X}_n$ and s_n^2. If $\{\mathbf{U}_n\}$ is a sequence of p-dimensional random vectors, we shall define what is meant by asymptotic independence of the components. Let the components of $\mathbf{U}_n$ be denoted by $U_{n1}, \cdots, U_{np}$. If there exists a p-dimensional random vector $\mathbf{U}$ with independent components, none of which is a constant, and if there exist numbers $b_{n1}, \cdots, b_{np}, a_{n1}, \cdots, a_{np}$ such that the joint distribution function of $b_{n1}U_{n1} + a_{n1}, \cdots, b_{np}U_{np} + a_{np}$ converges completely to that of $\mathbf{U}$ as $n \to \infty$, then we shall say that the components of $\mathbf{U}_n$ or of $\{\mathbf{U}_n\}$ are asymptotically independent.

First a word about notation. The symbols $\bar{X}_n$, s_n^2 and H_{nj} have been defined in Section 1. For $0 < \alpha \leqslant 2$, $\mathfrak{D}(\alpha)$ denotes the domain of attraction of the class of stable distributions of characteristic exponent α, and $\mathfrak{D}_{\mathfrak{N}}(\alpha)$ refers to the domain of normal attraction for α. If $\mathbf{Z}$ is a random vector, $F_{\mathbf{Z}}$ will always denote its distribution function.

THEOREM 3. *Let $\{X_n\}$ be a sequence of independent, identically distributed random variables with common symmetric distribution function F and such that the distribution function G of X_1^2 is in $\mathfrak{D}(\alpha)$ for some $\alpha \in (0, 2]$. Then $\bar{X}_n$ and s_n^2 are asymptotically independent if and only if $G \in \mathfrak{D}(\alpha)$ for some $\alpha \in [1, 2]$.*

PROOF. Let us assume that $G \in \mathfrak{D}(\alpha)$ for some $\alpha \in (1, 2]$. Then it is known that $E|X|^{2\delta} < \infty$ for $0 \leqslant \delta < \alpha$ (see [3], page 179), and thus $EX^2 < \infty$, which implies that $F \in \mathfrak{D}_{\mathfrak{N}}(2)$. It is also known (see Lemma 5 in [10]) that if $G \in \mathfrak{D}(\alpha)$, then there exists a slowly varying function $L(x)$ (where $L(x) \sim$ const. if and only if $G \in \mathfrak{D}_{\mathfrak{N}}(\alpha)$, and $L(x) \to \infty$ as $x \to \infty$ if $G \in \mathfrak{D}(2) \setminus \mathfrak{D}_{\mathfrak{N}}(2)$) such that for some sequence of constants $\{d_n\}$,

$$\frac{1}{n^{1/\alpha}L(n)}\Sigma_{i=1}^{n}X_i^2 + d_n$$

converges in law to a stable distribution with characteristic exponent α. Since F is symmetric and in $\mathfrak{D}_{\mathfrak{N}}(2)$, then the distribution of $n^{-\frac{1}{2}}\Sigma_{i=1}^{n}X_i$ converges completely to $\mathfrak{N}(0, \mu_2)$, where $\mu_2 = EX_1^2$. Let $p = 2$ now, $X_{nk;\,1} = X_k/n^{\frac{1}{2}}$, $X_{nk;\,2} = X_k^2/n^{1/\alpha}L(n)$ and $\mathbf{X}_{nk}$ be the vector random variable whose components are $X_{nk;\,1}$ and $X_{nk;\,2}$. We shall show first that the infinitesimal system $\{\{\mathbf{X}_{nk}, 1 \leqslant k \leqslant n\}\}$ obeys Theorem 1 and that the component summands are asymptotically independent. Let $\mathbf{a}_n = \begin{pmatrix} 0 \\ d_n \end{pmatrix}$; we shall first show that $\Sigma_{k=1}^{n}\mathbf{X}_{nk} + \mathbf{a}_n$ satisfies condition (i) of Theorem 1 as well as (i) in Theorem 2. We do it first for $\alpha = 2$. If $G \in \mathfrak{D}(2)$, then by the univariate central limit theorem $nP[|X_{n1;\,1}| \geqslant \varepsilon] \to 0$ and $nP(|X_{n1;\,2}| \geqslant \varepsilon] \to 0$ as $n \to \infty$ for every $\varepsilon > 0$, thus establishing (i) in both theorems when $\alpha = 2$. We shall next show that (i) is verified in Theorems 1 and 2. Indeed, for any $\varepsilon > 0$ (now $1 < \alpha < 2$),

