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COMPARISON THEOREMS FOR SAMPLE FUNCTION GROWTH

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The growth rate at 0 of a Lévy process is compared with the growth rate at a local minimum, m, of the process. For the lim inf it is found that the growth rate at m is the same as that on the set of "ladder points" following 0, parameterized by inverse local time; this result gives a precise meaning to the notion that a Lévy process leaves its minima "faster" than it leaves 0. A less precise result is obtained for the lim sup.

1. Introduction. Let $\{X_t,\ t \geq 0\}$ be a real process with stationary independent increments. Problems of local growth at 0—namely to evaluate $\limsup_{t\to 0} X_t/f(t)$ and $\liminf_{t\to 0} X_t/f(t)$ for various f and $\{X_t\}$—have a long and distinguished history. By the strong Markov property, these results give immediately the growth rates at any stopping time T: e.g., $\limsup_{t\to 0} X_t/f(t) = \limsup_{t\to 0} X(t+T) - X(T)/f(t)$ a.s. More recently ([7], [11], [12], [13]) attempts have been made to study the growth rates at random times T that are not stopping times. The problem here is much more difficult and so far there are results only for the case of stable processes and a selected few T. Moreover, these results have been obtained via intricate and painful calculation, which, while providing concrete and interesting results, nevertheless leave general structural questions unanswered. For example, such methods seem incapable of answering the general question whether an arbitrary process leaves T "faster" (or "slower") than it leaves 0; indeed, even exact rates at 0 are known for only relatively few processes. In an effort to get results that apply to a wide assortment of processes, one could attempt to evaluate the growth rate at a random time T in terms of functionals that concern only the behavior of the process near 0. By this kind of comparison, one avoids the very difficult problem of providing exact growth rates for every process, but yet should obtain a fairly good idea of the sample function behavior in question. In the case where T is the time of a local minimum, such an approach can actually be carried through. The main result of this paper can then be described as follows:

Restricting attention to processes whose local minima do not occur at optional times, let m be the time of a local minimum, let $X_t^+ = \sup_{s\leq t} X_s$ and T_t be the right continuous inverse of a local time at 0 for the process $\{X_t^+ - X_t\}$. The main result is an exact comparison theorem for the lower envelope at m: for any increasing positive f:

$$\liminf_{t\to 0} X(m+t) - X(m)/f(t) = \liminf_{t\to 0} X^+(T_t)/f(T_t). \tag{1.1}$$

Since the distribution of $(X^+(T_t), T_t)$ is known, this result leads in sufficiently nice cases to integral tests for f. In the general case it gives the precise sense in which processes leave their minima "faster" than they leave 0. Evidently (1.1) implies

$$\liminf_{t\to 0} X(m+t) - X(m)/f(t) \geq \liminf_{t\to 0} X^+(t)/f(t). \tag{1.2}$$

In fact strict inequality typically occurs in (1.2), and it can be argued that processes typically leave m an order of magnitude faster than they leave 0. Similar methods also establish a weaker comparison for the upper envelope:

$$\limsup_{t\to 0} X(m+t) - X(m)/f(t) \geq \limsup_{t\to 0} X^+(t)/f(t). \tag{1.3}$$

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Throughout this paper, notations of general Markov theory follow those of Reference [1].

2. Sample function growth at a local minimum. Throughout this paper, $\{X_t, t \geq 0\}$ will be a real Lévy process, i.e., a process with stationary independent increments constructed so as to have right continuous paths with left limits. The process then has the strong Markov property. Let P^x denote the measures on an appropriate probability space for the process starting at x; if $x = 0$, the superscript will usually be omitted. The transition functions of $\{X_t\}$ are $P_t(x, dy) = \mu_t(dy - x)$ where $\mu_t(dy)$ is the P^0 distribution of X_t. Assume throughout that

(2.1) $$0 \text{ is regular for } (0, \infty) \text{ and for } (-\infty, 0).$$

This means that if A is a Borel set and $T_A = \inf\{t > 0: X_t \in A\}$ then $P\{T_{(-\infty,0)} = 0\} = P\{T_{(0,\infty)} = 0\} = 1$. As indicated in Section 1, we intend to study the sample function behavior of X to the right of its local minima; if (2.1) were violated, then all local minima occur at optional times (see [8], Section 2) so the behavior at a local minimum is exactly the same as that at time 0. So assumption (2.1) merely restricts attention to the problem of real interest.

