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ON THE SHAPE OF A RANDOM STRING

BY RICHARD DURRETT

University of California, Los Angeles

We define a string of length m to be a sequence $z_0, z_1, \cdots, z_m$ of points in Z^2 which are chosen so that z_{i-1} and z_i are adjacent points on the lattice. If we consider the set of all strings from $z_0 = (0, 0)$ to $z_m = (n, 0)$ then we can introduce a probability distribution by assuming that all the strings are equally likely. In this paper we will prove some results which describe the shape of the random string when $m/n \to \lambda \in (1, \infty)$. The main result states that if we let (U_k, V_k) be the coordinate of z_k then $(U_{[m\cdot]}/n, V_{[m\cdot]}/n^{\frac{1}{2}})$ converges weakly to $(\cdot, \rho^{\frac{1}{2}} W_0(\cdot))$ where W_0 is a Brownian bridge and $\rho = (\lambda^2 - 1)/2\lambda$. A second set of results describes $B_n = \sup\{U_k - U_l : 0 \leqslant k \leqslant l \leqslant m\}$, a quantity which measures the amount of backtracking in the string. We find that this is $0(\log n)$ and there is a sequence of constants b_n so that $B_n - b_n$ approaches a double exponential distribution. The results described above were motivated by, and are related to, results of Abraham and Reed, and Gallovotti on the shape of the interface profile in the two dimensional Ising model.

1. Introduction. We define a string of length m to be a sequence $z_0, z_1, \cdots, z_m$ of points in Z^2 with $|z_{i,1} - z_{i-1,1}| + |z_{i,2} - z_{i-1,2}| = 1$ for $1 \leqslant i \leqslant m$. If we let $\Omega_{m,n}$ be the set of all strings $(z_0, z_1, \cdots, z_m)$ which have $z_0 = (0, 0)$ and $z_m = (n, 0)$ then we can introduce a probability distribution on $\Omega_{n,m}$ by supposing that all the strings in $\Omega_{m,n}$ are equally probable. To describe the shape of the random string we introduce the random variables $U_k = z_{k,1}$ and $V_k = z_{k,2}$ and write the string as the random function $(U_{[m\cdot]}, V_{[m\cdot]})$ where $[mt]$ denotes the largest integer $\leqslant mt$.

In this paper we will prove some results which describe the shape of the random string defined above when $m/n \to \lambda \in (1, \infty)$. In this case if we divide the x coordinate by n then the limit as $n \to \infty$ can be viewed as a sequence of discrete approximations to a continuous string of length λ from $(0, 0)$ to $(1, 0)$. By analogy with what happens when a long string is dropped on the floor we might expect that as $n \to \infty$, $(U_{[m\cdot]}/n, V_{[m\cdot]}/n)$ would converge weakly to a limit process which looks like a two-dimensional Brownian motion conditioned on $W(1) = (1, 0)$. In fact, however, the limit is quite different—the random string when properly normalized converges weakly to the graph of a (random) function. In Section 2 we show that

$$(1) \qquad \left(\frac{U_{[m\cdot]}}{n}, \frac{V_{[m\cdot]}}{n^{\frac{1}{2}}}\right) \Rightarrow \left(\cdot, \rho^{\frac{1}{2}} W_0(\cdot)\right)$$

where W_0 is the Brownian bridge, $\rho = (\lambda - 1)^2/2\lambda$, and $\Rightarrow$ denotes weak convergence in $D([0, 1], R^2)$ (see Billingsley (1968) for details).

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The result above shows that in the limit the movement in the x-direction is a linear drift and the fluctuations in the y-direction are like $n^{\frac{1}{2}}$ times a Brownian bridge. The reason for this becomes clear if we examine the situation in another way. Let Ω_m be the collection of all strings of length m with $z_0 = (0, 0)$. If we introduce a probability distribution on Ω_m by supposing all the elements of Ω_m are equally probable then z_m has the same distribution as a two-dimensional Bernoulli random walk $S_{[m\cdot]}$ so a uniform distribution on $\Omega_{m,n}$ corresponds to $(S_{[m\cdot]}|S_m = (n, 0))$. If we separate the coordinates of the random walk and consider $(S^1_{[m_1\cdot]}|S^1_{m_1} = n)$ and $(S^2_{[m_2\cdot]}|S^2_{m_2} = 0)$ where m_1 and m_2 are the number of horizontal and vertical steps then results of Durrett (1977) and Liggett (1968) suggest that the coordinates of the string should have the limits indicated in (1). This argument can be made rigorous by showing that m_2/n converges to ρ in probability and that the time substitutions which relate the two conditioned processes to the string coordinates converge to the identity—the details are given in Section 2.

The limit theorem described above shows that as $U_{[m\cdot]}$ converges to a deterministic drift. In Section 3 we will obtain some limit theorems for $B_n = \max\{U_k - U_l : 0 \leqslant k \leqslant l \leqslant m\}$, a quantity which measures how much the x coordinate of the string differs from a monotone function at stage n. To introduce these results we need some notation. Let $\sigma_1 = \inf\{k : U_k = j\}$ be the hitting time of j and let

$$B_{n,j} = j - \inf\{U_k : \sigma_j \leqslant k \leqslant m\}$$

be the amount the path backtracked after it hit j. Clearly $B_n = \max_{0 \leqslant j \leqslant n} B_{n,j}$.