$$nP\left[|X_1| \geqslant \varepsilon n^{\frac{1}{2}}, X_1^2 \geqslant \varepsilon n^{1/\alpha}L(n)\right] = nP\left[X_1^2 \geqslant \varepsilon^2 n, X_1^2 \geqslant \varepsilon n^{1/\alpha}L(n)\right].$$

By a theorem of Karamata ([5], page 59), if $Q(x)$ is a slowly varying function, $x^{\delta}Q(x) \to \infty$ as $x \to \infty$ for all $\delta > 0$. Since $1/L(x)$ is slowly varying, it follows that for all large n, $n^{1/\alpha}L(n)\varepsilon < \varepsilon^2 n$, and hence each side of the above equation is eventually equal to $nP[X_1^2 \geqslant \varepsilon^2 n]$. Since $G \in \mathfrak{D}(\alpha)$, $1 < \alpha < 2$, there is a slowly varying function $Q(x)$ such that

$$P\left[X_1^2 > x\right] \sim x^{-\alpha}Q(x).$$

(See [2].) Hence $nP(X_1^2 \geqslant \varepsilon^2 n] \sim \varepsilon^{-2\alpha}n^{-(\alpha-1)}Q(\varepsilon^2 n)$. By Karamata's theorem above and the fact that $\alpha > 1$, we observe that $n^{-(\alpha-1)}Q(\varepsilon^2 n) \to 0$ as $n \to \infty$, thus verifying (i) in both Theorems 1 and 2 for $1 < \alpha < 2$. We have yet to establish condition (ii) in both theorems for $\alpha \in (1, 2]$. By the symmetry of F, we know that condition (ii) is true in Theorem 2 without even taking limits. This symmetry also implies that the matrix of the quadratic form on the left side of the equation of condition (ii) of Theorem 1 is diagonal. Thus, it remains to prove that the iterated limits

$$\lim_{\varepsilon\downarrow 0} \overline{\underline{\lim}}_{n\to\infty}\Sigma_{j=1}^{n}\Sigma_{r=1}^{2}u_r^2 \operatorname{Var}\left(X_{nj;\,r}I_{[|X_{nj;\,r}|<\varepsilon]}\right)$$

exist and are equal; if they do, then automatically the limit is a nonnegative definite quadratic form. By the univariate central limit theorem, since $F_{X_1} \in D_{\mathfrak{N}}(2)$, it follows that

$$\lim_{\varepsilon\downarrow 0} \overline{\underline{\lim}}_{n\to\infty}\Sigma_{j=1}^{n} \operatorname{Var}\left(X_{nj;\,1}I_{[|X_{nj;\,1}|<\varepsilon]}\right) = \mu_2.$$

Also, since $F_{X_1^2} \in \mathfrak{D}(\alpha)$, then the same theorem implies that both limits of $\lim_{\varepsilon\downarrow 0} \overline{\underline{\lim}}_{n\to\infty}\Sigma_{j=1}^{n} \operatorname{Var}(X_{nj;\,2}I_{[|X_{nj;\,2}|\leqslant\varepsilon]})$ exist and are equal, being positive if and only if $\alpha = 2$. Thus we have shown that the joint distribution of

$$n^{\frac{1}{2}}\overline{X}_n \quad \text{and} \quad \frac{1}{n^{1/\alpha}L(n)}\Sigma_{i=1}^n X_i^2 + d_n$$

converges to a bivariate distribution of a random vector with independent components. We now observe that

$$\frac{1}{n^{1/\alpha}L(n)}\Sigma_{i=1}^n X_1^2 + d_n = \frac{n-1}{n^{1/\alpha}L(n)}s_n^2 + d_n + \frac{n^{1-1/\alpha}}{L(n)}\overline{X}_n^2$$

and

$$\frac{n^{1-1/\alpha}}{L(n)}\overline{X}_n^2 = \frac{1}{n^{1/\alpha}L(n)}\left(n^{-\frac{1}{2}}\Sigma_{i=1}^n X_i\right)^2.$$

