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ON THE MULTIVARIATE LAW OF THE ITERATED LOGARITHM

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A Hilbert space law of the iterated logarithm is proved which generalizes Kolmogorov's law for bounded random variables and which generalizes results of Teicher for unbounded random variables. The result for identically distributed random vectors is a consequence. The key idea is the requirement of the convergence of the average of the covariance operators.

1. Introduction. The best early result on the law of the iterated logarithm is due to A. N. Kolmogorov [2], whose hypotheses required bounded random variables growing at a rate depending on their variances. Recently, Teicher and others have significantly extended this result to unbounded random variables by giving sufficient conditions giving the law; see [12] and [1]. In another direction, the law of the iterated logarithm has been extended by Kuelbs and others, primarily by the use of the central limit theorem, to the case of random vectors with values in a Hilbert space or Banach space; see for example [5]. The present work is primarily an extension of Teicher's work to finite-dimensional and Hilbert space valued random vectors.

An interesting feature of the multidimensional law is the more subtle limit set, essentially an ellipsoid. In getting away from the identically distributed case, however, an equally interesting difficulty involves settling on appropriate requirements for the variances of the random vector components. A natural condition is one which generalizes a conditon on the covariance matrices yielding the finite-dimensional central limit theorem; see [8], page 25. This condition is that the average of the covariance matrices converge to a limit matrix; it is trivial in one dimension. The generalization used for the Hilbert space setting is that the average of the covariance operators converges to a limit operator (which then has an intimate relationship with the a.s. limit set). This condition is the fundamental idea in the paper, and seems to be previously unexplored. It would be interesting to determine the accessibility of the infinite-dimensional central limit theorem from this condition.

In the finite-dimensional case, the convergence of the average of the covariance matrices means convergence of matrices entrywise. In the Hilbert space case, however, convergence of the average of the covariance operators is operator convergence, which could refer to weak operator convergence, strong operator convergence, or norm convergence, among others. What is actually needed is the even more stringent trace class convergence. This is discussed briefly in Section 2.

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It will be evident from the proofs that weaker forms of convergence do not suffice, so that in some sense this is the right condition.

Section 2 begins with Theorem 1, a finite-dimensional generalization of Kolmogorov's law. It is in line with Sheu's work [11] and the proof uses many of the same ideas, particularly the reduction via a linear transformation to the one-dimensional case. It is a nontrivial generalization in its introduction of the condition of convergence for the covariance matrices, and more significantly, in its handling of the subtle difficulties arising from a singular limit matrix. Theorem 2 provides an extension of Kolmogorov's law to the Hilbert space setting, and relies on the finite-dimensional result for its proof.

In Section 3, results analogous to those of Teicher are proved for random vectors with values in a Hilbert space. Theorem 3 is the principle result here, and relies heavily on Theorem 2. Some immediate corollaries of this result yield results which improve those of the literature by demanding existence only of the second moment; see [5], page 397. Theorem 4 generalizes a necessary condition given by Teicher.

Upon completion of these results, the author was introduced to some recent work of Kuelbs [4] in which a result similar to Theorem 2 is proved. The similarity is in the hypothesis of boundedness for the random vectors, and both results depend on an exponential estimate due to Yurinskii [15]. However, instead of the covariance operator convergence used here, Kuelbs uses a condition essentially requiring uniform approximation by finite-dimensional random vectors. His condition seems generally different, but is clearly more restrictive in the finite-dimensional case. Moreover, his condition suggests the sufficient conditions for the covariance operator convergence presented in Theorem 5 here. Again this indicates that the convergence condition is closer perhaps to the proper requirements for the situation. However, it should also be noted that the results in [4] apply to the more general situation of Banach space valued random vectors.