It is easy to get an upper bound for B_n. The first result in Section 3 shows that if $K_n \to \infty$ then with a probability which approaches one

$$B_n \leqslant \frac{3/2}{\log(r^{-1})} \log n + K_n. \tag{2}$$

To improve this result takes some work. The first step is to show that as $n \to \infty$

$$P(B_{n,0} \geqslant k) \to r^k$$

where r is the constant which appears in (2). This result suggests that as $n \to \infty$, B_n is the maximum of n geometric random variables. If the variables $B_{n,j}$, $0 \leqslant j < n$ were independent then we could easily calculate the limit of the B_n. However the $B_{n,j}$'s are related by $B_{n,j-1} \geqslant B_{n,j} - 1$ so if we want to write B_n as a maximum of independent random variables we have to use another approach.

To introduce this new approach we begin by considering the (easier) problem of determining the amount of backtracking in a one-dimensional Bernoulli random walk S^*_k with $P(S^*_1 = 1) = p^* > \frac{1}{2}$. Let $\tau_k = \inf\{j : S_j = k\}$ be the hitting time of k and let

$$B^*_n = \sup_{0 \leqslant k \leqslant n}(k - \inf_{j \geqslant \tau_k} S^*_j)$$

be the maximum backtracking from a point $k \leqslant n$. If we break the time interval

$[0, \infty)$ into pieces $[\tau_0, \tau_1), [\tau_1, \tau_2), \cdots [\tau_{n-1}, \tau_n), [\tau_n, \infty)$ then we can write B_n^* as

(3) $$\sup_{0 \leqslant k < n}(k - \inf_{\tau_k \leqslant j < \tau_{k+1}} S_j^*) \vee (n - \inf_{j \geqslant \tau_n} S_j^*).$$

It is easy to see that the variables in the first term are i.i.d. and that as $n \to \infty$ the probability that the maximum is attained by the second term approaches 0.

Since the distribution of $\inf_{j \in [\tau_0, \tau_1)} S_j^*$ can be computed explicitly it is easy to use (3) to show that if we let $b_n = \log n / \log(p^*/1 - p^*)$, and pick x_n so that $b_n + x_n$ is an integer and x_n is a bounded sequence then

(4) $$P(B_n^* \leqslant b_n + x_n) - \exp\left(-\left(1 - \frac{q^*}{p}\right)\left(\frac{q^*}{p}\right)^{x_n}\right) \to 0$$

where $q^* = 1 - p^* = P\{S_1^* = -1\}$.

The last result gives the amount of backtracking in a random walk with positive drift. To use this result to obtain the corresponding limit theorem for the string we observe that the motion in the x-coordinate is (up to a random change of time) the same as a symmetric Bernoulli random walk conditioned on $S_{m_1}^1 = n$. Using the idea of associated random variables (see Feller (1970), pages 548–553) it is possible to reduce the problem to one concerning a random walk with positive drift and a less drastic conditioning. After the transformation it is easy to obtain the result for backtracking in the string from the result for the associated random walk. Our final result may be stated as the following.

THEOREM. *If* $b_n = \log n / \log(r^{-1})$ *then*

$$\sup_{z \in Z} P(B_n < z) - \exp(-r^{z - b_n}(1 - r)) \to 0.$$

The reader should note that the distribution of B_n does not converge but it does (in a sense made precise by the theorem) approach a double exponential distribution.

Having described our results we can now relate them to the Ising model. Let $\Lambda_n = [-n, n]^2 \cap Z^2$ and consider the spin system in Λ_n which corresponds to the Ising model with a boundary condition which is $+1$ at $z \in \Lambda_n^c$ with $z_1 \geqslant 0$ and -1 at $z \in \Lambda_n^c$ with $z_1 < 0$ (see [1] or [7] for precise definitions and the facts quoted below). It is easy to see that the spin configurations in Λ_n can be represented by giving the contour lines which separate $+$ and $-$ spins and that with the boundary condition given above there will be one contour line which goes from $(\frac{1}{2}, -n)$ to $(\frac{1}{2}, n)$. This contour line gives the interface profile and corresponds to our random string.

At first glance the distribution of the shape of the contour line looks much different from that of the string—in the Ising model the length of the contour is random and all shapes with a fixed length are not equally likely. However, as $n \to \infty$ these differences apparently disappear—the results which have been proved about the contour lines are almost exactly the same as those for our string. Abraham and Reed (1976) have shown that if $T < T_c$ (the temperature below which the Gibbs state is not unique) then the level at which the string crosses the line $z_2 = 0$ when divided by $n^{\frac{1}{2}}$ converges to a normal distribution, while Gallavotti (1972) has shown that if T is small enough (i.e., $T \ll T_c$) then the amount of backtracking in the contour line is $0(\log n)$. It seems likely that the normalized contour line converges weakly to a Brownian bridge and that the backtracking is $0(\log n)$ for all $T < T_c$ but neither of these results has been proved.

2. The shape of the string. In this section we will prove some results which describe the asymptotic shape of a random string when $m, n \to \infty$ and $m/n \to \lambda \in (1, \infty)$. The main result is

THEOREM 2.1. *If we let* (U_k, V_k) *be the coordinates of the* kth *vertex in the path then*

$$\left(\frac{U_{[m\cdot]}}{n}, \frac{V_{[m\cdot]}}{n^{\frac{1}{2}}}\right) \Rightarrow \left(\cdot, \rho^{\frac{1}{2}} W_0(\cdot)\right)$$

where W_0 *is the Brownian bridge and* $\rho = (\lambda^2 - 1)/2\lambda$.