Since the distribution of $n^{-\frac{1}{2}}\Sigma_{i=1}^n X_i$ converges to that of $N(0, \mu_2)$, it follows that $(n^{1/\alpha}L(n))^{-1}(n^{-\frac{1}{2}}\Sigma_{i=1}^n X_i)^2 \to_p 0$. Thus we obtain in the case $\alpha \in (1, 2]$ that the joint distribution of $n^{\frac{1}{2}}\overline{X}_n$ and $((n-1)/n^{1/\alpha}L(n))s_n^2 + d_n$ converges to a bivariate distribution function of independent random variables. We must next show asymptotic independence of $\overline{X}_n$ and s_n^2 when $\alpha = 1$, i.e., $G \in \mathfrak{D}(1)$. In this case there exist a slowly varying function $Q(x)$ and a sequence of constants $\{d_n\}$ such that $(nQ(n))^{-1}\Sigma_{i=1}^n X_i^2 + d_n$ converges in law to a stable distribution of characteristic exponent $\alpha = 1$. It is also known (see [3], page 176) that Q must satisfy $nP[X_1^2 \geqslant cnQ(n)] \sim K \neq 0$, where K and c are positive constants which determine each other. There is also a slowly varying function $R(x)$ such that $P[X_1^2 \geqslant x] \sim x^{-1}R(x)$, from which we obtain $P[X_1 \geqslant x] = P[X_1 \leqslant -x] \sim x^{-2}R(x^2)/2$. Now $R(x^2)$ is a slowly varying function of x; thus, by Theorem 4 in [9], $F \in \mathfrak{D}(2)$. By Lemma 5 of [9], there is a nondecreasing slowly varying function $S(x)$ such that $\{n^{\frac{1}{2}}S(n)\}$ serve as normalizing coefficients for F. (Note: since F is symmetric, no centering constants are needed.) We now denote $\mathbf{X}_{nk}$ as the vector whose components are $X_{nk;1} = X_k/n^{\frac{1}{2}}S(n)$ and $X_{nk;2} = X_k^2/nQ(n)$. We shall show that conditions (i) of Theorems 1 and 2 are satisfied. Since $F \in \mathfrak{D}(2)$, then by the univariate central limit theorem, $nP[|X_1| \geqslant \varepsilon n^{\frac{1}{2}}S(n)] \to 0$ as $n \to \infty$, or $nP[X_1^2 \geqslant \varepsilon^2 nS^2(n)] \to 0$ as $n \to \infty$. However $G \in \mathfrak{D}(1)$ implies $nP[X_1^2 \geqslant \varepsilon nQ(n)] \sim$ (some constant) $K \neq 0$. Hence for sufficiently large n, $nS^2(n)\varepsilon^2 > nQ(n)\varepsilon$, and therefore $nP[\|X_{n1}\| \geqslant \varepsilon] = nP[|X_1| \geqslant n^{\frac{1}{2}}S(n)\varepsilon,\ X_1^2 \geqslant nQ(n)\varepsilon] = nP[X_1^2 \geqslant nS^2(n)\varepsilon^2] \to 0$. Thus, conditions (i) of Theorems 1 and 2 are satisfied. The proof that conditions (ii) of Theorems 1 and 2 are satisfied is the same as in the case when $1 < \alpha \leqslant 2$, with obvious modification when $F \in \mathfrak{D}(2) \setminus \mathfrak{D}_{\mathfrak{N}}(2)$. Thus we have shown that the joint limiting distribution of $(n^{\frac{1}{2}}/S(n))\overline{X}_n$ and $(nQ(n))^{-1}\Sigma_{i=1}^n X_i^2 + d_n$ exists and is that of two independent random variables. One can easily verify that

$$\frac{1}{nQ(n)}\Sigma_{i=1}^n X_i^2 + d_n = \frac{n-1}{nQ(n)}s_n^2 + d_n + D_n,$$

where

$$D_n = \frac{1}{Q(n)}\bar{X}_n^2 = \frac{S^2(n)}{nQ(n)}\left(\left(n^{\frac{1}{2}}S(n)\right)^{-1}\Sigma_{i=1}^n X_i\right)^2.$$

