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THE OPTIONAL SAMPLING THEOREM FOR MARTINGALES INDEXED BY DIRECTED SETS[1]

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A natural generalization of the optional sampling theorem for martingales is given. For discrete valued stopping times the result holds for directed sets; for more general stopping times the result holds for lattices satisfying a type of separability condition. The discrete case improves a lemma of Chow. The general case depends upon a lemma showing that all martingales with respect to σ-algebras satisfying a "right continuity" condition have a modification which has a regularity property that is similar to, but weaker than, right continuity. A result of Wong and Zakai is obtained as a corollary.

1. INTRODUCTION. Work on martingales indexed by partially ordered sets has been primarily concerned with generalizations of the martingale convergence theorem, motivated in part by differentiation theory. (See Helms (1958), Krickeberg (1963), Chow (1960), Cairoli (1969) and Gut (1976).) In addition there have been extensions of Doob's Inequality (Cairoli (1969) and Shorack and Smythe (1976)), and extensive work on integration with respect to martingales by Wong and Zakai (1974, 1976) and Cairoli and Walsh (1975, 1977).

Various decomposition and representation theorems are included with the work on integration. Extensions of the optional sampling theorem include a lemma in Chow (1960) (see (2.3) below), some unpublished work of Bühler, and a result of Wong and Zakai (1976) for a special class of martingales and a notion of stopping time that does not immediately generalize the one dimensional concept.

In this paper we give a natural generalization of the optional sampling theorem for martingales using a definition of a stopping time that is identical to the linearly ordered definition. Our proof is similar to that of Chow (for discrete valued stopping times), but without an unnatural restriction on the definition of a stopping time.

In Section 2 we give the definitions and basic materials we need, prove the optional sampling theorem and give an example of Chow demonstrating that the optional sampling theorem does not hold in general for submartingales.

In Section 3 we discuss the relationship of our result to that of Wong and Zakai.

In a related paper (Kurtz (1980)) we apply the optional sampling theorem in the study of simultaneous random time changes of Markov processes. In particular we show that a large class of diffusions can be represented as multiple random time

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changes of Brownian motion and give a converse of a theorem of Knight (1970) concerning time changes of continuous orthogonal martingales.

In addition we anticipate applications in the study of point processes.

2. The optional sampling theorem. Let $\mathcal{G}$ be a directed set with partial ordering denoted by $t \leqslant s$. That is, $\mathcal{G}$ is partially ordered and for $t_1, t_2 \in \mathcal{G}$ there exists $t_3 \in \mathcal{G}$ such that $t_1 \leqslant t_3$ and $t_2 \leqslant t_3$. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_t\}$ be a family of σ-algebras ($\mathcal{F}_t \subset \mathcal{F}$) indexed by $\mathcal{G}$ and increasing in the sense that $\mathcal{F}_t \subset \mathcal{F}_s$ whenever $t \leqslant s$. A stochastic process $X(t)$ indexed by $\mathcal{G}$ is a martingale if

$$E(X(t)|\mathcal{F}_s) = X(s) \quad \text{for all} \quad t \geqslant s. \tag{2.1}$$

An $\mathcal{G}$ valued random variable τ is a *stopping time* if $\{\tau \leqslant t\} \in \mathcal{F}_t$ for every $t \in \mathcal{G}$. As in the linearly ordered case we define

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leqslant t\} \in \mathcal{F}_t \quad \text{for all } t. \tag{2.2}$$

The following lemma is similar to one found in Chow (1960). Chow, however, requires that both $\{\tau \leqslant t\}$ and $\{\tau \geqslant t\} \in \mathcal{F}_t$ for all t.