PROOF. The proof will be accomplished with five steps:

1. Let $H = \{i : z_i - z_{i-1} = (-1, 0) \text{ or } (1, 0)\}$ be the indices of the horizontal steps, let $I = \{1, \cdots, m\} - H$ be the indices of the vertical steps. In the first step in the proof we will show that

$$(1) \qquad |I|/n \to \rho \qquad \text{in probability.}$$

This result gives asymptotic formulas for $|H|$ and $|I|$ so the next step is to consider the distributions of the individual coordinates given $|H|$ and $|I|$.

2. Let $X_1, X_2 \cdots$ be the size of the first, second $\cdots$ horizontal steps and let $Y_1, Y_2, \cdots$ be the sequence of vertical steps. Anticipating the result given in the theorem we will let $T_k = X_1 + \cdots + X_k$ (time) and $S_k = Y_1 + \cdots + Y_k$ (the random walk which will produce the Brownian bridge). The sequences T_k and S_k can be related to the coordinates of the string (U_k, V_k) by a random change of time. If we let $M(k) = |H \cap \{1, \cdots, k\}|$ and $N(k) = |I \cap \{1, \cdots, k\}|$ then we have that

$$(U_{[m\cdot]}|\ |H| = k) =_d (T_{M(m\cdot)}|M(m) = k, T_k = n)$$
$$(V_{[m\cdot]}|\ |I| = k) =_d (S_{N(m\cdot)}|N(m) = k, S_k = 0).$$

To evaluate the limits of the terms on the right-hand side we will quote two results and then use a random change of time.

3. From Liggett (1968) we have that if $k \to \infty$

$$\left(k^{-\frac{1}{2}} S_{[k\cdot]}|S_k = 0\right) \Rightarrow W_0$$

where W_0 is the Brownian bridge.

4. From Durrett (1977) we have that if $k/n \to \lambda - \rho > 1$

$$\left(\frac{T_{[k\cdot]} - n\cdot}{k^{\frac{1}{2}}}\middle| T_k = n\right) \Rightarrow W_0$$

so

$$\left(\frac{T_{[k\cdot]} - n\cdot}{n} \middle| T_k = n \right) \Rightarrow 0.$$

5. The last ingredient in the proof is to show that if $k/n \to \rho > 0$ then for all $t \in (0, 1)$

$$\left(\frac{N(mt)}{k} \middle| N(m) = k \right) \to t \quad \text{in probability.} \tag{2}$$

With 3, 4, and 5 it is easy to prove the result. If we let

$$W_k(\cdot) = \left(k^{-\frac{1}{2}} S_{[k\cdot]} | S_k = 0 \right)$$

then we can write

$$\left(\frac{S_{N(m\cdot)}}{n^{\frac{1}{2}}} \middle| N(m) = k,\, S_k = 0 \right) = \left(\frac{k^{\frac{1}{2}}}{n^{\frac{1}{2}}} W_k\left(\frac{N(m\cdot)}{k} \right) \middle| N(m) = k \right).$$

From 3 we have that $W_k \Rightarrow W_0$. If $k/n \to \rho$ then it follows from 5 that $N(mt)/k \to t$ uniformly on [0, 1]. Combining these two results with Theorem 3 of Durrett and Resnick (1977) shows that if $k/n \to \rho$

$$\left(\frac{V_{[m\cdot]}}{n^{\frac{1}{2}}} \middle| |I| = k \right) \Rightarrow \rho^{\frac{1}{2}} W_0(\cdot).$$

A similar argument using 4 shows that if $l/n \to \lambda - \rho > 1$

$$\left(\frac{U_{[m\cdot]} - n\cdot}{n} \middle| |H| = l \right) \Rightarrow 0.$$

Combining the last two results with 1 gives the desired conclusion.

To complete the proof at this point we need to prove (1) and (2). To begin the proof of (1) we observe that the number of paths with $|I| = k$ is

$$m! \Big/ \frac{k}{2}!\, \frac{k}{2}!\, \frac{m-k-n}{2}!\, \left(m - k - \frac{m-k-n}{2} \right)! \tag{3}$$

From Stirling's formula (see Feller (1968), page 52), we have that $m! \sim (2\pi)^{\frac{1}{2}} m^{m+\frac{1}{2}} e^{-m}$ so if n is large $m/n = \lambda$, and $k/n = \rho < \lambda - 1$ the expression in (3) is

$$n(2\pi n)^{-\frac{3}{2}} \lambda^{\lambda n + \frac{1}{2}} \Big/ \left(\frac{\rho}{2} \right)^{\rho n + 1} \left(\frac{\lambda - \rho - 1}{2} \right)^{(\lambda-\rho-1)n+\frac{1}{2}} \left(\frac{\lambda + \rho + 1}{2} \right)^{(\lambda-\rho+1)n+\frac{1}{2}}.$$