By Karamata's theorem referred to above, $S^2(n)/nQ(n) \to 0$ as $n \to \infty$. Coupling this with the fact that $(n^{\frac{1}{2}}S(n))^{-1}\Sigma_{i=1}^n X_i$ converges in law to a normal distribution, we obtain $D_n \to_p 0$. Thus in the case $\alpha = 1$, the joint limiting distribution of $(n^{\frac{1}{2}}/S(n))\bar{X}_n$ and $((n-1)/nQ(n))s_n^2 + d_n$ exists and is that of two independent random variables. Conversely, let us suppose $\bar{X}_n$ and s_n^2 are asymptotically independent. We shall assume $\alpha < 1$ and arrive at a contradiction. Thus we assume $G \in \mathfrak{D}(\alpha)$, $0 < \alpha < 1$. This implies that there is a slowly varying function $K(x)$ such that $P[X_1^2 \geq x] \sim x^{-\alpha}K(x)$. By symmetry of F, $P[X_1 \leq -x] = P[X_1 \geq x] \sim \frac{1}{2}x^{-2\alpha}Q(x)$, where $Q(x) = K(x^2)$ is also a slowly varying function. Hence $F \in \mathfrak{D}(2\alpha)$. As was pointed out in a previous argument, normalizing coefficients $\{B_n\}$ for F are known to be of the form $B_n = n^{1/2\alpha}L(n)$, where $L(x)$ is a slowly varying function. This is also known to satisfy

$$P\left[|X_1| \geq cn^{1/2\alpha}L(n)\right] \sim K/n,$$

where positive constants K, c determine each other. Hence $P[X_1^2 \geq c^2n^{1/\alpha}L^2(n)] \sim K/n$, and thus $\{n^{1/\alpha}L^2(n)\}$ may serve as a sequence of normalizing coefficients for G. Let $\{d_n\}$ be a sequence of constants such that the distribution of $n^{-1/\alpha}L^{-2}(n)\Sigma_{i=1}^n X_i^2 + d_n$ converges to some stable distribution (α). Then

$$(n^{1/\alpha}L^2(n))^{-1}\Sigma_{i=1}^n X_i^2 + d_n = \frac{n-1}{n^{1/\alpha}L^2(n)}s_n^2 + d_n + \frac{n}{n^{1/\alpha}L^2(n)}\bar{X}_n^2.$$

But

$$\frac{n}{n^{1/\alpha}L^2(n)}\bar{X}_n^2 = n^{-1}\left(\left(n^{1/2\alpha}L(n)\right)^{-1}\Sigma_{i=1}^n X_i\right)^2,$$

and the quantity being squared on the right converges in law to a symmetric stable distribution of characteristic exponent 2α, from which it follows that $(n/n^{1/\alpha}L^2(n))\bar{X}_n^2 \to_p 0$. Thus we have shown that the joint distribution of

$$\frac{1}{n^{1/2\alpha}L(n)}\Sigma_{i=1}^n X_i \quad \text{and} \quad \frac{1}{n^{1/\alpha}L^2(n)}\Sigma_{i=1}^n X_i^2 + d_n$$

converges to that of two independent random variables. But

$$nP\left[|X_1| \geq \varepsilon n^{1/2\alpha}L(n), X_1^2 \geq \varepsilon n^{1/\alpha}L^2(n)\right]$$
$$= nP\left[|X_1| \geq \max\{\varepsilon, \varepsilon^{\frac{1}{2}}\}n^{1/2\alpha}L(n)\right] \sim K \neq 0,$$

and hence condition (i) in Theorem 2 is violated, yielding the contradiction promised, and thus concluding the proof of the theorem.

