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ON THE TAIL BEHAVIOUR OF RECORD-TIME DISTRIBUTIONS IN A RANDOM RECORD PROCESS

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Consider a sequence of independent and identically distributed random variables attached to the points of an independent point process P. The random record process is the epochs of successive maxima in this sequence. In this paper necessary and sufficient conditions are found to ensure a certain tail behaviour for the distributions of times to successive records, and interrecord times, when P is a renewal process with relatively stable partial sums for its intervals.

1. Introduction. Let P be a renewal process on $[0, \infty)$ with interval sequence $\{Y_i\}(i = 1, 2, \cdots)$, so $S_j = \sum_{i=1}^{j} Y_i (j = 1, 2, \cdots)$ is the jth point of P. The common distribution function of the Y_i is denoted by F, with mean $\mu \in (0, \infty]$. For convenience we assume P has a point at 0, of index 0, i.e., $S_0 = 0$; such an assumption is not essential. Associated with S_j is an independent real-valued random variable X_j. The $\{X_j\}(j = 0, 1, 2, \cdots)$ are assumed to be i.i.d. with common continuous distribution function G.

The (upper) records in the sequence $\{X_j\}$ are the successive maxima, so that X_j is a record if $X_j > X_k (k = 0, 1, \cdots, j-1)$. This definition is unambiguous by the continuity of G. The point process of epochs of records in the sequence $\{X_j\}$ is called the *random record process*.

Define, for $r = 1, 2, \cdots$,

$$T_r = \text{time to } r\text{th record;}$$

$$\tau_r = T_r - T_{r-1}, \text{ the } r\text{th interrecord time } (T_0 = 0).$$

Note that X_0 is *not* counted as a record.

The random record process was introduced by Gaver [3] and studied further by Westcott [7]. Many of its properties are simple consequences of results for sequences of record times, corresponding to the degenerate case $Y_i = 1$, due to Chandler [1]. In particular, it follows from his work that $E(\tau_r)$, hence $E(T_r)$, are infinite for each r whether or not $\mu < \infty$. Thus it is especially of interest to examine the tail behaviour of the distributions of τ_r and T_r.

The obvious attack is through transforms, but if we take the Laplace transform of $\Pr(\tau_1 > t)$, for example, it transpires that as its argument $s \to 0$ the transform is asymptotically $\mu \log(s^{-1})$ when $\mu < \infty$. This is precisely the situation when standard Tauberian theory (Feller [2], page 447; Seneta [6], page 59) breaks down.

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However it suggests the conjecture that $\Pr(\tau_1 > t) \sim \mu t^{-1}$ as $t \to \infty$, a result certainly true for every known example (cf. [3]). This conjecture was verified in [7] under the extra hypothesis that $E(Y^{1+\delta}) < \infty$ for some $\delta > 0$. It implies that, in some sense, $E(\tau_1)$ is "only just" infinite.

In this paper we derive a necessary and sufficient condition for such tail behaviour when $r = 1$, as a corollary of a more general theorem which treats the case of $\{S_j\}$ relatively stable (Gnedenko and Kolmogorov [4], page 139). This is the content of Section 2, and the results are extended to arbitrary r in Section 3. The situation when $1 - F(t)$ is of dominated or regular variation with index in $(-1, 0)$ is treated in [7]; in the case of regular variation the Tauberian theory is now applicable.

Since this work was completed, Dr. N. H. Bingham has pointed out to me that a recent extension by de Haan [5] of standard Tauberian theory to cover certain cases of -1-varying functions may be applicable to this problem. A discussion is given in Section 4.

2. Behaviour of $\Pr(\tau_1 > t)$. It is easy to show ([3], [7]) that

$$\Pr(\tau_1 > t) = \Sigma_{j=1}^{\infty}\{j(j+1)\}^{-1}\Pr(S_j > t). \tag{1}$$

In order to study the behaviour of (1) as $t \to \infty$ we recall some necessary machinery. Define $\mu(t) = \int_0^t y\, dF(y) \quad (t > 0)$.