The logarithm of the last expression is

$$\begin{aligned} &n \log\left(\lambda^\lambda \Big/ \left(\frac{\rho}{2} \right)^\rho \left(\frac{\lambda - \rho - 1}{2} \right)^{(\lambda-\rho-1)/2} \left(\frac{\lambda - \rho + 1}{2} \right)^{(\lambda-\rho+1)/2} \right) \\ &\quad - \frac{3}{2}\log(2\pi n) + \frac{1}{2}\log\left(\lambda \Big/ \left(\frac{\rho}{2} \right)^2 \left(\frac{\lambda - \rho - 1}{2} \right)\left(\frac{\lambda - \rho + 1}{2} \right) \right). \end{aligned} \tag{4}$$

If n is large the last expression has its largest value at the maximum of

$$\lambda \log \lambda - \rho \log(\rho/2) - \left(\frac{\lambda - \rho - 1}{2}\right)\log\left(\frac{\lambda - \rho - 1}{2}\right) - \left(\frac{\lambda - \rho + 1}{2}\right)\log\left(\frac{\lambda - \rho + 1}{2}\right).$$

Differentiating gives

$$\frac{d}{d\rho} = -\log(\rho/2) - 1 + \frac{1}{2}\log\left(\frac{\lambda - \rho - 1}{2}\right) + \frac{1}{2} + \frac{1}{2}\log\left(\frac{\lambda - \rho + 1}{2}\right) + \frac{1}{2}$$

so the maximum occurs when

$$-2\log\rho + \log(\lambda - \rho - 1) + \log(\lambda - \rho + 1) = 0$$

i.e.,

$$\rho^2 = (\lambda - \rho - 1)(\lambda - \rho + 1) = \lambda^2 - 2\lambda\rho + \rho^2 - 1.$$

Solving gives $\rho = (\lambda^2 - 1)/2\lambda$. As we might expect $\rho = 0$ when $\lambda = 1$ and $\rho - (\lambda/2) \to 0$ when $\lambda \to \infty$.

In the calculations above we have located the number of vertical steps which produces the maximum number of strings. To prove (1) we have to show that most of the contribution to the sum comes from terms near ρn. To do this we first consider the sum over $l \geqslant (\rho + 2\varepsilon)n$ and show that this is small when compared to the sum over $\rho n \leqslant l \leqslant (\rho + \varepsilon)n$. To estimate the sums we observe that from the proof of Stirling's formula (Feller (1968), page 54) we have that for all $k \geqslant 1$

$$e^{-1} \leqslant \frac{k!}{(2\pi)^{\frac{1}{2}} k^{k+\frac{1}{2}} e^{-k}} \leqslant 1.$$

From this it follows that

$$\Sigma_{l=(\rho+2\varepsilon)n+1}^{m-n-2} \leqslant n\int_{(\rho+2\varepsilon)}^{(m/n)-1} e^4 (2\pi n)^{-\frac{3}{2}} f_1(m/n, t)\,dt$$

where

$$f_1(\theta, t) = \theta^{\theta n + \frac{1}{2}} \Big/ \left(\frac{t}{2}\right)^{tn+1}\left(\frac{\theta - t - 1}{2}\right)^{((\theta - t - 1)n + 1)/2}\left(\frac{\theta - t + 1}{2}\right)^{((\theta - t + 1)n + 1)/2}$$

From (4) and the calculations which follow it

$$\frac{d}{dt}\log f_1(\theta, t) = n\left(-\log(\theta/2) + \frac{1}{2}\log\left(\frac{\theta - t - 1}{2}\right) + \frac{1}{2}\log\left(\frac{\theta - t + 1}{2}\right)\right) + 0(1).$$

So it follows from the calculations used to maximize $f_1(\lambda, t)$ that if n is sufficiently large $f_1(m/n, t)$ is a decreasing function of $t \geqslant \rho + 2\varepsilon$. Using this observation we have that the sum over $l < (\rho + 2\varepsilon)n$ is smaller than

$$ne^4(2\pi n)^{-\frac{3}{2}}\left(\frac{m}{n} - 1 - \rho - 2\varepsilon\right) f_1\left(\frac{m}{n}, \rho + 2\varepsilon\right).$$

By a similar argument we can show that the sum over $\rho n \leqslant l \leqslant (\rho + \varepsilon)n$ is larger than

$$ne^{-1}(2\pi n)^{-\frac{3}{2}}\varepsilon f_1\left(\frac{m}{n}, \rho + \varepsilon\right).$$

It is easy to check that

$$f_1\left(\frac{m}{n}, \rho + 2\varepsilon\right) \Big/ f_1\left(\frac{m}{n}, \rho + \varepsilon\right) \to 0.$$

Combining the last three observations shows that for all $\varepsilon > 0$, $P(|I|/n > \rho + 2\varepsilon) \to 0$. A similar argument shows that $P(|I|/n < \rho - 2\varepsilon) \to 0$ so the proof of (1) is complete.