LEMMA (2.3). *Let $X(t)$ be a martingale and let τ_1 and τ_2 be stopping times assuming countably many values and satisfying $\tau_1 \leqslant \tau_2$ a.s. If there exists a sequence $\{T_m\} \subset \mathcal{G}$ such that*

$$\lim_{m\to\infty} P\{\tau_2 \leqslant T_m\} = 1 \tag{2.4}$$

and

$$\lim_{m\to\infty} E\big(|X(T_m)|\chi_{\{\tau_2 \leqslant T_m\}^c}\big) = 0 \tag{2.5}$$

and $E(|X(\tau_2)|) < \infty$, then

$$E\big(X(\tau_2)|\mathcal{F}_{\tau_1}\big) = X(\tau_1). \tag{2.6}$$

PROOF. Define

$$\begin{aligned} \tau_{im} &= \tau_i && \text{on} \quad \{\tau_i \leqslant T_m\} \\ &= T_m && \text{on} \quad \{\tau_i \leqslant T_m\}^c. \end{aligned} \tag{2.7}$$

If $T_m \leqslant t$ then $\{\tau_{im} \leqslant t\} = \Omega \in \mathcal{F}_t$; if $T_m \nleqslant t$ then

$$\begin{aligned} \{\tau_{im} \leqslant t\} &= \{\tau_i \leqslant t\} \cap \{\tau_i \leqslant T_m\} \\ &= \bigcup_{\alpha \leqslant t;\, \alpha \leqslant T_m} \{\tau_i = \alpha\} \in \mathcal{F}_t. \end{aligned}$$

Therefore τ_{im} is a stopping time.

Let Γ_i denote the set of values assumed by τ_i. In order to verify (2.6) it is enough to show that

$$\int_{A\cap\{\tau_1=t\}} X(\tau_2)\, dP = \int_{A\cap\{\tau_1=t\}} X(\tau_1)\, dP = \int_{A\cap\{\tau_1=t\}} X(t)\, dP \tag{2.8}$$

for every $t \in \Gamma_1$ and $A \in \mathcal{F}_{\tau_1}$. Note that for any s either $t \leqslant s$ and hence $A \cap \{\tau_1 = t\} \cap \{\tau_{2m} = s\} \in \mathcal{F}_s$ or $A \cap \{\tau_1 = t\} \cap \{\tau_{2m} = s\} = \varnothing$. Let $t \in \Gamma_1$ and $t \leqslant T_m$.

Since $X(s) = E(X(T_m)|\mathcal{F}_s)$ for $s \leqslant T_m$ we may write $X(\tau_{2m}) = \Sigma E(X(T_m)|\mathcal{F}_s)\chi_{\{\tau_{2m}=s\}}$ and obtain

$$
\begin{aligned}
\int_{A\cap\{\tau_1=t\}} X(\tau_{2m})\, dP &= \int_{A\cap\{\tau_1=t\}} \Sigma E(X(T_m)|\mathcal{F}_s)\chi_{\{\tau_{2m}=s\}}\, dP \\
&= \Sigma \int_{A\cap\{\tau_1=t\}\cap\{\tau_{2m}=s\}} E(X(T_m)|\mathcal{F}_s)\, dP \\
&= \Sigma \int_{A\cap\{\tau_1=t\}\cap\{\tau_{2m}=s\}} X(T_m)\, dP \\
&= \int_{A\cap\{\tau_1=t\}} X(T_m)\, dP \\
&= \int_{A\cap\{\tau_1=t\}} X(t)\, dP.
\end{aligned} \tag{2.9}
$$

Finally

$$
\begin{aligned}
&\int_{A\cap\{\tau_1=t\}} X(\tau_{2m})\, dP \\
&\quad = \int_{A\cap\{\tau_1=t\}\cap\{\tau_2<T_m\}} X(\tau_2)\, dP + \int_{A\cap\{\tau_1=t\}\cap\{\tau_2<T_m\}^c} X(T_m)\, dP.
\end{aligned} \tag{2.10}
$$

By (2.5) we have

$$\lim_{m\to\infty} \int_{A\cap\{\tau_1=t\}} X(\tau_{2m})\, dP = \int_{A\cap\{\tau_1=t\}} X(\tau_2)\, dP, \tag{2.11}$$

and (2.8) follows from (2.9).

