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FURTHER LIMIT THEOREMS FOR THE RANGE OF A TWO-PARAMETER RANDOM WALK IN SPACE

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Let $\{X_{ij}: i \geq 1, j \geq 1\}$ be a double sequence of i.i.d. random variables taking values in the d-dimensional lattice E_d. Also let $S_{kl} = \Sigma_{i=1}^{k}\Sigma_{j=1}^{l}X_{ij}$. Then the range of random walk $\{S_{kl}: k \geq 1, l \geq 1\}$ up to time (m, n), denoted by R_{mn}, is the cardinality of the set $\{S_{kl}: 1 \leq k \leq m, 1 \leq l \leq n\}$, i.e., the number of distinct points visited by the random walk up to time (m, n). Let $r^{(l)}$ be the probability that the random walk never hits the origin on the time set $\{(i, l): i \geq 1\}$. In this paper a sufficient condition in terms of the characteristic function of X_{11} is given so that

$$\lim_{(m,n)\to\infty}\frac{mn - R_{mn}}{m + n} = \Sigma_{l=1}^{\infty}(1 - r^{(l)}) < \infty \qquad \text{a.s.}$$

1. Introduction. Let $\{X_{ij}: i \geq 1, j \geq 1\}$ be a double sequence of independent identically distributed (i.i.d.) random variables which take values in the d-dimensional integer lattice E_d. The double sequence $\{S_{mn}: m \geq 1, n \geq 1\}$ defined by $S_{mn} = \Sigma_{i=1}^{m}\Sigma_{j=1}^{n}X_{ij}$ is called a two-parameter random walk or simply a random walk when there is no danger of confusion.

Let R_{mn} denote the cardinality of the set $\{S_{ij}: 1 \leq i \leq m, 1 \leq j \leq n\}$; R_{mn} is called the range of the two-parameter random walk. In [1] we proved that for a random walk with "$d \geq 3$", $R_{mn}/mn \to 1$ almost surely. In [3] a stronger result is obtained by establishing a sufficient condition in terms of the characteristic function of X_{11}. In the present work after giving some notation and preliminary results in Section 2, we will introduce, in Section 3, a new double sequence of random variables T_{mn}^{*}. We will investigate its limit behavior as $(m, n) \to \infty$ and we will use the result, in Section 4, to show that under a suitable condition on the characteristic function of X_{11}, one has $(mn - R_{mn})/(m + n) \to \Sigma_{l=1}^{\infty}(1 - r^{(l)}) < \infty$ almost surely, where $r^{(l)} = P\{S_{il} \neq 0: i = 1, 2, \cdots\}$.

2. Notation and preliminaries. From the two-parameter random walk one can induce one-parameter random walks, which will be of considerable interest, as follows: let $\{X_{ij}: (i,j) \in I^{+} \times I^{+}\}$, where I^{+} is the set of positive integers, be the corresponding double sequence of i.i.d. random variables, defined on the probability space $(\Omega, \mathcal{F}, P)$, and

$$X_i^{(l)} = X_{i1} + X_{i2} + \cdots + X_{il}, \qquad (i, l) \in I^{+} \times I^{+}. \tag{2.1}$$

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Then fixing $l \in I^+$, the process $\{S_m^{(l)} : m \in I^+\}$ defined by

$$S_m^{(l)} = \Sigma_{i=1}^m X_i^{(l)}, \qquad m \in I^+, \tag{2.2}$$

will give us a one-parameter random walk.

Also, let $R_m^{(l)}$ be the range of the one-parameter random walk $\{S_k^{(l)} : k \in I^+\}$ up to time m, i.e., the cardinality of the set $\{S_k^{(l)} : 1 \leqslant k \leqslant m\}$. One of the early results concerning the limit behavior of $R_m^{(l)}$ is obtained in [5] page 38; it asserts that

$$\lim_{m\to\infty} R_m^{(l)}/m = r^{(l)} \quad \text{a.s.} \tag{2.3}$$

An extensive study of the range of one-parameter random walk has been done in a series of papers by Jain and Pruitt (for some of these see the references given in [1]).

To achieve our objective, we will need an estimate for the variance of $R_m^{(l)}$. This will be accomplished by a slight modification of some of the ideas given in [4].

For $x \in E_d$, $l \in I^+$, the notation $P_x^{(l)}\{\cdot\}$ will be used to denote probabilities of events related to the random walk $S_m^{(l)} = x + \Sigma_{i=1}^m X_i^{(l)}$; when $x = 0$, we will use $P^{(l)}\{\cdot\}$. Thus for $m \in I^+$, $x,y \in E_d$ we let

$$P^{m(l)}(x,y) = P_x^{(l)}\{S_m^{(l)} = y\} = P^{(l)}\{S_m^{(l)} = y - x\} \tag{2.4}$$

and note that $P^{m(l)}(x,y) = P^{m(l)}(0,y-x)$. $T_x^{(l)}$ will denote the first hitting time of the lattice point x on the time set $\{(i,l) : i \in I^+\}$, i.e.,

$$T_x^{(l)} = \min\{m \in I^+ : S_m^{(l)} = x\}; \tag{2.5}$$

if there are no integers m with $S_m^{(l)} = x$, then $T_x^{(l)} = \infty$. We also define

$$P_z^{m(l)}(x,y) = P_x^{(l)}\{S_m^{(l)} = y,\, T_z^{(l)} \geqslant m\}, \tag{2.6}$$