2. Bounded random vectors. Throughout the following, $\mathbb{R}^p$ will denote p-dimensional real Euclidean space with vectors $a = (a_1, \ldots, a_p)$, and H will denote a real separable Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$. Z_n will denote a random vector in $\mathbb{R}^p$ or H; that is, a measurable function from a probability space $(\Omega, \mathfrak{F}, P)$ to $(\mathbb{R}^p, \mathfrak{B})$ or $(H, \mathfrak{B})$ where $\mathfrak{B}$ is the σ-algebra of Borel sets. The expectation of Z_n is denoted $E(Z_n)$ and defined by $E(Z_n) = x$ where $E\langle Z_n, y\rangle = \langle x, y\rangle$ for all y. The covariance of Z_n is denoted by $\mathrm{Cov}(Z_n) = T_n$ and defined by $\langle T_n x, y\rangle = E\langle Z_n, x\rangle\langle Z_n, y\rangle - \langle E(Z_n), x\rangle\langle E(Z_n), y\rangle$ for all x and y. See [7] for details and properties. Also, $\log\log s_n^2$ will be abbreviated LLs_n^2.

For the limit matrix Σ in Theorem 1, let μ be any mean zero probability measure on $\mathbb{R}^n$ with covariance matrix Σ; for example, μ could be the mean zero Gaussian measure (see [7]). Then, following the notation of Lemma 2.1 of [3], μ generates a subspace H_μ of $\mathbb{R}^n$ consisting of all elements of the form $x\Sigma = \int_{\mathbb{R}^n}(xz')z \, d\mu(z)$, for $x \in \mathbb{R}^n$. The inner product and norm induced on H_μ by μ are respectively given by

$\langle x\Sigma, y\Sigma\rangle_\mu = \int_{\mathbb{R}^n}(xz^t)(zy^t)\,d\mu(z) = x\Sigma y^t$ and $\|x\Sigma\|_\mu = (x\Sigma x^t)^{\frac{1}{2}}$. Let K_μ be the unit ball of H_μ with regard to $\|\cdot\|_\mu$.

THEOREM 1. *If $\{Z_n\}$ are independent random vectors in $\mathbb{R}^p$ with $E(Z_n) = 0$ and $\mathrm{Cov}(Z_n) = \Sigma_n$, if for some positive constants $\{s_n^2\}$, $s_n^2 \uparrow \infty$, $s_{n+1}^2/s_n^2 \to 1$, ess sup $|Z_n| \leqslant \varepsilon_n s_n(LLs_n^2)^{-\frac{1}{2}}$ for some sequence $\varepsilon_n \downarrow 0$, and $(1/s_n^2)\Sigma_{j=1}^n \Sigma_j \to \Sigma$, and if μ is a mean zero probability measure on $\mathbb{R}^p$ with covariance matrix Σ, then the* a.s. *limit set D of $\{(2s_n^2LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^n Z_j\}$ is K_μ.*

PROOF. The first step justifies the use of Theorem 3.1 of [3]; it is shown that if $a \in \mathbb{R}^n$, then

$$\limsup_{n\to\infty}(2s_n^2LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^n aZ_j^t \leqslant \sup_{x\in K_\mu} ax^t \quad \text{a.s.} \tag{1}$$

Observe that $\mathrm{Var}(aZ_j^t) = a\Sigma_j a^t$ so that $(1/s_n^2)\Sigma_{j=1}^n \mathrm{Var}(aZ_j^t) \to a\Sigma a^t$. Suppose first that $a\Sigma a^t = 0$. It follows from the hypotheses that $s_n^2/s_{n+1}^2 \to 1$. Let $c > 1$ and let $n_1 < n_2 < \dots$ be such that $s_{n_i} \sim c^i$. Now ess sup $|aZ_n^t| \leqslant |a|\varepsilon_n s_n(LLs_n^2)^{-\frac{1}{2}}$ so $\mathrm{Var}(aZ_n^t) \leqslant |a|^2\varepsilon_n^2 s_n^2(LLs_n^2)^{-1}$. Let $b_n = s_n(LLs_n^2)^{\frac{1}{2}}$ and $t_i^2 = (1/b_{n_i^2})\Sigma_{j=n_{i-1}+1}^{n_i}\mathrm{Var}(aZ_j^t)$. Then ess sup $|aZ_n^t| = o(b_n/LLb_n)$ and $\Sigma_{i=1}^\infty e^{-\varepsilon^2/t_i^2} < \infty$. It follows from a result of Loève([6], page 258) that $(s_n^2LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^n aZ_j^t \to 0$ a.s. This of course yields (1) for this case.