REMARK. The following example shows that the above lemma cannot be extended to submartingales: let $\mathcal{I} = \{1, 2, 2', 3\}$ with $\leqslant$ given by $1 \leqslant 2 \leqslant 3$, $1 \leqslant 2' \leqslant 3$. Let $P\{Y = 1\} = P\{Y = -1\} = \frac{1}{2}$ and define $X_1 = 0$, $X_2 = Y$, $X_{2'} = -Y$, $X_3 = 1$, $\mathcal{F}_1 = \{\varnothing, \Omega\}$, $\mathcal{F}_2 = \mathcal{F}_{2'} = \mathcal{F}_3 = \sigma(Y)$. Then $\tau = 2$ on $\{Y = -1\}$ and $\tau = 2'$ on $\{Y = 1\}$ is a stopping time, X_k is a submartingale but $E(X_\tau|\mathcal{F}_1) = -1 < X_1$.

In order to extend the result in Lemma (2.3) to a more general class of stopping times we must place additional restrictions on the martingale $X(t)$ and the index set $\mathcal{I}$. The most convenient assumption for $\mathcal{I}$ is that it is a topological lattice. That is $\mathcal{I}$ is a Hansdorff space and for $t_1, t_2 \in \mathcal{I}$ there exist unique elements $t_1 \wedge t_2 \in \mathcal{I}$ and $t_1 \vee t_2 \in \mathcal{I}$ such that

$$\{s \in \mathcal{I} : s \leqslant t_1\} \cap \{s \in \mathcal{I} : s \leqslant t_2\} = \{s \in \mathcal{I} : s \leqslant t_1 \wedge t_2\}$$

and

$$\{s \in \mathcal{I} : s \geqslant t_1\} \cap \{s \in \mathcal{I} : s \geqslant t_2\} = \{s \in \mathcal{I} : s \geqslant t_1 \wedge t_2\},$$

and $t_1 \wedge t_2$ and $t_1 \vee t_2$ are continuous mappings of $\mathcal{I} \times \mathcal{I}$ onto $\mathcal{I}$ ($\mathcal{I} \times \mathcal{I}$ having the product topology). Note that this implies sets of the form $[t_1, t_2] = \{t : t_1 \leqslant t \leqslant t_2\}$ (called intervals) are closed and hence Borel measurable.

We will say that a topological lattice $\mathcal{I}$ is *separable from above* if there exists a sequence $\{t_k\} \subset \mathcal{I}$ (which we will call a *separating sequence*) such that for all $t \in \mathcal{I}$ $t = \lim_{n\to\infty} t^{(n)}$ where $t^{(n)}$ is defined by

$$t^{(n)} \equiv \min\{t_k : k \leqslant n,\ t_k \geqslant t\}. \tag{2.12}$$

(If necessary adjoin to $\mathcal{I}$ a point ∞ and define $\min\{\varnothing\} = \infty$.)

We will also make use of the hypothesis that every interval $[T_1, T_2] \equiv \{t \in \mathcal{G} : T_1 \leqslant t \leqslant T_2\}$ is separable from above. Note that if $\mathcal{G}$ is separable from above then every interval is separable from above (take $\{(T_2 \wedge t_k) \vee T_1\}$ as the separating sequence) but not conversely. For example consider the lattice of all positive continuous functions on $(-\infty, \infty)$ with the topology of uniform convergence on compact sets.