for $m, l \in I^+$ and $x,y,z \in E_d$. We will use $u_m^{(l)}$ for $P^{m(l)}(0,0)$, $f_m^{(l)}$ for $P_0^{m(l)}(0,0)$, $r_m^{(l)} = \Sigma_{k=m+1}^\infty f_k^{(l)}$. Note that $r^{(l)} = P^{(l)}\{T_0^{(l)} = \infty\}$ and $\Sigma_{k=1}^\infty f_k^{(l)} = P[T_0^{(l)} < \infty] = 1 - r^{(l)}$. For x,y in E_d we will also use $G_m^{(l)}(x,y) = \delta(x,y) + \Sigma_{k=1}^m P^{k(l)}(x,y)$, ($\delta(x,y) = 0$ if $x \neq y$ and one otherwise), $G^{(l)}(x,y) = \lim_{m\to\infty} G_m^{(l)}(x,y)$ and $F^{(l)}(x,y) = P_x^{(l)}\{T_y^{(l)} < \infty\}$. Finally we need a notation for the characteristic function of a random walk. For convenience we will use Greek letters to denote the element of R^d. A typical element will be $\theta = (\theta_1, \theta_2, \cdots, \theta_d)$ where each θ_i is a real number for $i = 1, 2, \cdots, d$. Now define the characteristic function of the random walk generated by X_{11} by

$$\varphi(\theta) = \Sigma_{x\in E_d} P\{X_{11} = x\} e^{ix\cdot\theta}, \tag{2.7}$$

where $x\cdot\theta = \Sigma_{i=1}^d x_i\theta_i$. To set up a convenient notation for integrations, let

$$C = \{\theta \in R^d : |\theta_i| \leqslant \pi \text{ for } i = 1, 2, \cdots, d\}. \tag{2.8}$$

Then for complex-valued functions $g(s)$ which are Lebesgue measurable on C, the integral over C is denoted by

$$\textstyle\int g(\theta)\,d\theta = \int_C g(\theta)\,d\theta = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} g(\theta)\,d\theta_1, \cdots, d\theta_d. \tag{2.9}$$

Thus $d\theta$ always denotes the volume element. Using this notation, the inversion formula for characteristic functions becomes

$$(2.10)\qquad P\{S_A = x\} = \frac{1}{(2\pi)^d}\int e^{-ix\cdot\theta}(\varphi(\theta))^{|A|}\,d\theta, \qquad x \in E_d,$$

where A is a finite time set in $I^+ \times I^+$ with cardinality $|A|$ and $S_A = \Sigma_{(i,j)\in A} X_{ij}$. Note that

$$(2.11)\qquad P\{S_A = x\} \leqslant \frac{1}{(2\pi)^d}\int |\varphi(\theta)|^{|A|}\,d\theta, \qquad x \in E_d.$$

General Remarks. (i) Throughout this paper c stands for an unimportant constant which may depend only on the distribution of X_{11} and may be different in various places.

(ii) The inequalities $1 - s^{n/2} \leqslant 1 - s^n \leqslant 2(1 - s^{n/2})$ and $1 - s^n \geqslant (n/2)s^{n/2}(1 - s)$, $n \in I^+$ and $s \in (0, 1)$ will be used frequently without giving any references.

Lemma 2.1. *For s in* (0, 1), *there exists c in* $(0, \infty)$ *such that*

(a) $\Sigma_{i=1}^\infty \Sigma_{j=1}^\infty \Sigma_{k=i+1}^\infty \Sigma_{l=j+1}^\infty \dfrac{s^{kl-ij}}{(ij)^{\frac{1}{2}}} \leqslant c\dfrac{1}{(1-s)^{\frac{3}{2}}}\log\dfrac{1}{1-s}$;

(b) $\Sigma_{i=1}^\infty \Sigma_{j=1}^\infty \Sigma_{k=1}^\infty \Sigma_{l=1}^\infty \dfrac{s^{kl+ij}}{(i+k)^{\frac{1}{2}}(l+j)^{\frac{1}{2}}} \leqslant c\dfrac{1}{(1-s)^{\frac{3}{2}}}\log\dfrac{1}{1-s}$.

Proof. Estimate the sum in (a) by an integral to obtain

$$\begin{aligned}
(2.12)\quad \Sigma_{i=1}^\infty \Sigma_{j=1}^\infty \Sigma_{k=i+1}^\infty \Sigma_{l=j+1}^\infty \frac{s^{kl-ij}}{(ij)^{\frac{1}{2}}} &= \Sigma_{i=1}^\infty \Sigma_{j=1}^\infty \frac{s^{-ij}}{(ij)^{\frac{1}{2}}} \Sigma_{k=i+1}^\infty \frac{s^{k(j+1)}}{1-s^k} \\
&\leqslant \Sigma_{i=1}^\infty \Sigma_{j=1}^\infty \frac{s^{-ij}}{(ij)^{\frac{1}{2}}} \int_i^\infty \frac{s^{x(j+1)}}{1-s^x}\,dx \\
(\text{use } s^x = s^i v)\qquad &= -\frac{1}{\log s}\Sigma_{i=1}^\infty \Sigma_{j=1}^\infty \frac{1}{(ij)^{\frac{1}{2}}}\int_0^1 \frac{s^i v^j}{1-s^i v}\,dv \\
\left(\text{use } \Sigma_{j=1}^\infty \frac{v^j}{j^{\frac{1}{2}}} \leqslant c\frac{1}{(1-v)^{\frac{1}{2}}}\right)\qquad &\leqslant -\frac{c}{\log s}\Sigma_{i=1}^\infty \frac{1}{i^{\frac{1}{2}}}\int_0^1 \frac{s^i}{(1-v)^{\frac{1}{2}}(1-s^i v)}\,dv \\
\text{use } 1 - v = u^2)\qquad &\leqslant -\frac{c}{\log s}\Sigma_{i=1}^\infty \left(\frac{s^i}{i(1-s^i)}\right)^{\frac{1}{2}} \tan^{-1}\left(\frac{s^i}{1-s^i}\right)^{\frac{1}{2}} \\
&\leqslant -\frac{c}{\log s}\frac{1}{(1-s)^{\frac{1}{2}}}\Sigma_{i=1}^\infty \frac{s^{i/4}}{i} \\
(-\log s \sim 1 - s \text{ as } s\uparrow 1)\qquad &\leqslant \frac{c}{(1-s)^{\frac{3}{2}}}\log\frac{1}{1-s^{\frac{1}{4}}} \\
\left[1 - s \leqslant 4(1 - s^{\frac{1}{4}})\right]\qquad &\leqslant \frac{c}{(1-s)^{\frac{3}{2}}}\log\frac{1}{1-s}.
\end{aligned}$$