The last step in the proof of Theorem 2.1 is to prove that if $k/n \to \rho > 0$ then for all $t \in (0, 1)$

$$\left(\frac{N(mt)}{k}\middle| N(m) = k\right) \to t \qquad \text{in probability.}$$

Given $N(m) = k$ the number of ways we can have $N(mt) = j$ is

$$(5) \qquad \frac{(mt)!}{j!\,(mt-j)!}\,\frac{(m-mt)!}{(k-j)!\,(m-mt-k+j)!}.$$

From Stirling's formula $m! \sim (2\pi)^{\frac{1}{2}} m^{m+\frac{1}{2}} e^{-m}$ so if $j/k = s \in (0, 1)$ the expression in (5) is

$$\sim \frac{(2\pi n)^{-1} C(\lambda, \rho, s, t)(\lambda t)^{\lambda tn}(t - \lambda t)^{(t-\lambda t)n}}{(s\rho)^{s\rho n}(\lambda t - s\rho)^{(\lambda t - s\rho)n}(\rho - s\rho)^{(\rho - s\rho)n}(\lambda - \lambda t - \rho + s\rho)^{(\lambda - \lambda t - \rho - s\rho)n}}$$

where $C(\lambda, \rho, s, t)$ is the constant (i.e., it is independent of n) which comes from the $m^{\frac{1}{2}}$ terms in Stirling's formula (see the last term in (4)).

Taking logarithms we see that the last expression has its largest value at the maximum of

$$\begin{aligned} &\lambda t \log \lambda t + (t - \lambda t)\log(t - \lambda t) - s\rho \log(s\rho) \\ &- (\lambda t - s\rho)\log(\lambda t - s\rho) - (\rho - s\rho)\log(\rho - s\rho) \\ &- (\lambda - \lambda t - \rho + s\rho)\log(\lambda - \lambda t - \rho + s\rho). \end{aligned}$$

Differentiating gives

$$\begin{aligned} \frac{d}{ds} = &-\rho \log(s\rho) - \rho + \rho \log(\lambda t - s\rho) + \rho \\ &+ \rho\log(\rho - s\rho) + \rho - \rho \log(\lambda - \lambda t - \rho + s\rho) - \rho, \end{aligned}$$

so the maximum occurs where

$$\log \frac{(\lambda t - s\rho)(\rho - s\rho)}{s\rho(\lambda - \lambda t - \rho + s\rho)} = 0.$$

i.e.,

$$(\lambda t - s\rho)(\rho - s\rho) = s\rho(\lambda - \lambda t - \rho + s\rho).$$

Solving gives $\lambda t - s\rho - s\lambda t + s^2\rho = s(\lambda - \lambda t - \rho) + s^2\rho$ which reduces to $s = t$ the answer we expected.

In the calculations above we have shown that of all the paths with $N(m) = k$ the ones with $N(mt) = kt$ are the most numerous. To prove (2) we have to show that most of the paths with $N(m) = k$ have $N(mt)/k$ near t. The proof of this statement is the same as the corresponding part of the proof of (1) so the details are omitted.

3. Backtracking. In Section 2 we showed that the x coordinate of the string is asymptotically (as $n \to \infty$) a linear function of the index and that the deviation from linearity is $0(n^{\frac{1}{2}})$. This implies of course that the deviation from monotonicity is at most $0(n^{\frac{1}{2}})$ but this is a crude estimate—we will show in this section that the deviation from monotonicity is $0(\log n)$.

To introduce these results we need some notation. Let T_k be the sum of the first k horizontal steps and let $N(m)$ be the number of horizontal steps in $\{1, \cdots, m\}$ (these quantities were defined in Section 2). Let

$$\sigma_j = \inf\{k : T_k = j\}$$

be the hitting time of j and let

$$B_{n,j} = j - \min\{T_k : \sigma_j \leq k \leq N(m)\}$$

be the amount the path backtracked after it hit j.

In the proof of Theorem 1 we observed that if there were k horizontal steps then the sums of the horizontal steps have the same distribution as a Bernoulli random walk conditioned to be at n at time k. Since $n/k \approx 1/(\lambda - \rho)$ this event has a very small probability (which converges to 0 exponentially as $n \to \infty$) so it is more convenient to study the transformed random walk S_k^* which takes steps with a distribution given by

$$P(X_1^* = 1) = \frac{1}{2}\frac{e^{\theta}}{(e^{\theta} + e^{-\theta})/2}$$

$$P(X_1^* = -1) = \frac{1}{2}\cdot\frac{e^{-\theta}}{(e^{\theta} + e^{-\theta})/2}$$

where θ is chosen so that $ES_k^* = n$. It is easy to check that if S_k is a symmetric Bernoulli random walk then

$$P(S_k^* = j) = P(S_k = j)\frac{e^{\theta j}}{((e^{\theta} + e^{-\theta})/2)^k}$$

and that

(1) $$(S^*_{[k\cdot]}|S_k^* = n) =_d (S_{[k\cdot]}|S_k = n)$$

(for more detail in a more general situation see Durrett (1977) Section 2).

The reason for considering the associated random walk is that the probability of $S_k^* = n$ is much larger—it follows from the local central theorem (see Stone (1965)) that $P(S_k^* = n) \sim C_\lambda^{-1}k^{-\frac{1}{2}}$ where $C_\lambda = (2\pi \text{ Var } S_1^*)^{\frac{1}{2}}$. Since the probability goes to 0 at a rate which is only a power of k we can obtain information from the following trivial estimate. Let A be a set of paths of length k.

(2) $$P(S^*_{[k\cdot]} \in A|S_k^* = n) \leq P(S^*_{[k\cdot]} \in A)/P(S_k^* = n).$$

If $k, n \to \infty$ and $k/n \to (\lambda - \rho)$ then we have

$$(3) \qquad P(S^*_{[k\cdot]} \in A \mid S_k^* = n) \leqslant 2C_\lambda k^{\frac{1}{2}} P(S_{[k\cdot]} \in A)$$

for all n sufficiently large.