Define

(2.2) $$X_t^- = \inf_{s \leq t} X_s, \qquad X_t^+ = \sup_{s \leq t} X_s.$$

The following proposition is known, under assumption (2.1)(cf. [8]).

(2.3) PROPOSITION. (a) *X is continuous at its local minima.*

(b) *If $t > 0$, X_t^+ has a continuous distribution.*

(c) *On any fixed interval* $[0, a]$ *there is exactly one time point M_a at which $X(M_a) = X^+(a)$; moreover, $0 < M_a < a$.*

(d) *If T is any optional time, $M_a \neq T$ with probability* 1.

For $\{X_t\}$ a real Lévy process, it is well known and easy to prove that the processes

(2.4) $$H_t^- = X_t - X_t^-$$
$$H_t^+ = X_t^+ - X_t$$

are Hunt processes. Indeed, a simple calculation ([10]) shows the vector process (X_t, X_t^-) is a Hunt process with transitions $Q_t((x, a), f) = E^x f(X_t, X_t^- \wedge a)$ (this much is true for any real Hunt process X_t); a simple application of the scale change theorem ([3], page 325) with $\gamma((x, y)) = y - x$ yields the result immediately. Under assumption (2.1), 0 is regular for $\{0\}$ for the processes (2.4). Let

(2.5) T_t be the right continuous inverse of a local time at 0 for $\{H_t^+\}$

Fristedt ([4]) has shown that the vector process

(2.6) $(X^+(T_t), T_t)$

is a truncated Lévy process with strictly increasing paths (co-ordinate wise).

Even more is true. Define

(2.7) $$M_t = \sup\{s \leq t: X_s = X_s^+ \text{ or } X_{s-} = X_s^+\}.$$

Then ([4], Chapter 9),

(2.8) the random sets in $R^2\{(X^+(T_t), T_t), t \geq 0\}$ and $\{(X^+(M_t), M_t), t \geq 0\}$ are the same and are traced out in the same order as t moves from 0 to ∞.

Fix $a > 0$. Let

(2.9) $$m = m^a = \sup\{t < a: X_t = X_a^-\}$$

so m is the time point in $[0, a]$ at which $\{X_t, 0 \le t \le a\}$ achieves its ultimate minimum. Since a will be fixed henceforth, it will be deleted from the notation.

(2.10) THEOREM. *Let f be a nonnegative increasing function, $f(0) = 0$. Then*

$$\liminf_{t\to 0} (X(m+t) - X(m))/f(t) = \liminf_{t\to 0} X^+(T_t)/f(T_t).$$

REMARK. By the Blumenthal zero one law, the quantity on the right in (2.10) is constant; by the zero one law of [8], the left side will be also. By changing the interval $[0, a]$ to $[a, b]$, this result gives the lower envelope at any local minimum.

PROOF. Define

(2.11) $$I_t = \inf\{X(m+s); t \le s \le a - m\}$$

so in particular $I_0 = X(m)$. Define also

(2.12) $$m_t = \sup\{s: t \le s \le a - m, \quad X(m+s) = I_t\}.$$

The random set $\cup_t\{m + m_t\} \subset [m, a]$ gives the points at which occur the "last minima" of the process $\{X(s), 0 \le s \le a\}$.

Let us establish first that

(2.13) $$\liminf_{t\to 0}(X(m+t) - X(m))/f(t) = \liminf_{t\to 0}(X(m+m_t) - X(m))/f(m_t).$$

It is obvious that the right side of (2.13) is at least as big as the left side, since the limit on the right is computed over fewer values. On the other hand, by the definition of m_t and the continuity of X at $m + m_t$:

$$X(m+t) - X(m) \ge X(m+m_t) - X(m).$$

Since f is increasing

$$f(t) \le f(m_t)$$

so

$$(X(m+t) - X(m))/f(t) \ge (X(m+m_t) - X(m))/f(m_t),$$

proving (2.13).