To see the validity of part (b) follow Lemma 2.3 in [3] and make use of the fact that

$$\Sigma_{k=1}^{n} k^{\frac{1}{2}} d_k \sim \tfrac{2}{3} n^{\frac{3}{2}} \log n \tag{2.13}$$

in order to obtain

$$\Sigma_{i=1}^{\infty} \Sigma_{j=1}^{\infty} (ij)^{\frac{1}{2}} s^{ij} = \Sigma_{k=1}^{\infty} k^{\frac{1}{2}} d_k s^k \leqslant c \frac{1}{(1-s)^{\frac{3}{2}}} \log \frac{1}{1-s}, \tag{2.14}$$

where d_k, $k \in I^+$, is the number of divisors of k. □

LEMMA 2.2. *Let $\varphi(\theta)$ be the characteristic function associated with the random walk. Then for i, l, m', m in I^+*

(i) $\Sigma_x P^{i(l)}(0,x)[G^{(l)}(u,x) + G^{(l)}(x,u)] \leqslant c(i\int |\varphi(\theta)|^{il} d\theta + \int |\varphi(\theta)|^{il}/(1 - |\varphi(\theta)|^l)\, d\theta)$,

(ii) $\Sigma_x P^{i(l)}(0,x) G^{(l)}(u,x) G^{(l)}(x,v) \leqslant c[(\int |\varphi(\theta)|^{il} d\theta) \Sigma_{j=0}^{i-1} \int |\varphi(\theta)|^{jl}/(1 - |\varphi(\theta)|^l)\, d\theta + (\int |\varphi(\theta)|^{il}/(1 - |\varphi(\theta)|^l)\, d\theta)^2]$,

(iii) $\Sigma_{x \neq 0} G_m^{(l)}(0,x)[G^{(l)}(u,x) + G^{(l)}(x,u)] \leqslant c \Sigma_{i=1}^{m} \int |\varphi(\theta)|^{il}/(1 - |\varphi(\theta)|^l)\, d\theta$,

(iv) $\Sigma_{x \neq 0} G_m^{(l)}(0,x) G^{(l)}(u,x) G^{(l)}(x,v) \leqslant c \Sigma_{i=1}^{m} (\int |\varphi(\theta)|^{il}/(1 - |\varphi(\theta)|^l)\, d\theta)^2$,

(v) $\Sigma_{x \neq 0} G_m^{(l)}(0,x) P_x^{(l)}\{m' < T_x^{(l)} < \infty,\ T_0^{(l)} < \infty\} \leqslant c(\int |\varphi(\theta)|^{(m'+1)l/2}/(1 - |\varphi(\theta)|^l)\, d\theta)(\Sigma_{i=1}^{m} \int |\varphi(\theta)|^{il}/(1 - |\varphi(\theta)|^l)\, d\theta)$,

for some $c > 0$ depending on d.

PROOF. If $\int 1/(1 - |\varphi(\theta)|)\, d\theta = \infty$, there is nothing left to prove. Otherwise just follow the proofs of the Lemmas 2 to 4 in [4] and use (2.11) instead of Lemma 1 in that paper. □

Now for k, l, m in I^+ define the following self-explanatory events for the random walk on the time set $\{(i,l) : i \in I^+\}$

$$F_k^{(l)} = \cap_{i=k+1}^{\infty} \{S_k^{(l)} \neq S_i^{(l)}\}, \tag{2.15}$$

$$F_k^{m(l)} = \cap_{i=k+1}^{m} \{S_k^{(l)} \neq S_i^{(l)}\},\ 1 \leqslant k < m;\ F_m^{m(l)} = \Omega, \tag{2.16}$$

$$H_k^{m(l)} = \cup_{j=m+1}^{\infty} \cap_{i=k+1}^{m} \{S_k^{(l)} \neq S_i^{(l)};\ S_k^{(l)} = S_j^{(l)}\}, \quad 1 \leqslant k < m. \tag{2.17}$$