Suppose now that $a\Sigma a^t \neq 0$. Then $\Sigma_{j=1}^n \mathrm{Var}(aZ_j^t/(a\Sigma a^t)^{\frac{1}{2}}) \sim s_n^2$ so that from Kolmogorov's law ([6], page 260),

$$\limsup_{n\to\infty}(2s_n^2LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^n aZ_j^t/(a\Sigma a^t)^{\frac{1}{2}} = 1 \quad \text{a.s.}$$

Therefore, with reference to Lemma 2.1 of [3],

$$\limsup_{n\to\infty}(2s_n^2LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^n aZ_j^t = (a\Sigma a^t)^{\frac{1}{2}} = \sup_{x\in K_\mu} ax^t \text{ a.s.,} \tag{2}$$

and (1) holds for this case also.

Theorem 3.1 of [3] now applies and asserts, since $\mathbb{R}^p$ may be considered compact by adding ∞, that

$$\lim_{n\to\infty} d\left((2s_n^2LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^n Z_j, K_\mu\right) = 0 \quad \text{a.s.,}$$

where $d(z, K_\mu) = \inf_{x\in K_\mu}|z - x|$. Thus $D \subset K_\mu$.

To show that $K_\mu \subset D$, one argues as in Proposition 2.7 of [11], using (2) and the fact that $|x\Sigma|^2 \leqslant Lx\Sigma x^t$ for some constant L and for all $x \in \mathbb{R}^p$. This completes the proof.

In this theorem, if $\Sigma \neq 0$, then it is necessarily true that $s_{n+1}^2/s_n^2 \to 1$. If $\Sigma = 0$, it is straightforward to see that lim sup $s_{n+1}^2/s_n^2 < \infty$ is sufficient. Although not so apparent, it is also true that lim inf $s_{n+1}^2/s_n^2 > 1$ suffices when $\Sigma = 0$.

In the ensuing results, convergence of symmetric, positive semidefinite, trace class operators on the Hilbert space H arises, and this convergence takes place in the trace class norm $\|\cdot\|_*$. If A is an operator on H, the absolute value of A is defined to be $|A| = (A^tA)^{\frac{1}{2}}$. An operator A is a trace class operator if

$\Sigma_k\langle|A|g_k, g_k\rangle < \infty$ for some cons$\{g_k\}$. In this case, $\|A\|_* \equiv \Sigma_k\langle|A|g_k, g_k\rangle$ is the trace class norm of A and is independent of the cons$\{g_k\}$. If A is a trace class operator, then $\mathrm{tr}(A) \equiv \Sigma_k\langle Ag_k, g_k\rangle$ converges for any cons$\{g_k\}$ and is independent of the choice of $\{g_k\}$. The trace class operators are complete under $\|\cdot\|_*$ and are closed under addition, scalar multiplication, and composition with any bounded operator. Moreover, if X is a bounded operator and A is a trace class operator, then $\|AX\|_* \leqslant \|X\|\|A\|_*$, $\|XA\|_* \leqslant \|X\|\|A\|_*$, $|\mathrm{tr}(A)| \leqslant \|A\|_*$, and $\|A\| \leqslant \|A\|_*$. In particular, trace class convergence implies norm convergence. These properties are the ones used in what follows; for proofs and further details, see [9] or [10].

For the limit operator T in Theorem 2, 3 and 4, let μ be any mean zero probability measure on H with covariance operator T; again, μ could be taken as the mean zero Gaussian measure with covariance operator T. With reference to Lemma 2.1 of [3] as before, μ generates a Hilbert space H_μ continuously embedded in H and consisting of all elements of the form $T(x) = \int_H\langle x, z\rangle z\, d\mu(z)$ for $x \in H$. The inner product and norm on H_μ are given by $\langle T(x), T(y)\rangle_\mu = \int_H\langle x, z\rangle\langle y, z\rangle\, d\mu(z)$ and $\|T(x)\|_\mu = \langle T(x), T(x)\rangle_\mu^{\frac{1}{2}}$. Let K_μ be the unit ball of H_μ with respect to $\|\cdot\|_\mu$ and recall that K_μ is a compact subset of H. See [3] for details.

