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# A NEW LOOK AT CONVERGENCE OF BRANCHING PROCESSES

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The classical almost sure convergence of the normed supercritical Galton-Watson branching process with finite mean is obtained by a new method which does not involve probability generating functions.

**1. Introduction.** From the most elementary level upwards, the behaviour of branching processes has customarily been studied using the analytic properties of probability generating functions and their functional iterates; these transforms are useful for their comparative ease of manipulation and their economy in summarising probability distributions in a closed mathematical form. However, results obtained by more direct probabilistic considerations usually shed more light on the intrinsic nature of the process; the martingale convergence theorem has been of particular value here.

Consider what may now be called the "classical" theory of almost sure convergence of the normed supercritical Galton-Watson process with finite mean, which may be found in Section I.10 of Athreya and Ney (1972). Convergence in distribution was originally discovered, using analytic considerations, by Seneta (1968), and subsequently strengthened to almost sure convergence by Heyde (1970), who used a martingale argument. Heyde's martingale, however, has little intuitive significance, being expressed in terms of inverses of probability generating functions, and the objective of this paper is to use a different martingale to obtain the same result by a more revealing and probabilistic method.

**2. Statement of results.** The key theorem in the proof of almost sure convergence is the following.

Theorem 1. *Let* $\{Z_n; n = 0, 1, 2, \cdots\}$ *and* $\{Z_n^*; n = 0, 1, 2, \cdots\}$ *be independent Galton-Watson branching processes with the same offspring distribution and arbitrary possibly distinct initial distributions. Let* $\mathcal{F}_n$ *be the* $\sigma$*-field generated by* $Z_0, \cdots, Z_n, Z_0^*, \cdots, Z_n^*$, *and let* $Y_n = Z_n^*/(Z_n + Z_n^*)$, *where in the event of extinction of both processes, we adopt the convention that the process* $\{Y_n\}$ *takes on the last "meaningful" value it had and retains that value for all subsequent* $n$.

*Then* $\{Y_n, \mathcal{F}_n\}$ *is a martingale and so, being bounded in* $[0, 1]$, $Y_n$ *converges almost surely to a limit random variable with values in* $[0, 1]$. *Hence* $Z_n^*/Z_n = Y_n/(1 - Y_n)$ *converges almost surely to a limit random variable taking values in the extended half-line* $[0, \infty]$.

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We now state our version of the "classical" theorem.

THEOREM 2. *Let $\{Z_n\}$ be a supercritical Galton-Watson process with finite mean family size m. Then there exists a sequence $\{c_n\}$ of constants with $c_n^{-1}c_{n+1} \to m$ as $n \to \infty$ such that $c_n^{-1}Z_n \to_{\text{a.s.}} W$ where W is finite and nonzero on the event of nonextinction.*

**3. Proofs.** We start the proof of Theorem 1 by showing that $\{Y_n, \mathcal{F}_n\}$ is a martingale.

$$\begin{aligned} E[Y_{n+1}|\mathcal{F}_n] &= E[Y_{n+1}|Z_n, Z_n^*] \text{ (by the Markov property)} \\ &= Y_n \quad \text{if} \quad Z_n = Z_n^* = 0 \text{ (by convention)} \\ &= E\left[\frac{\sum_{i=1}^{Z_n^*} X_i^{*(n)}}{\sum_{i=1}^{Z_n} X_i^{(n)} + \sum_{i=1}^{Z_n^*} X_i^{*(n)}} \middle| Z_n, Z_n^*\right] \text{ (otherwise)} \end{aligned}$$

where $X_1^{(n)}, \cdots, X_{Z_n}^{(n)}, X_1^{*(n)}, \cdots, X_{Z_n^*}^{*(n)}$ are the sizes of the families of the individual members of the $n$th generations of the two processes respectively, and if all these happen to be zero, the quotient is to be interpreted as $Y_n$ by convention.

But otherwise, on the event $F$ that one or other of $Z_{n+1}$ and $Z_{n+1}^*$ is nonzero, the random variables $X_1^{(n)}, \cdots, X_{Z_n}^{(n)}, X_1^{*(n)}, \cdots, X_{Z_n^*}^{*(n)}$ are exchangeable, as they are i.i.d. conditional on at least one being nonzero. Hence by symmetry,

$$\begin{aligned} &E\left[\frac{X_1^{(n)}}{\sum_{i=1}^{Z_n} X_i^{(n)} + \sum_{i=1}^{Z_n^*} X_i^{*(n)}} \middle| Z_n, Z_n^*, F\right] \\ &\qquad = \cdots = E\left[\frac{X_{Z_n^*}^{*(n)}}{\sum_{i=1}^{Z_n} X_i^{(n)} + \sum_{i=1}^{Z_n^*} X_i^{*(n)}} \middle| Z_n, Z_n^*, F\right] = \frac{1}{Z_n + Z_n^*} \end{aligned}$$

and so

$$\begin{aligned} E[Y_{n+1}|Z_n, Z_n^*, F] &= \sum_{j=1}^{Z_n^*} E\left[\frac{X_j^{*(n)}}{\sum_{i=1}^{Z_n} X_i^{(n)} + \sum_{i=1}^{Z_n^*} X_i^{*(n)}} \middle| Z_n, Z_n^*, F\right] \\ &= \frac{Z_n^*}{Z_n + Z_n^*} = Y_n. \end{aligned}$$