Clearly $F_k^{m(l)}$ is the union of two disjoint sets $F_k^{(l)}$ and $H_k^{m(l)}$, $1 \leqslant k < m$ and using $I(\cdot)$ as the indicator function we have

$$\begin{aligned} R_m^{(l)} &= \Sigma_{k=1}^{m} I(F_k^{m(l)}) \\ &= 1 + \Sigma_{k=1}^{m-1} I(F_k^{(l)}) + \Sigma_{k=1}^{m-1} I(H_k^{m(l)}) \\ &\doteq 1 + Z_m^{(l)} + W_m^{(l)}. \end{aligned} \tag{2.18}$$

THEOREM 2.1. *Let $\varphi(\theta)$ be the characteristic function associated with the random walk. Then for m, l in I^+ we have*

$$\operatorname{Var} R_m^{(l)} \leqslant cm \left[\int \frac{|\varphi(\theta)|^l}{1 - |\varphi(\theta)|^l} d\theta + \Sigma_{i=1}^{m} \left(\int \frac{|\varphi(\theta)|^{il/2}}{1 - |\varphi(\theta)|^l} d\theta \right)^2 \right]$$

where $c > 0$ depends on d.

PROOF. The theorem is trivially true if $\int 1/(1-|\varphi(\theta)|^l)\,d\theta=\infty$, otherwise from (2.18) we obtain

$$\text{(2.19)}\quad \operatorname{Var} R_m^{(l)} \leqslant 2\big(\operatorname{Var} Z_m^{(l)} + \operatorname{Var} W_m^{(l)}\big) \leqslant 2\Big[\operatorname{Var} Z_m^{(l)} + E\big(W_m^{(l)}\big)^2\Big].$$

Now follow the proof of Theorem 1 in [4] and use the previous lemma to get

$$\text{(2.20)}\quad E\big(W_m^{(l)}\big)^2 \leqslant 2\Sigma_{j=1}^{m-1}\Sigma_{i=1}^{j} EI\big(H_i^{m(l)} \cap H_j^{m(l)}\big)$$
$$\leqslant c\Sigma_{j=1}^{m-1}\bigg(\int \frac{|\varphi(\theta)|^{(m-j+1)l/2}}{1-|\varphi(\theta)|^l}\,d\theta\,\Sigma_{i=1}^{j}\int\frac{|\varphi(\theta)|^{il}}{1-|\varphi(\theta)|^l}\,d\theta\bigg).$$

Also

$$\text{(2.21)}\quad \operatorname{Var} Z_m^{(l)} \leqslant cm\int\frac{|\varphi(\theta)|^{l}}{1-|\varphi(\theta)|^l}\,d\theta + c\Sigma_{j=1}^{m-1}\Sigma_{i=1}^{j}\bigg(\int\frac{|\varphi(\theta)|^{il}}{1-|\varphi(\theta)|^l}\,d\theta\bigg)^2.$$

Utilizing (2.21) and (2.20) in (2.19) will give us the result.

In order to get the result mentioned in Section 1, we need to define some new events. Define

$$
\text{(2.22)}\quad
\begin{aligned}
F_{(i)}^{j} &= \cap_{l=j+1}^{\infty}\{S_{il}\neq S_{ij}\} \\
F_{(i)}^{nj} &= \cap_{l=j+1}^{n}\{S_{il}\neq S_{ij}\}, && 1\leqslant j<n,\ i\in I^+, \\
F_{ij}^{*mn} &= F_i^{m(j)}\cap F_{(i)}^{nj}, && 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n, \\
F_{ij}^{1mn} &= \cap_{k=i+1}^{m}\cap_{l=j+1}^{n}\{S_{kl}\neq S_{ij}\}, && 1\leqslant i<m,\ 1\leqslant j<n, \\
F_{ij}^{2mn} &= \cap_{k=1}^{i-1}\cap_{l=j+1}^{n}\{S_{kl}\neq S_{ij}\}, && 2\leqslant i\leqslant m,\ 1\leqslant j<n, \\
F_{ij}^{mn} &= F_{ij}^{1mn}\cap F_{ij}^{2mn}\cap F_{ij}^{*mn}, && 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n,
\end{aligned}
$$

and we understand the above events to be Ω for those i and j, $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n$, where the events are not specified. Now we can write

$$\text{(2.23)}\quad R_{mn} = \Sigma_{i=1}^{m}\Sigma_{j=1}^{n} I\big(F_{ij}^{mn}\big)$$
$$= \Sigma_{i=1}^{m}\Sigma_{j=1}^{n} I\big(F_{ij}^{*mn}\big) - \Sigma_{i=1}^{m}\Sigma_{j=1}^{n} I\Big[F_{ij}^{*mn}\cap\big(F_{ij}^{1mn}\cap F_{ij}^{2mn}\big)^c\Big].$$

We will denote the last two sums by T_{mn}^* and T_{mn} respectively, i.e.,

$$\text{(2.24)}\quad R_{mn} = T_{mn}^* - T_{mn}.$$

Our first concern is the limit behavior of T_{mn}^* which we will undertake in the following section.