The last inequality gives a way of relating the amount of backtracking in the string to the amount of backtracking in the random walk S_k^*, a quantity which can be easily calculated. Let $\tau_j = \inf\{k : S_k^* = j\}$ and let

$$B_{n,j}^* = j - \min\{S_k^* : k \geqslant \tau_j\}$$

(here the subscript n reminds us that the distribution of S_1^* depends upon n). It is easy to see that for any integers b and $j \geqslant 0$ $P(B_{n,j}^* \geqslant b) = r_n^b$ where $r_n = P(B_{n,0}^* \geqslant 1)$ so the $B_{n,j}^*$ $j \geqslant 0$ are identically distributed. It is also clear that $B_{n,j-1}^* \geqslant B_{n,j}^* - 1$ so the variables are not independent. To circumvent the dependence we use the following estimate

$$(4) \qquad P(\max_{0 \leqslant j < n} B_{n,j}^* \geqslant b) \leqslant nP(B_{n,0}^* \geqslant b).$$

Combining (1), (3) and (4) shows that as $n \to \infty$

$$P(\max_{0 \leqslant j < n} B_{n,j} \geqslant b) \leqslant 2C_\lambda k^{\frac{1}{2}} n r_n^b.$$

If $k/n \to (\lambda - \rho) \in (1, \infty)$ and we let

$$b_n = \frac{-3 \log n}{2 \log r_n} + K_n$$

be a sequence of integers with $K_n \to \infty$ then as $n \to \infty$

$$P(\max_{0 \leqslant j < n} B_{n,j} \geqslant b_n) \leqslant 2C_\lambda (\lambda - \rho)^{\frac{1}{2}} r_n^{K_n} \to 0$$

so with a probability which approaches 1 the amount of backtracking at stage n is smaller than

$$(5) \qquad \frac{3/2}{(\log r_n^{-1})} \log n + K_n .$$

The estimate above is fairly crude. If we could avoid the factor of $k^{\frac{1}{2}}$ from (2) we could replace the 3/2 in (5) by 1. In what follows we will show that it is possible to do this and that if we center by the resulting b_n then there is a nondegenerate limit.

The first step is to consider the amount of backtracking from zero. In the associated random walk $B_{n,0}^*$ has a geometric distribution with mean $1/r_n$ so on the basis of (1) we might guess that as $n \to \infty$ $B_{n,0}$ will converge to a geometric with mean $\lim_{n\to\infty} 1/r_n$—i.e., in the limit the conditioning on $S_k^* = n$ has no effect on the behavior of the string near time 0. To prove this observe that

$$P(B_{n,0} \geqslant b \mid |H| = k) = P(B_{n,0}^* \geqslant b \mid S_k^* = n)$$

$$P(B_{n,0}^* \geqslant b, S_k^* = n) = \textstyle\sum_{j=1}^k P(\tau_{-b} = j) P(S_k^* = n \mid S_j^* = -b).$$

As $k, n \to \infty$, $ES_1^* = n/k \to (\lambda - \rho)^{-1}$ so if b is fixed the distribution of τ_{-b} converges to the hitting time of $-b$ in a Bernoulli random walk with mean

$(\lambda - \rho)^{-1}$. If b and j are fixed then it follows from the local central limit (see Stone (1965)) that

$$P(S^*_{k-j} = n + b)/P(S^*_k = n) \to 1$$

and

$$\sup_k \sup_{j \leqslant k/2} P(S^*_{k-j} = n + b)/P(S^*_k = n) < \infty.$$

Combining the last three results give that as $n \to \infty$

$$\frac{P(B^*_{n,0} \geqslant b, S^*_k = n)}{P(S^*_k = n)} \to \Sigma_{j=1}^{\infty} P(\tau_{-b} = j) = r^b$$

where r is the limit of the r_n's. From this it follows that if $k/n \to (\lambda - \rho)$

$$P(B_{n,0} \geqslant b | \, |H| = k) \to r^b.$$

Combining this result with (1) from Section 2 gives

THEOREM 3.1. $P(B_{n,0} \geqslant b) \to r^b$.

Having analyzed the amount of backtracking from 0, we can now consider the maximum backtracking. The limit theorem for $B_{n,0}$ suggests that if n is large then B_n is almost the maximum of n exponential distributions. If the $B_{n,j}$'s were independent then it would be easy to calculate the limit distribution of the B_n's. Unfortunately, $B_{n,j-1} \geqslant B_{n,j} - 1$ so the $B_{n,j}$ are dependent and we need another approach to evaluate the limit of the B_n's. The first step is to solve the corresponding problem for the associated random walk. Let

$$B^*_n = \sup_{0 \leqslant m \leqslant n}(m - \inf_{l \geqslant \tau_m} S^*_l).$$

If we break the time interval $[0, \infty)$ into pieces $[\tau_0, \tau_1), [\tau_1, \tau_2), \cdots, [\tau_{n-1}, \tau_n), [\tau_n, \infty)$ then we can write B^*_n as

$$\sup_{0 \leqslant m < n}(m - \inf_{\tau_m \leqslant l < \tau_{m+1}} S^*_l) \vee (n - \inf_{l \geqslant \tau_n} S^*_l). \tag{6}$$

To see this observe that if the maximum backtracking occurs from $m < n$ then the minimum level was achieved before the process hit $m + 1$ (otherwise the amount of backtracking from $m + 1$ would be one more than the maximum).