LEMMA (2.13). *Let $\mathcal{G}$ be separable from above and let $X(t)$ be a martingale with respect to an increase family of σ-algebras $\{\mathcal{F}_t\}$ indexed by $\mathcal{G}$. Suppose that $\mathcal{F}_t = \bigcap_n \mathcal{F}_{t^{(n)}}$ ($t^{(n)}$ defined in (2.12)). Then there exists an adapted measurable modification $\bar{X}$ of X such that $\lim_{n\to\infty} \bar{X}(t^{(n)}, \omega) = \bar{X}(t, \omega)$ for every $\omega \in \Omega$ for which the limit exists.*

PROOF. (Define $\bar{X}(\infty, \omega) = 0$.) Since $X(t^{(n)})$ is a reversed (one parameter) martingale for n such that $t^{(n)} \neq \infty$, $\lim_{n\to\infty} X(t^{(n)})$ exists a.s. for each t. Define $\bar{X}(t, \omega) = \lim_{n\to\infty} X(t^{(n)}, \omega)$ if the limit exists; $\bar{X}(t, \omega) = 0$ otherwise. Since $(t^{(n)})^{(k)} = t^{(n)}$ for $k \geqslant n$, $\bar{X}(t^{(n)}, \omega) \equiv X(t^{(n)}, \omega)$ so that $\bar{X}(t, \omega) = \lim_{n\to\infty} \bar{X}(t^{(n)}, \omega)$ if the limit exists. Define $Y_m(t, \omega) \equiv X(t^{(m)}, \omega)$. $Y_m(t, \omega)$ is a measurable function on $(\mathcal{G} \times \Omega, \mathcal{B}(\mathcal{G}) \times \mathcal{F})$ and hence $\bar{X}(t, \omega)$ is measurable.

REMARK. The restriction of $Y_m(t, \omega)$ to $\{t : t \leqslant T\} \times \Omega$ is $\mathcal{B}\{t : \leqslant T\} \times \mathcal{F}_{T^{(n)}}$ measurable for all $m \geqslant n$. Unfortunately $\bigcap_n \mathcal{B}\{t : t \leqslant T\} \times \mathcal{F}_{T^{(n)}}$ need not equal $\mathcal{B}\{t : t \leqslant T\} \times \mathcal{F}_T$ so we cannot conclude that $\bar{X}(t, \omega)$ is progressively measurable, that is, the restriction of $\bar{X}$ to $\{t : t \leqslant T\} \times \Omega$ may not be $\mathcal{B}\{t : t \leqslant T\} \times \mathcal{F}_T$ measurable. (We thank C. Dellacherie for this observation.)

LEMMA (2.14). *Let $\mathcal{G}$ be separable from above and suppose $\mathcal{F}_t = \bigcap_n \mathcal{F}_{t^{(n)}}$. If τ is a stopping time such that $\tau \leqslant t_k$ for some $k \leqslant n$, then $\tau^{(n)}$ is a stopping time, and $\mathcal{F}_\tau = \bigcap_n \mathcal{F}_{\tau^{(n)}}$.*

PROOF. Let $\mathcal{S}^{(n)}$ denote the collection of elements of the form $t_{k_1} \wedge t_{k_2} \cdots \wedge t_{k_m}$, $k_i \leqslant n$. Then $\{\tau^{(n)} \leqslant t\} = \bigcup_{s\in\mathcal{S}^{(n)};\, s \leqslant t} \{\tau \leqslant s\}$. Since $s \leqslant t$ implies $\{\tau \leqslant s\} \in \mathcal{F}_t$, $\tau^{(n)}$ is a stopping time. If $A \in \bigcap_n \mathcal{F}_{\tau^{(n)}}$ then $A \cap \{\tau^{(n)} \leqslant t^{(n)}\} \in \mathcal{F}_{t^{(n)}}$ for all n and hence $A \cap \{\tau \leqslant t\} = \bigcap_n A \cap \{\tau^{(n)} \leqslant t^{(n)}\} \in \bigcap_n \mathcal{F}_{t^{(n)}} = \mathcal{F}_t$.

THEOREM (2.15).