3. Limit theorems for T^*_{mn}. To study the limit behavior of T^*_{mn} we first break it up as follows:

$$
\begin{aligned}
T^*_{mn} &= \Sigma_{i=1}^m \Sigma_{j=1}^n I\big(F_i^{m(j)} \cap F_{(i)}^{nj}\big) \\
&= \Sigma_{i=1}^m \Sigma_{j=1}^n I\big(F_i^{m(j)}\big) + \Sigma_{i=1}^m \Sigma_{j=1}^n I\big(F_{(i)}^{nj}\big) \\
&\quad + \Sigma_{i=1}^{m-1} \Sigma_{j=1}^{n-1} I\big(F_i^{m(j)} \cup F_{(i)}^{nj}\big)^c - mn.
\end{aligned} \tag{3.1}
$$

We will identify the last three sums as Q_{mn}, Q^*_{mn} and $T^{(3)}_{mn}$ respectively, i.e.,

$$T^*_{mn} = Q_{mn} + Q^*_{mn} + T^{(3)}_{mn} - mn. \tag{3.2}$$

Note that

$$Q_{mn} = \Sigma_{l=1}^n R_m^{(l)}, \tag{3.3}$$

and Q^*_{mn} is the "dual" of Q_{mn} in the sense that we interchange the role of m and n.

THEOREM 3.1. *Let $\varphi(\theta)$ be the characteristic function associated with the random walk. Then*

(a)
$$
\begin{cases}
\lim_{m\to\infty} T^{(3)}_{mn} = \Sigma_{j=1}^{n-1} \Sigma_{i=1}^{\infty} I\big(F_i^{(j)} \cup F_{(i)}^{nj}\big)^c & \text{a.s.,} \\
\lim_{n\to\infty} T^{(3)}_{mn} = \Sigma_{i=1}^{m-1} \Sigma_{j=1}^{\infty} I\big(F_i^{m(j)} \cup F_{(i)}^{j}\big)^c & \text{a.s.}
\end{cases}
$$

(b)
$$\lim_{(m,n)\to\infty} T^{(3)}_{mn} = \Sigma_{i=1}^{\infty} \Sigma_{j=1}^{\infty} I\big(F_i^{(j)} \cup F_{(i)}^{j}\big)^c \quad \text{a.s.,}$$

and the limits in (a) *and* (b) *are finite almost surely provided that* $\int 1/(1 - |\varphi(\theta)|)\, d\theta$ *and* $\int 1/(1 - |\varphi(\theta)|) \log 1/(1 - |\varphi(\theta)|)\, d\theta$ *are finite respectively.*

PROOF. Since

$$I\big(F_i^{m(j)}\big)^c \uparrow I\big(F_i^{(j)}\big)^c, \qquad I\big(F_{(i)}^{nj}\big)^c \uparrow I\big(F_{(i)}^{j}\big)^c, \tag{3.4}$$

and

$$I\big(F_i^{m(j)} \cup F_{(i)}^{nj}\big)^c \uparrow I\big(F_i^{(j)} \cup F_{(i)}^{j}\big)^c,$$

as $(m, n) \to \infty$. Therefore by the monotone convergence theorem (a) and (b) follow. To conclude the theorem, it is enough to show that the expectations of the sums in (a) and (b) are finite. Now we have

$$
\begin{aligned}
\Sigma_{i=1}^{\infty} P\big(F_{(i)}^{nj}\big)^c &= \Sigma_{i=1}^{\infty} \Sigma_{l=1}^{n-j} f_l^{(i)} \leqslant \Sigma_{l=1}^{n-j} \Sigma_{i=1}^{\infty} u_i^{(l)} \\
&\leqslant \Sigma_{l=1}^{n-j} \Sigma_{i=1}^{\infty} u_i^{(1)} \leqslant c(n-j) \int \frac{1}{1 - |\varphi(\theta)|}\, d\theta < \infty.
\end{aligned} \tag{3.5}
$$

Similarly one can show $\Sigma_{j=1}^{\infty} P(F_i^{m(j)})^c < \infty$. Furthermore, since $I(F_i^{(j)})^c$ and $I(F_{(i)}^{j})^c$ are independent we obtain

$$\Sigma_{i=1}^{\infty} \Sigma_{j=1}^{\infty} P\big(F_i^{(j)} \cup F_{(i)}^{j}\big)^c = \big(\Sigma_{i=1}^{\infty} \Sigma_{j=1}^{\infty} f_i^{(j)}\big)^2. \tag{3.6}$$

Now

$$
\begin{aligned}
\Sigma_{i=1}^{\infty} \Sigma_{j=1}^{\infty} f_i^{(j)} &\leqslant \Sigma_{i=1}^{\infty} \Sigma_{j=1}^{\infty} u_i^{(j)} \leqslant c \Sigma_{j=1}^{\infty} \Sigma_{i=1}^{\infty} \int |\varphi(\theta)|^{ij}\, d\theta \\
&\leqslant c \Sigma_{j=1}^{\infty} \int \frac{2|\varphi(\theta)|^{j/2}}{j(1 - |\varphi(\theta)|)}\, d\theta \leqslant c \int \frac{1}{1 - |\varphi(\theta)|} \log \frac{1}{1 - |\varphi(\theta)|}\, d\theta < \infty;
\end{aligned} \tag{3.7}
$$

completes the proof. □

LEMMA 3.1. *Let $\varphi(\theta)$ be the characteristic function associated with the random walk. Then*

(a) $\int 1/(1 - |\varphi(\theta)|)\, d\theta < \infty \Rightarrow \lim_{(m,n)\to\infty} ET^*_{mn}/mn = 1$;

(b) $\int 1/(1 - |\varphi(\theta)|)\log 1/(1 - |\varphi(\theta)|)\, d\theta < \infty \Rightarrow \lim_{(m,n)\to\infty}(mn - ET^*_{mn})/(m + n) = \Sigma_{l=1}^{\infty}(1 - r^{(l)}) < \infty$.