The following result due to Yurinskii is used; see [15], page 491.

THEOREM (Yurinskii). *If $\{\xi_n\}$ are independent random vectors in H which satisfy $E(\xi_i) = 0$, $E\|\xi_i\|^k \leqslant k!b_i^2K^{m-2}/2$ for $m \geqslant 2$, and $B_n^2 = b_1^2 + \cdots + b_n^2$, then*

$$P[\|\xi_1 + \cdots + \xi_n\| \geqslant xB_n] \leqslant 2\exp\{-(x^2/2)/(1 + 1.62xK/B_n)\}.$$

THEOREM 2. *If $\{Z_n\}$ are independent random vectors in H with $E(Z_n) = 0$ and $\mathrm{Cov}(Z_n) = T_n$ such that for some positive constants $\{s_n^2\}$, $s_n^2\uparrow\infty$, $s_{n+1}^2/s_n^2 \to 1$, ess $\sup\|Z_n\| \leqslant \varepsilon_n s_n(LLs_n^2)^{-\frac{1}{2}}$ for some sequence $\varepsilon_n\downarrow 0$, and $(1/s_n^2)\Sigma_{j=1}^n T_j \to T$ in $\|\cdot\|_*$, and if μ is any mean zero probability measure with covariance operator T, then $\{(2s_n^2LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^n Z_j\}$ is* a.s. *relatively compact and the* a.s. *limit set D is K_μ.*

PROOF. Recall from the hypotheses that $s_n^2/s_{n+1}^2 \to 1$. Let $\beta > 1$ and let $n_1 < n_2 < \ldots$ be such that $s_{n_r}^2 \sim \beta^r$.

Let $\{g_k\}$ be a cons for H and let Q_m be the orthogonal projection onto $\overline{Sp}\{g_{m+1}, g_{m+2}, \ldots\}$. It will be shown that for any $\varepsilon > 0$ there exists M so that $m \geqslant M$ implies

$$P\Big[\|(2s_n^2LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^n Q_m(Z_j)\| \geqslant \varepsilon \text{ i.o.}\Big] = 0. \tag{3}$$

Let $D_r = [\|\Sigma_{j=1}^n Q_m(Z_j)\| \geqslant \varepsilon(2s_{n_r}^2LLs_{n_r}^2)^{\frac{1}{2}}$ for some $n_r < n \leqslant n_{r+1}]$. To show (3), it is sufficient to show $P(D_r \text{ i.o.}) = 0$. For $d_r \equiv \sup_{n_r<n\leqslant n_{r+1}} P[\|\Sigma_{j=1}^{n_{r+1}} Q_m(Z_j) - \Sigma_{j=1}^n Q_m(Z_j)\| > (\varepsilon/2)(2s_{n_r}^2LLs_{n_r}^2)^{\frac{1}{2}}]$, it follows from Chebyshev's inequality that $d_r \to 0$ uniformly in m. Furthermore,

$$P(D_r) \leqslant (1 - d_r)^{-1}P\left[\left\|\frac{1}{s_{n_{r+1}}}\Sigma_{j=1}^{n_{r+1}} Q_m(Z_j)\right\| \geqslant (\varepsilon/2)(s_{n_r}/s_{n_{r+1}})(2LLs_{n_r}^2)^{\frac{1}{2}}\right].$$

From $(1/s_n^2)\Sigma_{j=1}^n T_j \to T$ in $\|\cdot\|_*$ and properties of $\|\cdot\|_*$, it follows that

$$E\left\|\frac{1}{s_{n_{r+1}}}\Sigma_{j=1}^{n_{r+1}} Q_m(Z_j)\right\|^2 = \frac{1}{s_{n_{r+1}}^2}\Sigma_{j=1}^{n_{r+1}}\Sigma_{i=1}^{n_{r+1}}(\Sigma_{k>m}E\langle Z_j, g_k\rangle\langle Z_i, g_k\rangle)$$

$$= \Sigma_{k>m}\langle\frac{1}{s_{n_{r+1}}^2}\Sigma_{j=1}^{n_{r+1}} T_j g_k, g_k\rangle \to \Sigma_{k>m}\langle Tg_k, g_k\rangle \qquad \text{as} \quad r\to\infty.$$