4. Pairwise independence implies independence. The result of this section is related to Section 2 in that the intention is to show that within Theorem 1, pairwise asymptotic independence implies asymptotic independence. This is implied by showing that pairwise independence within a multivariate infinitely divisible distribution implies independence. This was proved by Pierre [7] under the assumption that finite second moments exist. It is proved here without that assumption.

THEOREM 4. *If $X_1, \cdots, X_n$ are random variables with a multivariate infinitely divisible distribution, and if X_i and X_j are independent for all $i < j$, then $X_1, \cdots, X_n$ are independent.*

PROOF. If $F_{\mathbf{X}}(\mathbf{x})$ denotes the joint distribution function of $X_1, \cdots, X_n$, then its characteristic function is given by

$$\hat{F}_{\mathbf{X}}(\mathbf{u}) = \exp\left\{ i\gamma'\mathbf{u} - \tfrac{1}{2}\mathbf{u}'C\mathbf{u} + \int_{\mathbf{x}\neq\mathbf{0}}\left(e^{i\mathbf{u}'\mathbf{x}} - 1 - \frac{\mathbf{u}'\mathbf{x}}{1+\mathbf{x}'\mathbf{x}}\right)M(d\mathbf{x})\right\}, \tag{6}$$

where $\gamma \in \mathbb{R}^n$ is a constant vector, $\mathbf{u} \in \mathbb{R}^n$, $C = (c_{ij})$ is an $n \times n$ nonnegative definite matrix, and M is the Lévy spectral measure. If $1 \leqslant i < j \leqslant n$, and if one sets $u_k = 0$ for $k \neq i, k \neq j, 1 \leqslant k \leqslant n$, then we obtain from (6) the joint characteristic function of X_i and X_j as

$$\hat{F}_{X_i, X_j}(u_i, u_j) = \exp\left\{ i(\gamma_i u_i + \gamma_j u_j) - \tfrac{1}{2}\left(u_i^2 c_{ii} + 2c_{ij}u_iu_j + u_j^2 c_{jj}\right) + \int\left(e^{i(u_ix_i + u_jx_j)} - 1 - \frac{i(u_ix_i + u_jx_j)}{1 + \sum_{l=1}^n x_l^2}\right)M(d\mathbf{x})\right\}.$$

The finiteness of

$$\int\left|\frac{x_1}{1+\sum_{l=1}^n x_l^2} - \frac{x_1}{1+\sum_{l=1}^2 x_l^2}\right|M(d\mathbf{x})$$

is verified by noting that if $I(x_1, x_2)$ denotes the integrand, then for $\varepsilon > 0$

$$\int_{\|\mathbf{x}\|\leqslant\varepsilon} I(x_1, x_2)M(d\mathbf{x}) = \int_{|\mathbf{x}|\leqslant\varepsilon}|x_1|\frac{\sum_{l=3}^n x_l^2}{\left(1+\sum_{l=1}^n x_l^2\right)\left(1+\sum_{l=1}^2 x_l^2\right)}M(d\mathbf{x}) \leqslant \varepsilon\int_{\|\mathbf{x}\|\leqslant\varepsilon}\|\mathbf{x}\|^2M(d\mathbf{x}) < \infty,$$

and

$$\int_{\|\mathbf{x}\|>\varepsilon} I(x_1, x_2)M(d\mathbf{x}) \leqslant 2M\{\mathbf{x} : \|\mathbf{x}\| > \varepsilon\} < \infty.$$

Thus, for $k = i, j$, and denoting

$$\alpha_k = \int \left(\frac{x_k}{1 + \|\mathbf{x}\|^2} - \frac{x_k}{1 + x_i^2 + x_j^2} \right) M(d\mathbf{x}),$$

which is finite by this last argument, we have

$$\begin{aligned} \hat{F}_{X_i, X_j}(u_i, u_j) = \exp\Big\{ & i((\gamma_i - \alpha_i)u_i + (\gamma_j - \alpha_j)u_j) \\ & - \tfrac{1}{2}\left(u_i^2 c_{ii} + 2u_i u_j c_{ij} + u_j^2 c_{jj}\right) \\ & + \int \left(e^{i(u_i x_i + u_j x_j)} - 1 - \frac{i(u_i x_i + u_j x_j)}{1 + x_i^2 + x_j^2} \right) M_{ij}(dx_i, dx_j) \Big\}, \end{aligned} \tag{7}$$

