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NOTE ON A SQUARE FUNCTION INEQUALITY

BY A. O. PITTENGER
University of Maryland, Baltimore County

Let X be an L_p martingale, $3 \leqslant p < \infty$. Let $M = \sup|X_k|$ and $V^2 = \Sigma(X_k - X_{k-1})^2$. We show that $\|X\|_p \leqslant (p-1)\|V\|_p$ and, consequently, that $\|M\|_p \leqslant p\|V\|_p$.

Let $p > 1$ and let $(X_n, \mathbb{F}_n, n \geqslant 0)$ with $X_0 = 0$ be an L_p martingale: $\sup(\|X_n\|_p) < \infty$. Let $M = \sup|X_k|$ be the usual maximal function, and let $V^2 = \Sigma(X_k - X_{k-1})^2$ be the usual square function. It is well known (e.g., [3] and [4]) that $X_n \to X$ a.s. and in L_p and that there exists a constant C_p, independent of X, such that $\|X\|_p \leqslant C_p\|V\|$. Also

$$\|M\|_p \leqslant p(p-1)^{-1}\|X\|_p \tag{1}$$

is a standard result. In [2] Klincsek showed that for $p \geqslant 2$

$$\|M\|_p \leqslant A_p\|V\|_p \tag{2}$$

with $A_p < (p+1)$, $A_p/p \to 1$ as $p \to \infty$ and $A_p = p$ for integer p. He conjectured that (2) was valid with $A_p = p$ for all $p \geqslant 2$. In this note we refine the techniques of [2] to establish $\|X\|_p \leqslant (p-1)\|V\|_p$ for $p \geqslant 3$, thus confirming the conjecture in $[3, \infty)$.

The proof is based on the following result which can be easily verified by elementary calculus:

LEMMA. *Let $p \geqslant 2$. Then for any real a, b*

$$|a|^p - |b|^p - p(a-b)\operatorname{sign}(b)\cdot|b|^{p-1} \leqslant p(p-1)(a-b)^2\int_0^1 s((1-s)|a| + s|b|)^{p-2}\,ds$$

with equality if a and b have the same sign.

THEOREM. *Let $p \geqslant 3$. Then $\|X\|_p \leqslant (p-1)\|V\|_p$ and, consequently, $\|M\|_p \leqslant p\|V\|_p$.*

PROOF. Let $V_n^2 = \Sigma_1^n(X_k - X_{k-1})^2$. By the lemma,

$$E\big[|X_n|^p - |X_{n-1}|^p\big] \leqslant p(p-1)E\Big[\big(V_n^2 - V_{n-1}^2\big)\int_0^1 s((1-s)|X_n| + s|X_{n-1}|)^{p-2}\,ds\Big].$$

Since $p - \geqslant 1$, we have a.s.

$$\begin{aligned}\big[(1-s)|X_n| + s|X_{n-1}|\big]^{p-2} &\leqslant \big[E(((1-s)|X| + sM)|\mathbb{F}_n)\big]^{p-2} \\ &\leqslant E\big(\big[(1-s)|X| + sM\big]^{p-2}|\mathbb{F}_n\big).\end{aligned} \tag{3}$$

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Hence, using (3) and summing over n,

$$E[|X|^p] \leqslant p(p-1)\int_0^1 sE[V^2((1-s)|X| + sM)^{p-2}]\,ds,$$

and by Hölder's and Minkowski's inequalities

$$\begin{aligned}\|X\|_p^p &\leqslant p(p-1)\int_0^1 s\|V\|_p^2\|(1-s)|X| + sM\|_p^{p-2}\,ds\\ &\leqslant p(p-1)\int_0^1 s\|V\|_p^2((1-s)\|X\|_p + s\|M\|_p)^{p-2}\,ds\\ &\leqslant p(p-1)\|V\|_p^2\|X\|_p^{p-2}\int_0^1 s((1-s)+sq)^{p-2}\,ds\end{aligned}$$

where $q = p(p-1)^{-1}$. Another application of the lemma gives

$$\begin{aligned}\|X\|_p^2 &\leqslant (p-1)^2\|V\|_p^2\{1 - q^p - p(1-q)q^{p-1}\}\\ &= (p-1)^2\|V\|_p^2,\end{aligned}$$

completing the proof.

The above proof fails at (3) if $p < 3$. However, if we replace $|X_n|$ and $|X_{n-1}|$ by M in the inequality preceding (3), we can obtain $\|X\|_p^p \leqslant (\frac{p}{2})\|V\|_p^2\|M\|_q^{p-2}$ or

$$\|X\|_p \leqslant (p-1)\left[\frac{1}{2}\left(\frac{p}{p-1}\right)^{p-1}\right]^{\frac{1}{2}} \cdot \|V\|_p, \tag{4}$$

a result obtained in [1] by a different argument. This suggests of course that for all $p \geqslant 2$ we have $\|X\|_p \leqslant (p-1)\|V\|_p$.

Note that if we relax (4) slightly we have the more concise relation $\|X\|_p \leqslant (p-1)(e/2)^{\frac{1}{2}}\|V\|_p$.

Added in proof. In *Proc. Amer. Math. Soc.* **68** pages 337–338, Dubins and Gilat construct an example which shows the constant $p(p-1)^{-1}$ in equation (1) is sharp. That same example can be used to show that $p - 1 \leqslant Cp$, and hence $p - 1$ is sharp for $3 \leqslant p$.

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Department of Mathematics
University of Maryland, Baltimore County
5401 Wilkens Avenue
Baltimore, Maryland 21228