(a) *Let $\mathcal{G}$ be separable from above with separating set $\{t_k\}$, $\mathcal{F}_t = \bigcap_n \mathcal{F}_{t^{(n)}}$ for all t, and let $X(t)$ be a martingale satisfying*

$$\lim_{n\to\infty} X(t^{(n)}, \omega) = X(t, \omega) \tag{2.16}$$

for all (t, ω) for which the limit exists. Let τ_1 and τ_2 be $\mathcal{G}$-valued stopping times such that $\tau_1 \leqslant \tau_2$ a.s. Suppose there exist $T_m \in \{t_k\}$ such that

$$\lim_{m\to\infty} P\{\tau_2 \leqslant T_m\} = 1,$$

and

$$\lim_{m\to\infty} E\big(|X(T_m)|\chi_{\{\tau_2 \leqslant T_m\}^c}\big) = 0, \tag{2.17}$$

and that $E(|X(\tau_2)|) < \infty$. Then

$$E\big(X(\tau_2)|\mathcal{F}_{\tau_1}\big) = X(\tau_1) \quad \text{a.s.} \tag{2.18}$$

(b) *Suppose all intervals in* $\mathcal{I}$ *are separable from above and that* $\lim_{n\to\infty} X(s_n, \omega) = X(t, \omega)$ *for any sequence* $\{s_n\}$ *such that* $s_{n+1} \geqslant s_n \geqslant t$ *and* $\lim_{n\to\infty} s_n = t$ *and all* ω. *Let* τ_1 *and* τ_2 *be* $\mathcal{I}$*-valued stopping times such that* $\tau_1 \leqslant \tau_2$ a.s. *Suppose there exist* S_m *and* $T_m \in \mathcal{I}$ *such that*

$$\lim_{m\to\infty} P\{S_m \leqslant \tau_i \leqslant T_m\} = 1 \tag{2.19}$$

and

$$\lim_{m\to\infty} E\big(|X(T_m)|\chi_{\{\tau_2 \leqslant T_m\}^c}\big) = 0, \tag{2.20}$$

and that $E(|X(\tau_2)|) < \infty$. *Then*

$$E\big(X(\tau_2)|\mathcal{F}_{\tau_1}\big) = X(\tau_1). \tag{2.21}$$

PROOF. Note that in (b) $\cup_m[\min_{k\leqslant m} S_k, \max_{k\leqslant m} T_k]$ is separable from above. By the "right continuity" assumption (2.16) will hold for any separating set and we may assume $\{T_m\} \subset \{t_k\}$. Consequently the proof of (b) reduces to the proof of (a).

As in the discrete case, let

$$\begin{aligned} \tau_{im} &= \tau_i \quad \text{on} \quad \{\tau_i \leqslant T_m\} \\ &= T_m \quad \text{on} \quad \{\tau_i \leqslant T_m\}^c. \end{aligned} \tag{2.22}$$

For n suffuciently large $T_m \in \{t_1 t_2 .. t_n\}$ and hence $\tau_{im}^{(n)} \leqslant T_m$.

Since $\tau_{im}^{(n)} \geqslant \tau_{im}^{(n+1)}$, Lemma (2.3) implies $X(\tau_{im}^{(n)})$ is a reverse martingale and hence $\lim_{n\to\infty} X(\tau_{im}^{(n)})$ exists a.s. and by the continuity hypothesis must equal $X(\tau_{im})$ a.s.

Let $A \in \mathcal{F}_{\tau_1}$. Since $A \cap \{\tau_1 \leqslant T_m\} \cap \{\tau_{1m} \leqslant t\} = A \cap \{\tau_1 \leqslant T_m \wedge t\} \in \mathcal{F}_t$, $A \cap \{\tau_1 \leqslant T_m\} \in \mathcal{F}_{\tau_{1m}} \subset \mathcal{F}_{\tau_{1m}^{(n)}}$ and Lemma (2.3) gives

$$\int_{A\cap\{\tau_1 \leqslant T_m\}} X\big(\tau_{2m}^{(n)}\big)\, dP = \int_{A\cap\{\tau_1 \leqslant T_m\}} X\big(\tau_{1m}^{(n)}\big)\, dP. \tag{2.23}$$