(i) The sequence $\{S_j\}$ is *relatively stable* if there exist constants $\{a_j\}$ such that $a_j^{-1}S_j \to 1$ in probability as $j \to \infty$. A necessary and sufficient condition for this ([2], page 236) is that $\mu(t)$ vary slowly at $+\infty$ or, equivalently,

$$\mu(t)/t\{1 - F(t)\} \to \infty \qquad \text{as } t \to \infty. \tag{2}$$

Defining $\sigma_t = \inf\{\sigma : t\mu(s)s^{-1} \leqslant 1,\ s \geqslant \sigma\}(t > 0)$, we may take $a_j = \sigma_j$ $(j = 1, 2, \cdots)$; note that

$$\sigma_t = t\mu(\sigma_t). \tag{3}$$

(ii) It is easy to see that σ_t is regularly varying of index 1 at $+\infty$, i.e.,

$$\sigma_t = tL(t)$$

where $L(t)$ is slowly varying at $+\infty$. Now for any such L there is a conjugate slowly varying L^* such that, as $t \to \infty$,

(a) $$L(t)L^*\{tL(t)\} \to 1,$$

(b) $$L^*(t)L\{tL^*(t)\} \to 1,$$

(c) $L^*(t)$ is asymptotically unique

(Seneta [6], page 25). Since $t\mu(tL(t)) = tL(t)$, from (3), we may take $L^*(t) = \{\mu(t)\}^{-1}$ by (a) and (c).

(iii) If

$$Y_i' = Y_i \quad (Y_i \leqslant t)$$
$$= 0 \quad (Y_i > t)$$

then for $\eta > 0$, $t > 0$, $j = 1, 2, \cdots$

(4) $\Pr\{|S_j - jE(Y')| > \eta\} \leqslant 2j\eta^{-2}\int_0^t y\{1 - F(y)\}\, dy - j(t^2\eta^{-2} - 1)\{1 - F(t)\}$

([2], page 235).

THEOREM 1. *If, in the random record process, P is a renewal process and* $\{S_j\}$ *is relatively stable, then as* $t \to \infty$,

(5) $$\Pr(\tau_1 > t) \sim \mu(t)t^{-1}$$

if and only if

(6) $$t\{\mu(t)\}^{-1}(\log\{t/\mu(t)\})\{1 - F(t)\} \to 0.$$

PROOF. For any $\varepsilon > 0$, split the range of summation in (1) into

$$(1, [tL^*(t)/(1 + \varepsilon)]), ([tL^*(t)/(1 + \varepsilon)] + 1, [tL^*(t)/(1 - \varepsilon)]),$$
$$([tL^*(t)/(1 - \varepsilon)] + 1, \infty)$$

where $[\cdot]$ denotes integer part. Call the sums $\Sigma_1, \Sigma_2, \Sigma_3$, respectively, and define $u_+ = tL^*(t)/(1 + \varepsilon)$, $u_- = tL^*(t)/(1 - \varepsilon)$.

Σ_3: Clearly

(7) $$\Pr(S_{[u_-]} > t)[u_-]^{-1} < \Sigma_3 < [u_-]^{-1}.$$

But

(8) $$\Pr(S_{[u_-]} > t) = \Pr\left(\frac{S_{[u_-]}}{\sigma_{u_-}} > \frac{(1 - \varepsilon)}{L^*(t)L(u_-)}\right)$$

and as $t \to \infty$ the right side of (8) $\to 1$ by (ii) (b) and the relative stability of $\{S_j\}$. Thus as $t \to \infty$,

(9) $$t\Sigma_3/\mu(t) \to 1.$$

Σ_2: We have

$$\Sigma_2 \leqslant u_+^{-1} - (u_- + 1)^{-1}$$

whence

(10) $$t\Sigma_2/\mu(t) < 3\varepsilon$$

for t sufficiently large.