Then for $\eta > 0$, there exist N and M such that $r \geqslant N$ and $m \geqslant M$ imply the following: $E\|\Sigma_{j=1}^{n_{r+1}} Q_m(Z_j)\|^2 \leqslant \eta^2 s_{n_{r+1}}^2$, $d_r < \frac{1}{2}$, $s_{n_r}/s_{n_{r+1}} \geqslant \frac{1}{2}$, and $1 + 1.62(\varepsilon 2^{\frac{1}{2}}/4\beta\eta^2)\varepsilon_{n_{r+1}}(LL\beta^r/LL\beta^{r+1})^{\frac{1}{2}} \leqslant 2$. The result of Yurinskii then applies and asserts, for $r \geqslant N$ and $m \geqslant M$,

$$P\left[\left\|\frac{1}{s_{n_{r+1}}^2}\Sigma_{j=1}^{n_{r+1}} Q_m(Z_j)\right\| \geqslant (\varepsilon/2)(s_{n_r}/s_{n_{r+1}})(2LLs_{n_r}^2)^{\frac{1}{2}}\right]$$

$$\leqslant 2(\log\beta)^{-\varepsilon^2/32\beta^2\eta^2} r^{-\varepsilon^2/32\beta^2\eta^2}.$$

Thus for η chosen so that $\varepsilon^2/32\beta^2\eta^2 > 1$, $\Sigma_r P(D_r) < \infty$ and by the Borel-Cantelli lemma, $P(D_r \text{ i.o.}) = 0$. This establishes (3).

Let P_m be the orthogonal projection onto $Sp\{g_1, \cdots, g_m\}$. Then $\mathrm{Cov}(P_m(Z_n)) = P_m T_n P_m$ which is finite-dimensional. Moreover,

$$\frac{1}{s_n^2}\Sigma_{j=1}^n T_j \to T \text{ in } \|\cdot\|_* \text{ implies } \frac{1}{s_n^2}\Sigma_{j=1}^n P_m T_j P_m \to P_m T P_m.$$

Theorem 1 now applies to $\{(2s_n^2 LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^n P_m(Z_j)\}$. The proof of Theorem 2 is completed using Theorem 1 and (3) just as in Theorem 3.1 of [3] and its proof.

3. Unbounded random vectors. The same notational conventions are used in this section as in the preceding one, particularly in regard to the measure μ associated with the limit operator T and the limit set K_μ coming from μ; again see Lemma 2.1 of [3] for details.

THEOREM 3. *If $\{Z_n\}$ are independent random vectors in H with distributions $\{\mu_n\}$ and with $E(Z_n) = 0$ and* $\mathrm{Cov}(Z_n) = T_n$, *if for some positive constants $\{s_n^2\}$, $s_n^2\uparrow\infty$, $s_{n+1}^2/s_n^2 \to 1$, and $(1/s_n^2)\Sigma_{j=1}^n T_j \to T$ in $\|\cdot\|_*$, if μ is a mean zero probability measure on H with covariance operator T, and if for some $\delta > 0$ and for all $\varepsilon > 0$,*

(1) $\Sigma_{j=1}^\infty P[\|Z_j\| > \delta s_j(LLs_j^2)^{\frac{1}{2}}] < \infty$,

(2) $\frac{1}{s_n^2}\Sigma_{j=1}^n \int_{[\|z\| > \varepsilon s_j(LLs_j^2)^{-\frac{1}{2}}]}\|z\|^2\, d\mu_j(z) = o(1)$, *and*