Since $X(\tau_{im}^{(n)}) = E(X(T_m)|\mathcal{F}_{\tau_{im}^{(n)}})$, they are uniformly integrable and letting $n \to \infty$ we have

$$\int_{A\cap\{\tau_1 \leqslant T_m\}\cap\{\tau_2 \leqslant T_m\}} X(\tau_2)\, dP + \int_{A\cap\{\tau_1 \leqslant T_m\}\cap\{\tau_2 \leqslant T_m\}^c} X(T_m)\, dP = \int_{A\cap\{\tau_1 \leqslant T_m\}} X(\tau_1)\, dP. \tag{2.24}$$

Letting $m \to \infty$ we obtain the desired result.

3. Theorem of Wong and Zakai. We will not introduce all of the notation and terminology of Wong and Zakai (1976) or of the theory of stochastic integration in the plane. It should be clear, however, to someone familiar with their work that their Proposition (5.2) (Wong and Zakai (1976), page 583) is a special case of the result we give below.

Let $\mathcal{I}_0$ be the positive quarter plane with the usual partial ordering and let $\mathcal{F}_z$ be an increasing family of σ-algebras satisfying $\mathcal{F}_z = \cap_n \mathcal{F}_{z_n}$ for any sequence $z_n \geqslant z_{n+1} \geqslant z$ such that $\lim_{n\to\infty} z_n = z$. Note that if $z_1 \leqslant z_2$ then the "interval" $[z_1, z_2]$ is a rectangle.

Let $\mathcal{G}_1$ be the collection of all finite unions of rectangles of the form $[0, z]$, ordered by inclusion, and let $\mathcal{G}_2$ be the collection of all compact subsets K of the quarter plane with the property that $z \in K$ implies $[0, z] \subset K$. $\mathcal{G}_2$ is the collection of all sets that can be obtained as intersections of sets in $\mathcal{G}_1$. If we identify z with $[0, z]$, then $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \mathcal{G}_2$, but note that while $\mathcal{G}_1$ is a sublattice of $\mathcal{G}_2$, $\mathcal{G}_0$ is not a sublattice of $\mathcal{G}_1$. The definition of $\wedge$ is consistent but not that of $\vee$.

For a subset A of the plane define $A_\varepsilon = \{y : |x - y| < \varepsilon \text{ for some } x \in A\}$ and for A, $B \in \mathcal{G}_2$ define

$$\rho(A, B) = \inf\{\varepsilon : A \subset B_\varepsilon, B \subset A_\varepsilon\}. \tag{3.1}$$

For the topology on $\mathcal{G}_1$ and $\mathcal{G}_2$ we take the topology determined by this metric. Note, for example, that if $\{K_n\}$ is a decreasing sequence in $\mathcal{G}_2$ then $\lim_{n\to\infty} K_n = K$ if and only if $K = \cap_{n=1}^\infty K_n$. For $K \in \mathcal{G}_2$, define

$$\mathcal{F}_K = \cap_{\varepsilon>0} \mathsf{V}_{z\in K_\varepsilon} \mathcal{F}_z. \tag{3.2}$$

If $K = [0, z]$, then $\mathcal{F}_K = \mathcal{F}_z$. If $K_n \supset K$ and $\lim_{n\to\infty} K_n = K$ then

$$\mathcal{F}_K = \cap_n \mathcal{F}_{K_n}.$$

Finally, notice that $\mathcal{G}_2$ is separable from above. In particular we will take the separating sequence to be an ordering $\{C_k\}$ of the collection of all finite unions of rectangles $[0, z]$ where z has rational coordinates. The rectangles $[0, z]$, z rational, form a subsequence $\{C_{k_i}\}$ of $\{C_k\}$. For the separating sequence in $\mathcal{G}_0$ we take the sequence $\{y_i\}$, where $[0, y_i] = C_{k_i}$.