Σ_1: Here,

(11) $$\Sigma_1 \leqslant \Sigma_{j=1}^{[u_+]}\{j(j + 1)\}^{-1} \Pr\{|S_j - j\mu(t)| > t - j\mu(t)\}$$
$$\leqslant \Sigma_{j=1}^{[u_+]}\{j(j + 1)\}^{-1} \Pr\{|S_j - j\mu(t)| > \varepsilon t/(1 + \varepsilon)\}.$$

So, from (iii),

$$\Sigma_1 \leqslant \Sigma_{j=1}^{[u_+]}(j+1)^{-1}\left\{2(1+\varepsilon)^2(\varepsilon t)^{-2}\int_0^t y\{1-F(y)\}\,dy - \left\{\left(\frac{1+\varepsilon}{\varepsilon}\right)^2 - 1\right\}\{1-F(t)\}\right\} \tag{12}$$

$$\leqslant C(\log\{t/\mu(t)\})t^{-2}\int_0^t y\{1-F(y)\}\,dy \tag{13}$$

for some positive constant C, since the second term in (12) is negative. Now if (6) holds, then for any $\delta > 0$ the integrand in (13) is dominated by $\delta\mu(y)(\log\{y/\mu(y)\})^{-1}$ for $y \geqslant t_0(\delta)$. Thus the upper bound in (13) is asymptotically bounded by $C\delta\mu(t)t^{-1}$ as $t \to \infty$ ([2], page 281; [6], page 53), whence

$$t\Sigma_1/\mu(t) \leqslant \varepsilon \tag{14}$$

for t sufficiently large. Then (9), (10) and (14) establish the desired sufficiency of (6).

Further, since the $\{Y_i\}$ are nonnegative and independent,

$$\Pr\{S_j > t\} \geqslant j\{1-F(t)\}\{F(t)\}^{j-1},$$

which implies

(15)

$$\begin{aligned} t\Sigma_1/\mu(t) &\geqslant t\{1-F(t)\}\{\mu(t)\}^{-1}\Sigma_{j=1}^{[u_+]}(j+1)^{-1}F^{j-1}(t) \\ &\geqslant t\{1-F(t)\}\{\mu(t)\}^{-1}\log(u_+)\{F(t)\}^{u_+} \\ &\geqslant t\{1-F(t)\}\{\mu(t)\}^{-1}(\log\{t/\mu(t)\})(1-t\{1-F(t)\}\{\mu(t)(1+\varepsilon)\}^{-1}) \end{aligned}$$

since $(1-x)^n \geqslant 1 - nx$. By (2) the last bracket $\to 1$ as $t \to \infty$. Combining this with (9) and (10) shows that (5) necessarily leads to (6), which completes the proof of the theorem.

COROLLARY 1. *If* $\mu < \infty$, *then for* $\Pr(\tau_1 > t) \sim \mu t^{-1}$ *it is necessary and sufficient that* $t(\log t)\{1-F(t)\} \to 0$ *as* $t \to \infty$.

COROLLARY 2. *For* $\Pr(\tau_1 > t) \sim \mu t^{-1}$ *it is sufficient that* $E(Y\log^+ Y) < \infty$, *where* $\log^+ y = \max(0, \log y)$.

For

$$\begin{aligned} E(Y\log^+ Y) < \infty &\Rightarrow \int_t^\infty y\log y\,dF(y) \to 0 \quad \text{as } t \to \infty, \\ &\Rightarrow t(\log t)\{1-F(t)\} \to 0 \quad \text{as } t \to \infty. \end{aligned}$$

Corollary 2 weakens the "extraneous" moment condition imposed in [7]. This is very nearly the ultimate possible reduction; an example follows to show that (6) is not implied by $E\{(Y\log^+ Y)/\{\log_{(k)}{}^+ Y\}^{1+\delta}\} < \infty$ ($\delta > 0$, k any positive integer), where $f_{(k)}$ means the k-fold iterate of a function f, and *a fortiori* is not implied by $E\{Y(\log^+ Y)^\alpha\} < \infty$ ($\alpha < 1$).