(3) $\Sigma_{j=1}^\infty (s_j^2 LLs_j^2)^{-1}\int_{[\varepsilon s_j(LLs_j^2)^{-\frac{1}{2}} < \|z\| \leqslant \delta s_j(LLs_j^2)^{\frac{1}{2}}]}\|z\|^2\, d\mu_j(z) < \infty$, *then* $\{(2s_n^2 LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^n Z_j\}$ *is* a.s. *relatively compact, and the* a.s. *limit set D is K_μ.*

PROOF. Let $Z_n' = Z_n I([\|Z_n\| \leqslant \varepsilon_n s_n(LLs_n^2)^{-\frac{1}{2}}])$ where $I(A)$ is the indicator function of the set A and where $\varepsilon_n \downarrow 0$. Let $T_n' = \operatorname{Cov}(Z_n')$. Then for any $y \in H$,

$$\begin{aligned}\langle T_n'y, y\rangle &= E\langle Z_n' - E(Z_n'), y\rangle^2 \leqslant E\langle Z_n', y\rangle^2 \\ &= E\Big(\langle Z_n, y\rangle^2 I\Big(\Big[\|Z_n\| \leqslant \varepsilon_n s_n\big(LLs_n^2\big)^{-\frac{1}{2}}\Big]\Big) \leqslant E\langle Z_n, y\rangle^2 = \langle T_n y, y\rangle.\end{aligned}$$

Thus, in addition to being symmetric, $T_n - T_n'$ is positive semidefinite and in particular $|\Sigma_{j=1}^n T_j - T_j'| = \Sigma_{j=1}^n T_j - T_j'$. Then from condition (2), noting that some $\varepsilon_n \downarrow 0$ may replace ε as in [1],

$$\begin{aligned}\left\|\frac{1}{s_n^2}\Sigma_{j=1}^n T_j - \frac{1}{s_n^2}\Sigma_{j=1}^n T_j'\right\|_* &= \frac{1}{s_n^2}\operatorname{tr}(\Sigma_{j=1}^n T_j - T_j') \\ &= \frac{1}{s_n^2}\Sigma_{j=1}^n \Sigma_k(\langle T_j g_k, g_k\rangle - \langle T_j' g_k, g_k\rangle) \\ &= \frac{1}{s_n^2}\Sigma_{j=1}^n E\big(\Sigma_k\langle Z_j, g_k\rangle^2 - \Sigma_k\langle Z_j', g_k\rangle^2\big) \\ &\quad + \frac{1}{s_n^2}\Sigma_{j=1}^n \Sigma_k\langle E(Z_j) - E(Z_j'), g_k\rangle^2 \\ &\leqslant 2\frac{1}{s_n^2}\Sigma_{j=1}^n E\left\|Z_j I\Big(\Big[\|Z_j\| > \varepsilon_j s_j\big(LLs_j^2\big)^{-\frac{1}{2}}\Big]\Big)\right\|^2 \to 0.\end{aligned}$$

This in turn implies that $(1/s_n^2)\,\Sigma_{j=1}^n T_j' \to T$ in $\|\cdot\|_*$. Theorem 2 now applies to $\{Z_j' - E(Z_j')\}$ and asserts that $\{(2s_n^2 LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^n(Z_j' - E(Z_j'))\}$ is a.s. relatively compact with a.s. limit set equal to K_μ.

Finally $\{Z_n - (Z_n' - E(Z_n'))\}$ is dealt with exactly as in Theorem 1 of [12], except that the norm $\|\cdot\|$ replaces the absolute value $|\cdot|$ and an appeal must be made to the general Toeplitz-Kronecker lemma; see [14]. This completes the proof of the theorem.

Exactly as in Theorem 1, when $T \neq 0$, $s_{n+1}^2/s_n^2 \to 1$ necessarily holds, and when $T = 0$, either lim sup $s_{n+1}^n/s_n^2 < \infty$ or lim inf $s_{n+1}^2/s_n^2 > 1$ actually suffices. Moreover, when $T = 0$, condition (2) is automatically satisfied. Notice also that conditions (1) and (3) are implied by the following single condition: for some $\alpha \in (0, 2]$, and for all $\varepsilon > 0$,

$$\Sigma_{j=1}^\infty\big(s_j^2 LLs_j^2\big)^{-\frac{\alpha}{2}}\int_{\left[\|z\| > \varepsilon s_j(LLs_j^2)^{-\frac{1}{2}}\right]}\|z\|^\alpha\, d\mu_j(z) < \infty. \tag{4}$$

Thus conditions (1), (2), (3) could be replaced by conditions (2), (4).