Let $X(z)$ be a martingale with respect to $\{F_z\}$ such that $X(z)$ vanishes on the coordinate axes. By the continuity assumption on $\{F_z\}$, it follows that for any decreasing sequence $\{z_n\}$ with $\lim_{n\to\infty} z_n = z$ we have $\lim_{n\to\infty} X(z_n) = X(z)$ a.s. Consequently, we will assume (see Lemma (2.13)) that $X(z)$ is measurable and $\lim_{n\to\infty} X(z^{(n)}) = X(z)$ whenever the limit exists. (Recall $z^{(n)} = \min\{y_k : k \leqslant n, y_k \geqslant z\}$.)

For $z_1 = (t_1, s_1) \leqslant z_2 = (t_2, s_2)$, let $(z_1, z_2]$ denote the rectangle

$$[0, z_2] - [0, (t_1, s_1)] - [0, (t_2, s_1)].$$

If we think of $X(z)$ as the "measure" of $[0, z]$ then the "measure" of $(z_1, z_2]$ is

$$X(z_1, z_2] = X(z_2) - X(t_1, s_2) - X(t_2, s_1) + X(z_1). \tag{3.3}$$

Since every set $K \in \mathcal{G}_1$ is essentially a finite union of disjoint rectangles of the form $(z_1, z_2]$ (we can disregard the coordinate axes, as they have "zero measure"), we can define

$$X(K) = \Sigma_{j=1}^m X(z_{1j}, z_{2j}] \tag{3.4}$$

where $\cup_{j=1}^m (z_{1j}, z_{2j}] = K \cap (0, \infty) \times (0, \infty)$ and the $(z_{1j}, z_{2j}]$ are disjoint.

$X(K)$ will be a martingale with respect to $\{\mathcal{F}_K\}$ if and only if $X(z)$ is a strong martingale in the sense of Cairoli and Walsh (1975); that is,

$$E\big(X(z_1, z_2] \,|\, \vee_{z \ngeq z_1} \mathcal{F}_z\big) = 0. \tag{3.5}$$

This observation is essentially Theorem 5.6 of Wong and Zakai (1976).

If $X(K)$ is a martingale on $\mathcal{G}_1$, then we can immediately extend X to a martingale on $\mathcal{G}_2$ by

(3.6) $$X(K) = \lim_{n\to\infty} X(K^{(n)})$$

where $K^{(n)} = \min\{C_k : C_k \geqslant K\}$.

Wong and Zakai define stopping times as indicator functions of random sets γ with the following properties:

(3.7) $$z \in \gamma \quad \text{implies} \quad [0, z] \subset \gamma,$$

(3.8) $$\chi_\gamma(z) \quad \text{is} \quad \mathcal{F}_z \quad \text{measurable.}$$

We observe that $\tau = [0, z] \cap \bar{\gamma}$ ($\bar{\gamma}$ the closure of γ) is a stopping time in our sense. Proposition (5.2) of Wong and Zakai (1976) follows from

(3.9) $$E(X(z)|\mathcal{F}_\tau) = X([0, z] \cap \bar{\gamma})$$

and the fact that in their setting

(3.10) $$X([0, z] \cap \bar{\gamma}) = \int_{[0, z]} \chi_{\bar{\gamma}}\, dX = \int_{[0, z]} \chi_\gamma\, dX \text{ a.s.}$$

REMARK. (a) If $[0,z] \cap \gamma$ is a stopping time for every z, then χ_γ is a stopping time in the sense of Wong and Zakai.

(b) Cairoli and Walsh (1978) have made a further study of Wong-Zakai stopping times.

(c) Even in the case of the Brownian sheet there does not exist a modification that is right continuous on $\mathcal{G}_2$.

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