EXAMPLE. F is discrete with

$$\Pr(Y = j) = C\{\exp_{(k)}(n)\exp_{(k-1)}(n)\}^{-1} \text{ if } j = \exp_{(k)}(n) \quad n = 0, 1, 2, \cdots,$$
$$= 0 \qquad \text{otherwise.}$$

Clearly the above moment is finite, yet

$$1 - F(\exp_{(k)}(n) -) \geqslant \Pr(Y = \exp_{(k)}(n))$$

so that $t \log t\{1 - F(t)\} \geqslant C$ when $t = \{\exp_{(k)}(n) - \}$ and (6) cannot hold.

3. Behaviour of $\Pr(\tau_r > t), \Pr(T_r > t)$. We have, for $r = 1, 2, \cdots, t > 0$,

$$\Pr(\tau_r > t) = \Sigma_{j=1}^{\infty} q_{r,j} \Pr(S_j > t), \tag{16}$$

where $q_{r,j}$ is the probability that the difference of the indices of the $(r-1)$st and rth records in $\{X_j\}$ equals j. Repeating the previous proof on (16) rather than (7) it is clear that $q_{r,j}$ enters only through $Q_{r,j} = \Sigma_{k=j}^{\infty} q_{r,k}$ (at (7)) and $Q'_{r,j} = \Sigma_{k=1}^{j} k q_{r,k}$ (at (12) and (15)). It is known that, as $j \to \infty$,

$$Q_{r,j} \sim (\log j)^{r-1}/j(r-1)! \tag{17}$$

([7]), and hence

$$Q'_{r,j} \sim (\log j)^{r}/r!. \tag{18}$$

Further, for $\Pr(T_r > t)$ the probability corresponding to $Q_{r,j}$ gives the same asymptotic behaviour as at (17), (18). This proves

THEOREM 2. *If, in the random record process, P is a renewal process and $\{S_j\}$ is relatively stable, then as $t \to \infty$,*

$$\Pr(\tau_r > t) \sim \{\log(t/\mu(t))\}^{r-1}\mu(t)\{t(r-1)!\}^{-1}$$

if and only if

$$t\{\mu(t)\}^{-1}(\log\{t/\mu(t)\})^{r}\{1 - F(t)\} \to 0.$$

The theorem remains true if T_r replaces τ_r.

COROLLARY. *If $\mu < \infty$, then for $\Pr(\tau_r > t) \sim (\log t)^{r-1}\mu/\{t(r-1)!\}$ it is necessary and sufficient that $t(\log t)^r\{1 - F(t)\} \to 0$ as $t \to \infty$. This is ensured by $E\{Y(\log^+ Y)^r\} < \infty$. The same is true if T_r replaces τ_r.*

4. Generalizations and discussion. Independence of the $\{Y_i\}$ is important in both the necessity and sufficiency parts of the theorems. Uncorrelated would do for the sufficiency but this is perhaps not much improvement.

To relax the assumption of identical distributions, we would seem to need the Gnedenko and Kolmogorov version of the relative stability theorem ([4], page 141), which covers this extension. We conjecture that the tail behaviour is like σ_t^{-1}, where σ_t is derived from the appropriate norming constants in a manner analogous to (3), though possibly further conditions will be required.

As mentioned in the introduction, the recent paper by de Haan [5], which gives a necessary and sufficient condition for -1-variation of the upper tail of a distribution function in terms of its Laplace transform is in principle relevant to our problem since $\mu(t)$ is slowly varying in (5). However, although in the special case $\mu < \infty$ the sufficiency of (6) is fairly easy to establish by this route, the general problem appears to require a formidable amount of analysis. Further, the expression for the transform of $\Pr(\tau_r > t)$ for general r involves an integral (cf. [7], equation (6)), making the proof of Theorem 2 relatively unwieldy, while generalizations as previously discussed will be very complicated in a transform setting. For these reasons, it seems most useful to follow the methods of this paper.

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