PROOF. First of all note that the convergence of $\Sigma_{l=1}^{\infty}(1 - r^{(l)})$ is already obtained in (3.7). Secondly for k, l in I^+ with $1 \leqslant k < m$ and by (2.18) we have

$$(3.8) \qquad m - ER_m^{(l)} = \Sigma_{k=1}^{m-1} P\big(F_k^{m(l)}\big)^c = \Sigma_{k=1}^{m-1}\Sigma_{i=1}^{m-k} f_i^{(l)} = m\Sigma_{i=1}^{m-1} f_i^{(l)} - \Sigma_{i=1}^{m-1} i f_i^{(l)},$$

and consequently (3.3) implies that

$$(3.9) \qquad mn - EQ_{mn} = m\Sigma_{l=1}^{n}\Sigma_{i=1}^{m-1} f_i^{(l)} - \Sigma_{l=1}^{n}\Sigma_{i=1}^{m-1} i f_i^{(l)}.$$

Similarly

$$(3.10) \qquad mn - EQ^*_{mn} = n\Sigma_{l=1}^{m}\Sigma_{i=1}^{n-1} f_i^{(l)} - \Sigma_{l=1}^{m}\Sigma_{i=1}^{n-1} i f_i^{(l)}.$$

Now if $\int(1 - |\varphi(\theta)|)^{-1} d\theta < \infty$, then by the dominated convergence theorem we get

$$(3.11) \qquad m\Sigma_{l=1}^{\infty} f_m^{(l)} \leqslant m\Sigma_{l=1}^{\infty} u_m^{(l)} \leqslant cm\int \frac{|\varphi(\theta)|^m}{1 - |\varphi(\theta)|^m}\, d\theta \leqslant c\int \frac{|\varphi(\theta)|^{m/2}}{1 - |\varphi(\theta)|}\, d\theta \to 0$$

as $m \to \infty$. Therefore

$$(3.12) \qquad \frac{\Sigma_{l=1}^{n}\Sigma_{i=1}^{m-1} i f_i^{(l)} = \Sigma_{l=1}^{m}\Sigma_{i=1}^{n-1} i f_i^{(l)}}{m + n} \leqslant \frac{\Sigma_{i=1}^{m} i\Sigma_{l=1}^{\infty} f_i^{(l)}}{m} + \frac{\Sigma_{i=1}^{n} i\Sigma_{l=1}^{\infty} f_i^{(l)}}{n} \to 0$$

as $(m, n) \to \infty$. This, the preceding theorem and an easy calculation yields the result claimed in (b). Next observe that

$$(3.13) \qquad ET_{mn}^{(3)} \leqslant \Sigma_{k=1}^{m}\Sigma_{l=1}^{n}(1 - r^{(k)})(1 - r^{(l)}).$$

Since $r^{(n)} \to 1$ as $n \to \infty$ provided that $\int(1 - |\varphi(\theta)|)^{-1} d\theta < \infty$ (see (3.10) and (3.11) in [3]), $ET_{mn}^{(3)}/mn \to 0$ as $(m, n) \to \infty$. Now to complete the proof consult Theorem 3.1 in [3]. □

THEOREM 3.2. *Let $\varphi(\theta)$ be the characteristic function associated with the random walk. Then*

$$\text{(a)} \quad \int \frac{1}{1 - |\varphi(\theta)|}\log\frac{1}{1 - |\varphi(\theta)|}\, d\theta < \infty \Rightarrow \lim_{(m,n)\to\infty}\frac{T^*_{mn}}{mn} = 1 \quad \text{a.s.}$$

$$\text{(b)} \qquad \Rightarrow \lim_{n\to\infty}\frac{mn - T^*_{mn}}{m + n} = \Sigma_{l=1}^{m}(1 - r^{(l)}) \quad \text{a.s.}$$

(c) $$\Rightarrow \lim_{m\to\infty} \frac{mn - T^*_{mn}}{m+n} = \Sigma_{l=1}^{n}(1 - r^{(l)}) \qquad \text{a.s.}$$

(d) $$(\exists \varepsilon > 0) \int \frac{1}{(1 - |\varphi(\theta)|)^{1+\varepsilon}}\, d\theta < \infty \Rightarrow \lim_{(m,n)\to\infty} \frac{mn - T^*_{mn}}{m+n} = \Sigma_{l=1}^{\infty}(1 - r^{(l)}) < \infty \qquad \text{a.s.}$$

PROOF. Parts (a), (b) and (c) follow immediately from Remark 3.3, Theorem 3.2 in [3] and Theorem 3.1. By using the Schwarz inequality and Theorem 2.1 we obtain

$$\operatorname{Var} Q_{mn} = \operatorname{Var} \Sigma_{l=1}^{n} R_m^{(l)} \leqslant \left(\Sigma_{l=1}^{n}(\operatorname{Var} R_m^{(l)})^{\frac{1}{2}}\right)^2 \tag{3.14}$$

$$\leqslant cm\left\{\left[\Sigma_{l=1}^{n}\left(\int \frac{|\varphi(\theta)|^{l}}{1 - |\varphi(\theta)|^{l}}\, d\theta\right)^{\frac{1}{2}}\right]^2 + \left[\Sigma_{l=1}^{n}\left\{\Sigma_{i=1}^{m}\left(\int \frac{|\varphi(\theta)|^{il/2}}{1 - |\varphi(\theta)|^{l}}\, d\theta\right)^2\right\}^{\frac{1}{2}}\right]^2\right\}.$$