The reason for our interest in (4) is that the variables

$$\nu_m = m - \inf_{\tau_m \leqslant l < \tau_{m+1}} S^*_l$$

are independent and identically distributed and the distribution of the second term is independent of n. Since the distribution of ν_0 is unbounded we have that as $n \to \infty$ the probability that the second term in (4) is bigger than the first approaches 0, so to investigate the limit behavior of S^*_n it suffices to consider

$$M^*_n = \sup_{0 \leqslant m < n} \nu_m.$$

To compute the limit distribution of M^*_n we observe that $P(M^*_n < b) = (1 - P(\nu_0 \geqslant b))^n$ and that from an elementary result about birth and death processes (see Hoel, Port, and Stone, pages 30–31) we have

$$(7) \qquad P(\nu_0 \geqslant b) = P(\tau_{-b} < \tau_1) = \left(\frac{q_n}{p_n}\right)^b \left(\frac{p_n - q_n}{p_n}\right)\left(1 - \left(\frac{q_n}{p_n}\right)^{b+1}\right)^{-1}$$

where $p_n = P(X_1^* = 1)$ and $q_n = P(X_1^* = -1)$. If we let $b_n = \log n / \log(p_n/q_n)$ and then pick x_n so that $b = b_n + x_n$ is an integer then we have

$$nP(\nu_0 \geqslant b) = \left(\frac{q_n}{p_n}\right)^{x_n} \left(\frac{p_n - q_n}{p_n}\right)\left(1 - \left(\frac{q_n}{p_n}\right)^{b+1}\right)^{-1}$$

From this it follows that if x_n is any bounded sequence

$$(8) \qquad P(M_n^* < b_n + x_n) - \exp\left(-\left(\frac{q}{p}\right)^{x_n}\left(\frac{p-q}{p}\right)\right) \to 0$$

where p and q are the limits of p_n and q_n.

Having obtained a limit theorem for the amount of backtracking in a random walk we will now consider the problem for the string. The final result (Theorem 3.2) states that (as $n \to \infty$) the maximum backtracking is the same as in the associated random walk. To prove this we will use the same outline as in the random walk case—we will first show that the backtracking from n is negligible and then write the backtracking in terms of the maxima of independent random variables.

To begin the first step we observe that since the conditioned random walk is symmetric $(k - \tau_n | S_k^* = n)$ has the same distribution as $(\tau_0' | S_k^* = n)$ where τ_0' is the time of the last exit from 0. To compute the limiting distribution of this quantity we observe that

$$P(\tau_0' \geqslant j, S_k^* = n) = P(\tau_0 \circ \theta_j < \infty) P(S_k^* = n | \tau_0 \circ \theta_j < \infty)$$

where $\tau_0 \circ \theta_j$ denotes $\inf\{k \geqslant j : S_k^* = 0\}$. As $n \to \infty$, $P(\tau_0 \circ \theta_j < \infty)$ converges to the corresponding quantity in the limiting associated random walk. To evaluate the limit of the second term we write

$$P(S_k^* = n | \tau_0 \circ \theta_j < \infty) = \Sigma_{i=j}^k P(\tau_0 = i | \tau_0 \circ \theta_j < \infty) P(S_k^* = n | S_i^* = 0).$$

If j is fixed then as $n \to \infty$ the sequence of distributions $(\tau_0 | \tau_0 \circ \theta_j < \infty)$ converges so it follows from the local central limit theorem (see Stone (1965)) that

$$P(S_k^* = n | \tau_0 \circ \theta_j < \infty) / P(S_k^* = n) \to 1$$

and we have shown

$$(9) \qquad P(\tau_0' \geqslant j | S_k^* = n) \to P(\tau_0 \circ \theta_j < \infty).$$

Now that we know that the distribution of $k - \tau_n$ converges we can condition on $k - \tau_n = i_n$ and consider the distribution of

$$(\sup_{0 \leqslant m < n} \nu_m | \tau_n = k_n)$$

where $k_n = k - i_n$. It is intuitively clear that since $k_n = k + 0(1)$ and

$$k = \frac{n}{ES_1^*} = nE\tau_1 = E\tau_n$$

that conditioning on the value of τ_n will not effect the limit law for the maximum. The rest of this section is devoted to proving that this is the case.