Two special cases of Theorem 3 are particularly worth noticing. The first is the case where H is finite dimensional; that is, $H = \mathbb{R}^p$ for some p. In this case, the convergence of the covariance operators becomes just entrywise convergence of the covariance matrices, and the results give conditions under which normalized sums of finite-dimensional, possibly unbounded, random vectors converge to a possibly degenerate ellipsoid. The second is the case in which $T_n \equiv T$ for all n. Then $s_n^2 = n$ and the convergence of the covariance operators is trivial. Second moments are the highest moments required in the hypotheses of this case, and in this respect, the results improve the results in the literature; see, for example, [5], page 397.

The weighted independent identically distributed case provides another instance of the above results. This is a generalization of the careful study in the random variable case made by Teicher [12]. The idea is to start with independent identically distributed random vectors $\{Z_n\}$ in H with common distribution μ and with constants $\{\sigma_n\}$ letting $s_n^2 = \Sigma_{j=1}^n \sigma_j^2$. Theorem 3 is then applied to the sequence $\{\sigma_n Z_n\}$; again the convergence of the covariance operators is trivial, and conditions (1), (2), (3) become restrictions relating $\{\sigma_n\}$ and μ. Teicher's work in the weighted independent identically distributed case depends on considerations of the rate of growth of the σ_n's, and his results are readily extended to the Hilbert space setting considered here. The most important case of independent identically distributed random vectors, with $\sigma_n = 1$ for all n, is in particular directly accessible from Theorem 3, although this result for the Hilbert space case has been previously established by other means.

In generalizing conditions (1), (2), (3) to the multivariate case, various possibilities arise which are not present in one dimension. In particular, by considering the cons$\{g_k\} = \{e_1, f_1, e_2, f_2, \cdots\}$ where $H_\mu \subset \overline{Sp}\{e_k\}$ and $H_\mu^\perp \subset \overline{Sp}\{f_l\}$, it is reasonable to try to replace $\|Z_j\|$ in each condition by $\langle Z_j, g_k\rangle$ for $k = 1, 2, \cdots$. It is readily seen that the conditions with $\|Z_j\|$ imply those involving $\langle Z_j, g_k\rangle$ while the converse holds if $\{g_k\}$ is a finite system. Alternatively, the norm $\|Z_j\|$ in the conditions could be replaced by the special seminorm $\|Z_j\|_\mu$ coming from the limit operator T. Here, the conditions with $\|Z_j\|$ imply those with $\|Z_j\|_\mu$ if $\{e_k\}$ is a finite system, while the reverse implications hold if $\{e_k\}$ is a complete system.

The next theorem generalizes Teicher's work in the other direction by providing a necessary condition for the Hilbert space iterated logarithm law.

THEOREM 4. *If $\{Z_n\}$ are independent random vectors in H with $E(Z_n) = 0$ and $\mathrm{Cov}(Z_n) = T_n$ such that for some positive constants $\{s_n^2\}$, $s_n^2 \uparrow \infty$ and $(1/s_n{}^2)\Sigma_{j=1}^n T_j \to T$ in $\|\cdot\|_*$, and if $\{(2s_n^2 LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^n Z_j\}$ is* a.s. *relatively compact with* a.s. *limit set D equal to K_μ where μ is a mean zero probability measure with covariance operator T, then $\Sigma_{n=1}^\infty P[\|Z_n\| > \gamma(s_n^2 LLs_n^2)^{\frac{1}{2}}] < \infty$ for all large γ; moreover, if $\{e_k\}$ is a finite system, then $\Sigma_{n=1}^\infty P[\|Z_n\|_\mu > \eta(s_n^2 LLs_n^2)^{\frac{1}{2}}] < \infty$ for all $\eta > 2^{\frac{1}{2}}$.*