Now apply Hölder's inequality with $p = 1 + \varepsilon$ to get

$$\int \frac{|\varphi(\theta)|^{l}}{1 - |\varphi(\theta)|^{l}}\, d\theta \leqslant \left[\int\left(\frac{|\varphi(\theta)|^{l/2}}{1 - |\varphi(\theta)|^{l}}\right)^{1+\varepsilon} d\theta\right]^{1/(1+\varepsilon)} \left(\int |\varphi(\theta)|^{l/2\cdot(1+\varepsilon)/\varepsilon}\, d\theta\right)^{\varepsilon/(1+\varepsilon)}$$

$$\leqslant \frac{c}{l}\left(\int\left(\frac{1}{1 - |\varphi(\theta)|}\right)^{1+\varepsilon} d\theta\right)^{1/(1+\varepsilon)} \left(\int |\varphi(\theta)|^{l/2}\, d\theta\right)^{\varepsilon/(1+\varepsilon)} \tag{3.15}$$

$$\leqslant \frac{c}{l}\left(\int |\varphi(\theta)|^{l/2}\, d\theta\right)^{\varepsilon/(1+\varepsilon)}.$$

Again using Hölder's inequality with $p = 2(1 + \varepsilon)/\varepsilon$, one has

$$\left[\Sigma_{l=1}^{n}\left(\int \frac{|\varphi(\theta)|^{l}}{1 - |\varphi(\theta)|^{l}}\, d\theta\right)^{\frac{1}{2}}\right]^2 \leqslant c\left[\Sigma_{l=1}^{n} \frac{1}{l^{\frac{1}{2}}}\left(\int |\varphi(\theta)|^{l/2}\, d\theta\right)^{\varepsilon/2(1+\varepsilon)}\right]^2$$

$$\leqslant c\left[\Sigma_{l=1}^{n}\left(\frac{1}{l^{\frac{1}{2}}}\right)^{2(1+\varepsilon)/(2+\varepsilon)}\right]^{(2+\varepsilon)/(1+\varepsilon)} \tag{3.16}$$

$$\times\left[\Sigma_{l=1}^{n} \int |\varphi(\theta)|^{l/2}\, d\theta\right]^{\varepsilon/(1+\varepsilon)}$$

$$\leqslant cn^{1/(1+\varepsilon)}.$$

Now follow an argument similar to (3.15) and then use Hölder's inequality with

$p = (1 + \varepsilon)/(1 - \varepsilon)$ to obtain

$$\Sigma_{l=1}^{n}\left[\Sigma_{i=1}^{m}\left(\int \frac{|\varphi(\theta)|^{il/2}}{1 - |\varphi(\theta)|^{l/2}}\, d\theta\right)^{2}\right]^{\frac{1}{2}} \leqslant c\Sigma_{l=1}^{n}\left[\Sigma_{i=1}^{m}\frac{1}{l^{2}}\left(\int |\varphi(\theta)|^{il/4}\, d\theta\right)^{2\varepsilon/(1+\varepsilon)}\right]^{\frac{1}{2}}$$

$$\text{(3.17)} \qquad \leqslant c\Sigma_{l=1}^{n}\frac{1}{l}\left[m^{(1-\varepsilon)/(1+\varepsilon)}\left(\Sigma_{i=1}^{m}\int |\varphi(\theta)|^{il/4}\, d\theta\right)^{2\varepsilon/(1+\varepsilon)}\right]^{\frac{1}{2}}$$

$$\leqslant cm^{(1-\varepsilon)/2(1+\varepsilon)}\Sigma_{l=1}^{n}\left(\frac{1}{l}\right)^{1+\varepsilon/(1+\varepsilon)} \leqslant cm^{(1-\varepsilon)/2(1+\varepsilon)}.$$

Hence by (3.14)

$$\text{(3.18)} \qquad \operatorname{Var} Q_{mn} \leqslant cm\left(n^{1/(1+\varepsilon)} + m^{(1-\varepsilon)/(1+\varepsilon)}\right).$$

Now for $a > 1$, let $m_k = [a^k]$, $n_l = [a^l]$. Then an easy calculation shows $\Sigma_{l=1}^{\infty}\Sigma_{k=1}^{\infty} \operatorname{Var} Q_{m_k n_l}/(m_k + n_l)^2 < \infty$. Therefore by Chebyshev's inequality and the Borel-Cantelli lemma

$$\text{(3.19)} \qquad \lim_{(k,l)\to\infty}\frac{EQ_{m_k n_l} - Q_{m_k n_l}}{m_k + n_l} = 0 \qquad \text{a.s.}$$

Since a similar result is also true for Q^*_{mn}, from Lemma 3.1, Theorem 3.1 and (3.2) we can conclude that

$$\text{(3.20)} \qquad \lim_{(k,l)\to\infty}\frac{m_k n_l - T^*_{m_k n_l}}{m_k + n_l} = \Sigma_{i=1}^{\infty}(1 - r^{(i)}) < \infty \qquad \text{a.s.}$$

But $mn - T^*_{mn}$ is an "increasing process" in the sense that for $m_2 \geqslant m_1$ and $n_2 \geqslant n_1$ we have

$$\text{(3.21)} \qquad m_1 n_1 - T^*_{m_1 n_1} \leqslant m_2 n_2 - T^*_{m_2 n_2}.$$