Next, notice that for each ω, the points of the random set $\cup_t\{m(\omega) + m_t(\omega)\}$ are the points of "new maxima" for the process $\{X(a) - X(a-t), 0 \le t \le a\}$, but listed in reverse order. More succinctly, if $Z(\cdot)$ is any real function on $[0, a]$ with left and right limits at each point, define for $0 \le t \le a$:

(2.14) $$M_t(Z) = \sup\{s \le t: Z(s+) = Z^+(t) \quad \text{or} \quad Z(s-) = Z^+(t)\}$$

where, as usual, $Z^+(t) = \sup\{Z(s), 0 \le s \le t\}$. Clearly $M_t(Z) = M_a(Z)$ for $M_a(Z) \le t \le a$. If

$$Y(t) = X(a) - X(a-t)$$

then

(2.15) $$s = m + m_t \quad \text{if and only if} \quad a - s = M_{a+t-m}(Y).$$

Since Y_t is a Lévy process, results given in [4], Chapter 9 and summarized above guarantee the existence of a subordinator $\{S_t\}$ and a random time ζ such that

(2.16) $$\{M_t(Y): 0 \le t \le a\} = \{S_t: 0 \le t < \zeta\};$$

i.e., the set of new maxima of Y is the range of a subordinator S_t. Here $\zeta = \inf\{t: S_t > a\}$.

Of course

$$S_{\zeta-} = a - m \tag{2.17}$$

and

$$\{m_t\} = \{S_{\zeta-} - S_{\zeta-t}\}$$

(up to closure). Hence

$$\begin{aligned}\{(X(m + m_t) - X(m), m_t)\} &= \{(X(a - S_{\zeta-t}) - X(a - S_{\zeta-}), S_{\zeta-} - S_{\zeta-t})\} \\ &= \{(Y(S_{\zeta-}) - Y(S_{\zeta-t}), S_{\zeta-} - S_{\zeta-t})\}.\end{aligned} \tag{2.18}$$

By time reversal, the process $\{(Y(S_{\zeta-}) - Y(S_{\zeta-t}), S_{\zeta-} - S_{\zeta-t})\}$ has the same distribution as $\{(Y(S_t), S_t), t < \zeta\}$. But $Y(t) = X(a) - X(a - t)$ has the same distribution as $\{X(t), 0 \le t \le a\}$, so again by time reversal $\{(Y(S_t), S_t), t < \zeta\}$ has the same distribution as $\{(X(T_t), T_t), t < \zeta\} = \{(X^+(T_t), T_t), t < \zeta\}$, where T_t has the definition (2.5). Accordingly

$$\liminf_{t\to 0} (X(m + m_t) - X(m))/f(m_t) = \liminf_{t\to 0} X^+(T_t)/f(T_t) \tag{2.19}$$

and this, together with (2.13) gives the result. □

REMARK 1. The random sets $\{X^+(T_t), t \ge 0\}$ and $\{X^+(t), t \ge 0\}$ are the same; moreover $X^+(u)$ is constant on $T_{t-} \le u < T_t$ and continuous at T_{t-}. It follows that

$$\liminf X^+(t)/f(t) = \liminf X^+(T_{t-})/f(T_t). \tag{2.20}$$

The quantity $X^+(T_{t-})/f(T_t)$ will be small when T_t makes a big jump compared to the size of T_{t-}, and this happens often by [8], Section 5. Consequently it seems likely that typically

$$\liminf X^+(t)/f(t) \ll \liminf X^+(T_t)/f(T_t).$$

This phenomenon can be checked explicitly in the stable case.

REMARK 2. If X_t has no upward jumps, then

$$X^+(T_t) = ct,$$

c a positive constant. In this case

$$\liminf X^+(T_t)/f(T_t) = c \liminf t/f(T_t)$$

and, since the Laplace transform of T_t is known, the results of [5] can be applied to produce the best f's for this case.