We begin by observing that

$$\text{(10)} \quad P(\sup_{0 \leqslant m < n} \nu_m < b, \tau_n = k_n) \\ = (1 - P\{\nu_0 \geqslant b\})^n P(\tau_n = k_n | \sup_{0 \leqslant m < n} \nu_m < b).$$

If we let $b_n = \log n / \log(p_n/q_n)$ and then pick a bounded sequence x_n so that $b_n + x_n$ is an integer then it follows from the computations above (8) that

$$\text{(11)} \qquad (1 - P\{\nu_0 \geqslant b_n + x_n\})^n - \exp\left(-\left(\frac{q}{p}\right)^{x_n}\left(\frac{p-q}{p}\right)\right) \to 0.$$

To evaluate the limit of the other term we will use the local central limit theorem. To do this we observe that conditioned on $\sup_{m<n} \nu_m < b$, τ_n is the sum of n independent random variables with the same distribution as $(\tau_1 | \nu_1 < b)$. If $b \to \infty$

$$\operatorname{Var}(\tau_1 | \nu_1 < b) \to \operatorname{Var}(\tau_1)$$

and if $b \geqslant 2$ we have

$$E(\tau_1^3 | \nu_1 < b) \leqslant E\tau_1^3 / P(\tau_1 = 1) < \infty$$

so it follows from the classical proof of the local central limit theorem (see Gnedenko and Kolmogorov (1944), pages 233–235) that if $\mu_n = E(\tau_1 | \nu_1 < b_n)$, $\sigma_n^2 = \operatorname{Var}(\tau_1 | \nu_1 < b_n)$, and y_n is a bounded sequence then

$$\text{(12)} \qquad \sigma_n n^{\frac{1}{2}} P\left(\tau_n = \mu_n + y_n \sigma_n n^{\frac{1}{2}} | \sup_{0 \leqslant m < n} \nu_m < b\right) - \varphi(y_n) \to 0$$

where φ is the standard normal density.

To use (12) we have to compute μ_n. To do this we begin by observing that

$$E(\tau_1 | \tau_1 > \tau_{-b}) = E(\tau_{-b} | \tau_1 > \tau_{-b}) + E(\tau_1 | S_0^* = -b)$$

and

$$E(\tau_1 | S_0^* = -b) = (b+1) E^0 \tau_1.$$

To evaluate the other term we write

$$E(\tau_{-b} | \tau_1 > \tau_{-b}) = \Sigma_{i=1}^b E(\tau_{-i} | S_0^* = -i + 1, \tau_{-i} < \tau_1)$$

and observe that as $i \to \infty$, $E(\tau_{-i} | S_0^* = -i + 1, \tau_{-i} < \tau_1)$ converges to $E(\tau_{-1} | S_0^* = 0, \tau_{-1} < \infty)$ so

$$E(\tau_{-b} | \tau_1 > \tau_{-b}) \sim bE(\tau_{-1} | \tau_{-1} < \infty).$$

To compute $E(\tau_1 | \tau_1 < \tau_{-b})$ we observe that

$$E\tau_1 = P(\tau_1 > \tau_{-b}) E(\tau_1 | \tau_1 > \tau_{-b}) + P(\tau_1 < \tau_{-b}) E(\tau_1 | \tau_1 < \tau_{-b})$$

so we have

$$(13)\qquad E(\tau_1|\tau_1 < \tau_{-b}) = \frac{E\tau_1}{P(\tau_1 < \tau_{-b})} - \frac{P(\tau_1 > \tau_{-b})}{P(\tau_1 < \tau_{-b})} E(\tau_1|\tau_1 > \tau_{-b}).$$

To evaluate the limit of the right-hand side we observe that

$$\frac{nE\tau_1}{P(\tau_1 < \tau_{-b})} = nE\tau_1 + nE\tau_1 P(\tau_1 > \tau_{-b}) + nE\tau_1 \Sigma_{k=2}^{\infty} P(\tau_1 > \tau_{-b})^k.$$

From this formula we see that if $b = b_n + 0(1)$

$$nE\tau_1 P(\tau_1 < \tau_{-b}) = 0(1)$$

and

$$nE\tau_1 \Sigma_{k=2}^{\infty} P(\tau_1 < \tau_{-b})^k \to 0$$

so the first term in (13) is $E\tau_1 + 0(n^{-1})$. We will now show that the second is $0(n^{-1}\log n)$. To do this we observe that if $\varepsilon > 0$ then

$$E(\tau_1|\tau_1 > \tau_{-b}) \leqslant (b+1)E\tau_1 + b(E(\tau_{-1}|\tau_{-1} < \infty) + \varepsilon)$$

for n sufficiently large so

$$\frac{nP(\tau_1 < \tau_{-b})}{P(\tau_1 < \tau_{-b})} E(\tau_1|\tau_1 > \tau_{-b}) = 0(\log n).$$

Combining the last two results with (13) shows that $\mu_n = nE\tau_1 + 0(\log n)$ so it follows from (12) that

$$P(\tau_n = k_n|\sup_{0 \leqslant m < n} \nu_m < b) \sim \varphi(0)/\sigma n^{\frac{1}{2}}$$

where $\sigma^2 = \operatorname{Var}(\tau_1)$.

Since the result above is the same as the asymptotic formula that the local central limit theorem gives for $P(\tau_n = k_n)$ we have shown that the second term in (8) converges to 1 so we have proved that $(\sup_{m<n} \nu_m|\tau_n = k_n)$ has the same limiting behavior as the unconditioned sequence. Combining this observation with (9) above and (1) of Section 2 we have

THEOREM 3.2. *If* $b_n = \log n/\log(p_n/q_n)$ *then*

$$\sup_{z \in Z} P(B_n < z) - \exp\left(-\left(\frac{q}{p}\right)^{z-b_n}\left(\frac{p-q}{p}\right)\right) \to 0.$$

To compare this result with (5) and to obtain the result in the form given in the introduction observe that $r = q/p$.

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Department of Mathematics
University of California, Los Angeles
Los Angeles, California 90024