PROOF. Notice that there exist constants K and L such that $\|\cdot\| \leqslant K\|\cdot\|_\mu$ on H_μ, and $\|\cdot\|_\mu \leqslant L\|\cdot\|$ if $\{e_k\}$ is finite. Now it follows by hypothesis that $P[\|\Sigma_{j=1}^n Z_j\|_\mu \geqslant \eta(2s_n^2 LLs_n^2)^{\frac{1}{2}}\text{i.o.}] = 0$ for all $\eta > 1$ and $P[\|\Sigma_{j=1}^n Z_j\| > \gamma(2s_n^2 LLs_n^2)^{\frac{1}{2}}$ i.o.$] = 0$ for all $\gamma > K$. Using Chebyshev's inequality, independence, and the convergence of $(1/s_n{}^2)\Sigma_{j=1}^n T_j$ to T in $\|\cdot\|_*$, a straightforward calculation shows $(2s_n^2 LLs_n^2)^{-\frac{1}{2}}\Sigma_{j=1}^{n-1} Z_j \to_p 0$. The proof is completed by putting these ideas together exactly as in the proof of Theorem 1 of [13].

The final result gives a set of conditions guaranteeing the trace class convergence of the covariance operators which has figured so prominently in all previous theorems.

THEOREM 5. *If* $\{Z_n\}$ *are independent random vectors in* H *with* $E(Z_n) = 0$ *and* $\mathrm{Cov}(Z_n) = T_n$, *and if* T *is a symmetric, positive semidefinite, trace class operator and* $\{s_n^2\}$ *are positive constants with* $s_n^2 \uparrow \infty$ *and* $n = 0(s_n^2)$ *such that*

(a) $\frac{1}{s_n^2}\Sigma_{j=1}^n E\langle Z_j, x\rangle\langle Z_j, y\rangle \to \langle Tx, y\rangle$ *for all* $x, y \in H$ i.e., $\frac{1}{s_n^2}\Sigma_{j=1}^n T_j \to T$ *weakly*;

(b) *for every* $\delta > 0$ *there exists a* $\mathrm{cons}\{g_k\}$ *and a projection* P_m *onto* $Sp\{g_1, \cdots, g_m\}$ *such that* $E\|Z_n - P_m(Z_n)\|^2 \leqslant \delta$ *for all* n; *and*

(c) *there exists* M *such that* $E\|Z_n\|^2 \leqslant M$ *for all* n, *then* $(1/s_n^2)\Sigma_{j=1}^n T_j \to T$ *in* $\|\cdot\|_*$.

PROOF. As noted before, $T_n = \mathrm{Cov}(Z_n)$ implies $P_m T_n P_m = \mathrm{Cov}(P_m(Z_n))$. A direct calculation shows that $\|T_j - P_m T_j P_m\|_* \leqslant 2(E\|Z_j - P_m(Z_j)\|^2)^{\frac{1}{2}}(E\|Z_j\|^2)^{\frac{1}{2}}$. Also $P_m T P_m \to T$ in $\|\cdot\|_*$ as $m \to \infty$. Then for $\varepsilon > 0$, there exist, by conditions (b) and (c), $\{g_k\}$ and P_m so that $\|(1/s_n^2)\Sigma_{j=1}^n T_j - (1/s_n^2)\Sigma_{j=1}^n P_m T_j P_m\|_* \leqslant (n/s_n^2)n^{-1}\Sigma_{j=1}^n\|T_j - P_m T_j P_m\|_* < \varepsilon/3$ for all n and $\|P_m T P_m - T\|_* < \varepsilon/3$. For this m, condition (a) implies that $\|(1/s_n^2)\Sigma_{j=1}^n P_m T_j P_m - P_m T P_m\|_* < \varepsilon/3$ for all large n. Finally, breaking up $\|(1/s_n^2)\Sigma_{j=1}^n T_j - T\|_*$ as suggested by the above, it follows that $\|(1/s_n^2)\Sigma_{j=1}^n T_j - T\|_* < \varepsilon$ for all large n. This completes the proof.

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