Therefore,

$$\begin{aligned} E[Y_{n+1}|Z_n, Z_n^*] &= P[F|Z_n, Z_n^*]E[Y_{n+1}|Z_n, Z_n^*, F] \\ &\quad + P[F^c|Z_n, Z_n^*]E[Y_{n+1}|Z_n, Z_n^*, F^c] = Y_n. \end{aligned}$$

This concludes the proof that $\{Y_n, \mathcal{F}_n\}$ is a martingale; the rest of Theorem 1 follows from the martingale convergence theorem.

To prove Theorem 2, it is easy to see that if $\{Z_n\}$ and $\{Z_n^*\}$ are independent copies of the process, then for any integer $k \geqslant 0$, $Z_n^*/Z_{n+k}$ converges almost surely as $n \to \infty$ on the event $\{Z_k \neq 0\}$, and therefore certainly on the event $\{Z_n \to \infty\}$, to a $[0, \infty]$-valued random variable, since $\{Z_{n+k}; n = 0, 1, 2, \cdots\}$ is simply a

Galton-Watson process with the same offspring distribution as $\{Z_n^*\}$, and so Theorem 1 applies.

Now

$$Z_{n+k} = \Sigma_{i=1}^{Z_n} X_i^{(n,k)}$$

where $X_1^{(n,k)}, \cdots, X_{Z_n}^{(n,k)}$ are i.i.d. random variables with mean $m^k$, representing the numbers of members of the $(n+k)$th generation which are descended from the different members of the $n$th generation. Moreover, their distribution does not depend on $n$, and so using the weak law of large numbers, this decomposition tells us that $Z_{n+k}/Z_n \to m^k$ in probability on the event $\{Z_n \to \infty\}$ as $n \to \infty$ while $k$ is kept fixed.

Hence

$$P\left[\left(\frac{Z_n^*}{Z_n} \to \infty\right)\Delta\left(\frac{Z_n^*}{Z_{n+k}} \to \infty\right), Z_n \to \infty\right] = 0$$

where $\Delta$ denotes the symmetric differences between two events.

But

$$P\left[\frac{Z_n^*}{Z_{n+k}} \to \infty, Z_n \to \infty\right] = E\left\{P\left[\frac{Z_n^*}{Z_{n+k} + Z_n^*} \to 1 \middle| Z_k\right]; Z_n \to \infty\right\}$$

$$\leqslant E\left\{E\left[\lim_{n\to\infty} \frac{Z_n^*}{Z_{n+k} + Z_n^*} \middle| Z_k\right]; Z_n \to \infty\right\}$$

(since the limit random variable lies in [0, 1])

$$= E\left\{\frac{Z_0^*}{Z_k + Z_0^*}; Z_n \to \infty\right\}$$

using the martingale property and bounded convergence $\to 0$ as $k \to \infty$, by bounded convergence events. Hence

$$P\left[\frac{Z_n^*}{Z_n} \to \infty, Z_n \to \infty\right] = 0.$$

Similarly, by symmetry,

$$P\left[\frac{Z_n^*}{Z_n} \to 0, Z_n^* \to \infty\right] = 0.$$

Hence in the event of nonextinction of both processes, $Z_n^*/Z_n$ converges almost surely to a proper, positive random variable and therefore, by Fubini's theorem on this event, almost any sample path $\{Z_n\}$ will suffice as a sequence of norming constants $\{c_n\}$ for $\{Z_n^*\}$ on this event; but $\{Z_n\}$ and $\{Z_n^*\}$ are independent and so $\{c_n\}$ suffices for $\{Z_n^*\}$ on the whole sample space. The fact that $c_{n+1}/c_n \to m$ follows from the result, already obtained, that $Z_{n+1}^*/Z_n^* \to m$ in probability, and the obvious fact that $Z_{n+1}^*/Z_n^* \sim c_{n+1}/c_n$ almost surely, on the event $\{Z_n^* \to \infty\}$. This concludes the proof of Theorem 2.

**4. Concluding remarks.** Theorem 1 works equally well for subcritical processes and supercritical ones with infinite mean. In the latter case, the consequences on almost sure convergence results will be studied elsewhere.

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