REMARK 3. In general the joint distribution of $(X^+(T_t), T_t)$ is 'known' ([4], Chapter 9), and in certain cases it is possible to apply this information to study $\liminf X^+(T_t)/f(T_t)$. In particular, it should be possible in the stable case.

(2.21) COROLLARY.

$$\limsup_{t\to 0} (X(m + t) - X(m))/f(t) \ge \limsup_{t\to 0} X^+(t)/f(t).$$

PROOF. From the definitions it is obvious that

$$\limsup (X(m + t) - X(m))/f(t) \ge \limsup (X(m + m_t) - X(m))/f(m_t).$$

Hence by (2.18), et seq.

$$\limsup (X(m + t) - X(m))/f(t) \ge \limsup X^+(T_t)/f(T_t). \tag{2.22}$$

If $T_{t-} \le u < T_t$, then

$$X^+(u) = X^+(T_{t-}).$$

Therefore

(2.23) $$\limsup X^+(u)/f(u) \le \limsup X(T_{t-})/f(T_{t-})$$

since f is increasing. Since the process $X^+(T_{t-})/f(T_{t-})$ is separable (it is left continuous on $(0, \infty)$), there exists a sequence $t_{1,n} \downarrow 0$ such that

$$\limsup X^+(T_{t_{1,n}}-)/f(T_{t_{1,n}}-) = \limsup X(T_{t-})/f(T_{t-})$$

(see [2] page 555, Theorem 2.3); for the same reason there exists $t_{2,n} \downarrow 0$ so that

$$\limsup X^+(T_{t_{2,n}})/f(T_{t_{2,n}}) = \limsup X(T_t)/f(T_t).$$

Let t_n be the sequence obtained by combining the $t_{i,n}$ into a single decreasing sequence. Then

$$\limsup X^+(T_{t_n}-)/f(T_{t_n}-) = \limsup X(T_{t-})/f(T_{t-})$$

and

$$\limsup X^+(T_{t_n})/f(T_{t_n}) = \limsup X(T_t)/f(T_t).$$

However, $T_{t_n} = T_{t_n}-$ a.s., so

$$\limsup X(T_t)/f(T_t) = \limsup X(T_{t-})/f(T_{t-}).$$

This with (2.22), (2.23) proves the corollary.

REMARK. If X_t is Brownian motion, then the ideas in the proof of (2.21) together with a bit of excursion theory show, without calculation that

$$\limsup(X(m + t) - X(m))/f(t) = \limsup(2X^+(t) - X(t))/f(t).$$

The argument depends critically on symmetry and continuity of path, so this exact comparison does not extend to other processes.

REFERENCES

[1] BLUMENTHAL, R. M. AND GETOOR, R. K. (1968). *Markov Processes and Potential Theory.* Academic, New York.

[2] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[3] DYNKIN, E. B. (1965). *Markov Processes.* I. Academic, New York.

[4] FRISTEDT, B. (1973). Sample functions of stochastic processes with stationary independent increments. *Advances in Probability* III. 241-396. Dekker, New York.

[5] FRISTEDT, B. (1967). Sample function behavior of increasing processes with stationary independent increments, *Pac. J. Math.* **21** 21-33.

[6] MILLAR, P. W. (1975). First passage distributions of processes with independent increments, *Ann. Probability* **3** 215-233.

[7] MILLAR, P. W. (1976). Sample functions at a last exit time, *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **34** 91-111.

[8] MILLAR, P. W. (1977a). Zero-one laws and the minimum of a Markov process, *Trans. Amer. Math. Soc.* **266** 365-391.

[9] MILLAR, P. W. (1977b). Germ sigma fields and the natural state space of a Markov process, *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **39** 85-101.

[10] MILLAR, P. W. (1978). A path decomposition for Markov processes. *Ann. Probability* **6** 345-348.

[11] MONRAD, D. (1977). Stable processes: sample function growth at a last exit time, *Ann. Probability* **5** 455-462.

[12] MONRAD, D. (1978). Asymmetric Cauchy processes: sample functions at last zero. *Ann. Probability* **6** 771-787.

[13] MONRAD, D. AND SILVERSTEIN, M. L. Stable processes: sample function growth at a local minimum. Unpublished manuscript.

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