Thus fixing ω not in the exceptional set corresponding to (3.20) and using a standard argument we obtain

$$\text{(3.22)} \qquad \frac{1}{a}\Sigma_{l=1}^{\infty}(1 - r^{(l)}) \leqslant \liminf\frac{mn - T^*_{mn}}{m + n}$$

$$\leqslant \limsup\frac{mn - T^*_{mn}}{m + n} \leqslant a\Sigma_{l=1}^{\infty}(1 - r^{(l)}),$$

and since $a > 1$ is arbitrary, we are through. □

If we define the genuine dimension of the two-parameter random walk $\{S_{mn} : (m,n) \in I^+ \times I^+\}$ to be the genuine dimension of the one-parameter random walk $\{S_m^{(1)} : m \in I^+\}$, then we have:

COROLLARY 3.1. *Let $d \geqslant 3$ be the genuine dimension of the one-parameter random walk. Then the results of Theorem 3.2 are true.*

PROOF. Let the random walk be generated by X_{11}. Consider the symmetrized random walk generated by $X_{11} - X'_{11}$, where X'_{11} is independently identically distributed as X_{11}. If the genuine dimension of this random walk is not d, then one can show, see [1] page 842, that $T^*_{mn} = mn$ almost surely and we are through. Otherwise if we let $\varphi(\theta)$ be the characteristic function of X_{11}, then $|\varphi(\theta)|^2$ is the characteristic function of the symmetrized random walk. Now the proof follows from Proposition 5 in [5] page 70, and the fact that

$$1 - |\varphi(\theta)| \leqslant 1 - |\varphi(\theta)|^2 \leqslant 2(1 - |\varphi(\theta)|). \tag{3.23}$$

4. A limit theorem for R_{mn}. In this section we impose a condition on the characteristic function of the random walk so that $T_{mn}/(m + n)$, see (2.24), converges to zero almost surely. This will make R_{mn} behave as T^*_{mn} whose limit behavior in this particular case is already known.

LEMMA 4.1. *Let $\varphi(\theta)$ be the characteristic function associated with the random walk and*

$$T^{(1)}_{mn} = \Sigma_{i=1}^{m-1}\Sigma_{j=1}^{n-1} I\left(\cup_{k=i+1}^{m} \cup_{l=j+1}^{n}\{S_{kl} = S_{ij}\}\right).$$

Then

$$\int \frac{1}{(1 - |\varphi(\theta)|)^{\frac{3}{2}}} \log \frac{1}{1 - |\varphi(\theta)|}\, d\theta < \infty \Rightarrow \lim_{(m,n)\to\infty} \frac{T^{(1)}_{mn}}{(mn)^{\frac{1}{2}}} = 0 \quad \text{a.s.}$$

PROOF. Follow Lemma 3.1 in [3] and use Lemma 2.1. □

LEMMA 4.2. *Let $\varphi(\theta)$ be the characteristic function associated with the random walk and*

$$T^{(2)}_{mn} = \Sigma_{i=2}^{m}\Sigma_{j=1}^{n-1} I\left(\cup_{k=1}^{i-1} \cup_{l=j+1}^{n}\{S_{kl} = S_{ij}\}\right).$$

Then

$$\int \frac{1}{(1 - |\varphi(\theta)|)^{\frac{3}{2}}} \log \frac{1}{1 - |\varphi(\theta)|}\, d\theta < \infty$$

$$\Rightarrow \lim_{(m,n)\to\infty} \frac{T^{(2)}_{mn}}{(mn)^{\frac{1}{2}}} = 0 \quad \text{a.s.}$$

PROOF. Follow Lemma 3.2 in [3] and use Lemma 2.1. □

THEOREM 4.1. *Let $\varphi(\theta)$ be the characteristic function associated with the random walk. Then*

(a) $$\int \frac{1}{1 - |\varphi(\theta)|} \log \frac{1}{1 - |\varphi(\theta)|}\, d\theta < \infty \Rightarrow \lim_{m\to\infty} \frac{mn - R_{mn}}{m + n}$$

$$= \Sigma_{l=1}^{n}(1 - r^{(l)}) \quad \text{a.s.}$$

(b) $$\Rightarrow \lim_{n\to\infty} \frac{mn - R_{mn}}{m+n} = \Sigma_{l=1}^{m}(1 - r^{(l)}) \quad \text{a.s.}$$

(c) $$\int \frac{1}{(1 - |\varphi(\theta)|)^{\frac{3}{2}}} \log \frac{1}{1 - |\varphi(\theta)|}\, d\theta < \infty \Rightarrow \lim_{(m,n)\to\infty} \frac{mn - R_{mn}}{m+n}$$
$$= \Sigma_{l=1}^{\infty}(1 - r^{(l)}) \quad \text{a.s.}$$

PROOF. Note that $m + n \geqslant (mn)^{\frac{1}{2}}$, and

(4.1) $$T_{mn} \leqslant T_{mn}^{(1)} + T_{mn}^{(2)}.$$

Now (a), (b) follows from Remark 3.3 in [3]. For part (c) use Theorem 3.2 and the last two lemmas. □

COROLLARY 4.1. *Let d be the genuine dimension of the symmetrized random walk generated by X_{11}. Then with $d \geqslant 3$ part* (a) *and* (b) *and with $d \geqslant 4$ part* (c) *of the preceding theorem hold true.*

PROOF. Use Proposition 5 in [5] page 70 and (3.23) to estimate the integral. □

Further limit theorems on the range of the random walk when the dimension of the random walk is either one or two can be obtained in the author's thesis written at the University of Minnesota. We also invite the reader to consult [2] for more results concerning two-parameter random walks.

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