where $M_{ij}(A) = M\{\mathbf{x} : (x_i, x_j) \in A\}$ for $A \in \mathfrak{B}^{(2)}$. If we let $u_i = 0$ or $u_j = 0$ in (7), we can show in the same way as above that, for $l = i, j$, the quantity β_l defined by

$$\beta_l = \int \left(\frac{x_l}{1 + x_i^2 + x_j^2} - \frac{x_l}{1 + x_l^2} \right) M_{ij}(dx_i, dx_j)$$

is finite and that, for $l = i, j$,

$$\hat{F}_{X_l}(u_l) = \exp\left\{ iu_l(\gamma_l - \alpha_l - \beta_l) - \tfrac{1}{2}u_l^2 c_{ll} + \int \left(e^{iu_l x_l} - 1 - \frac{iu_l x_l}{1 + x_l^2} \right) M_l(dx_l) \right\} \tag{8}$$

where $M_i(A) = M_{ij}\{(x_i x_j) : x_i \in A\}$, $M_j(A)$ being likewise defined, for $A \in \mathfrak{B}^{(1)}$. Because of the independence of X_i and X_j, we have

$$\begin{aligned} \hat{F}_{X_i, X_j}(u_i, u_j) &= \hat{F}_{X_i}(u_i)\hat{F}_{X_j}(u_j) \\ &= \exp\Big\{ i[u_i(\gamma_i - \alpha_i - \beta_i) + u_j(\gamma_j - \alpha_j - \beta_j)] \\ & \quad - \tfrac{1}{2}\left(u_i^2 c_{ii} + u_j^2 c_{jj}\right) \\ & \quad + \int \left(e^{iu_i x_i} - 1 - \frac{iu_i x_i}{1 + x_i^2} \right) M_i(dx_i) \\ & \quad + \int \left(e^{iu_j x_j} - 1 - \frac{iu_j x_j}{1 + x_j^2} \right) M_j(dx_j) \Big\}. \end{aligned} \tag{9}$$

For $A \in \mathfrak{B}^2$ let us define

$$M_{ij}^*(A) = M_i\{x_i : (x_i, 0) \in A\} + M_j\{x_j : (0, x_j) \in A\}.$$

Then (9) becomes

$$\hat{F}_{X_i, X_j}(u_i, u_j) = \exp\Big\{ i\big[u_i(\gamma_i - \alpha_i - \beta_i) + u_j(\gamma_j - \alpha_j - \beta_j)\big] - \tfrac{1}{2}\big(u_i^2 c_{ii} + u_j^2 c_{jj}\big) + \int \left(e^{i(u_i x_i + u_j x_j)} - 1 - \frac{i(u_i x_i + u_j x_j)}{1 + x_i^2 + x_j^2} \right) M_{ij}^*(d(x_i, x_j)) \Big\}.$$

By the uniqueness of the canonical representation of the characteristic function of infinitely divisible distribution functions, we have $M_{ij}^* = M_{ij}$, $c_{ji} = c_{ij} = 0$ for $1 \leqslant i < j \leqslant n$, and $\beta_j = 0$, $1 \leqslant j \leqslant n$. Thus the covariance matrix of the Gaussian component, C, is diagonal, and we have shown that the projection of the M-mass in $\mathbb{R}^n$ onto the (x_i, x_j)-plane places all mass along the x_i- and x_j-axes for $1 \leqslant i < j \leqslant n$. This implies that all of the M-mass is concentrated along the n axes. Thus, we obtain

$$\hat{F}_{\mathbf{x}}(\mathbf{u}) = \Pi_{k=1}^n \exp\left\{ iu_k(\gamma_k - \alpha_k) - \tfrac{1}{2}\sigma_{kk} u_k^2 + \int \left(e^{iu_k x_k} - 1 - \frac{iu_k x_k}{1 + x_k^2} \right) M_k(dx_k) \right\}.$$

The kth member of this product has been shown above to be the (marginal) characteristic function of X_k. Thus we have shown independence. □

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