The Annals of Probability
1981, Vol. 9, No. 3, 529–532

MULTIPLE INTEGRALS OF A HOMOGENEOUS PROCESS WITH INDEPENDENT INCREMENTS

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Let $X(t)$ be a homogeneous process with independent increments having the representation $X(t) = W(t) + \int_{x\neq 0} x\nu^*(t, dx)$, where $W(t)$ is a Wiener process with parameter σ^2 and $\nu^*(t, dx) = \nu(t, dx) - t\mu(dx)$, where $\nu(t, dx)$ is a Poisson random measure with mean measure $t\mu(dx)$. If the mth absolute mean of $X(t)$ is finite, then $\int_0^t dX(t_1) \int_0^{t_1} dX(t_2) \cdots \int_0^{t_{m-1}} dX(t_m) = \{\partial^m/\partial u^m \exp\{uW(t) + \int_{x\neq 0} \log(1 + ux)\nu^*(t, dx) - \frac{1}{2}tu^2\sigma^2 - t\int_{x\neq 0} [ux - \log(1 + ux)]\mu(dx)\}\}_{u=0}/m!$.

1. Introduction. Let $X(t)$, $t \geqslant 0$, be a homogeneous process of independent increments. Multiple integrals of $X(t)$ had been studied in [4]. We study here a different type of multiple integral which is closely related to the stochastic integral equation

$$Z(t) = 1 + u\int_0^t Z(s-)\, dX(s). \tag{1.1}$$

Equation (1.1) has been extensively studied. For examples, see [6], page 36, [2], page 448 and [8], page 453.

Let $f(t)$, $t \geqslant 0$, be a process such that $\{X(v), f(v): 0 \leqslant v \leqslant s\}$ is independent of $X(t) - X(s)$ for $s < t$. Define the integral

$$\int_0^t f(s-)\, dX(s) = \lim_{n\to\infty} \textstyle\sum_{n=0}^{T} f(t_k)(X(t_{k+1}) - X(t_k)) \tag{1.2}$$

when the limit on the right-hand side of (1.2) exists in some sense, where $T = T_n = [2^n t]$ (the integer part of $2^n t$), $t_k = t_{n,k} = k2^{-n}$, $0 \leqslant k \leqslant T$, $t_{T+1} = t$. If $f(t)$ is $L^2(\Omega)$-continuous in t and $EX^2(t) < \infty$, then the integral (1.2) will exist in $L^2(\Omega)$ sense. Let $I^0(t) = 1$ and suppose that $I^n(t) = \int_0^t I^{n-1}(s-)\, dX(s)$ exists for each $n \geqslant 1$. We shall derive the explicit formulas for $I^n(t)$, $n \geqslant 1$. In the case that $X(t)$ is a Wiener process with parameter $\sigma^2 = 1$, $I^n(t) = H_n(t, W(t))$, where $H_n(t, x)$ is the nth Hermite polynomial (see [3] or [6], page 38). If $X(t) = P(t) - t$ is a mean centered Poisson process, then $I^n(t) = K_n(t, P(t))$, where $K_n(t, x)$, $n \geqslant 0$, are polynomials whose generating function is $(1 + u)^x e^{-ut}$, $u > -1$, (see [5]). In both cases, $I^n(t)$ is a polynomial of $X(t)$. However, this is not true in general.

2. Multiple integrals. Let $X(t)$, $t \geqslant 0$, be a stochastic continuous homogeneous process with independent increments. To evaluate the m-multiple integral ($m \geqslant 2$) of $X(t)$, we shall assume that $E|X(t)|^m < \infty$. There exist a Poisson random measure $\nu((t, dx)$ on $R' = R - \{0\}$ (see [1], Chapter VI) and a measure $\mu(dx)$ on R' such that

$$\int_{0<|x|\leqslant 1} x^2\mu(dx) + \int_{|x|>1} \mu(dx) < \infty.$$

For each Borel set A of R', $\nu(t, A)$ is a Poisson r.v. with parameter $t\mu(A)$. Indeed, $\nu(t, A)$

Received September 25, 1978; revised October 1979.

AMS 1970 *subject classifications.* Primary 60J30; secondary 60H05, 60H20.

Key words and phrases. Homogeneous process, independent increments, Wiener process, Poisson random measure, stochastic integral equation.

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is the number of jumps of $X(s)$, $0 \leq s < t$, which fall in A. Let us denote $\nu^*(t, dx) = \nu(t, dx) - t\mu(dx)$ and assume, for convenience, that $EX(t) \equiv 0$. Then $X(t)$ has a representation

$$X(t) = W(t) + \int_{R'} x\nu^*(t, dx)$$

where $W(t)$ is a Wiener process with some parameter σ^2. The characteristic function of $X(t)$ is

$$Ee^{i\theta X(t)} = \exp\{-\tfrac{1}{2}t\sigma^2\theta^2 + t\int_{R'} (e^{i\theta x} - 1 - i\theta x)\mu(dx)\}. \tag{2.1}$$

From (2.1), we obtain that $EX^2(t) = t(\sigma^2 + K)$ where $K = \int_{R'} x^2\mu(dx)$. Since $E|X(t)|^m < \infty$, K is finite.

For each n, let $A_n = \{x: 0 < |x| < n\}$, $A_n^c = R' - A_n$ and let

$$X_n(t) = W(t) + \int_{A_n} x\nu^*(t, dx).$$

Then it is clear that

$$E|X(t) - X_n(t)|^2 = t\int_{A_n^c} x^2\mu(dx). \tag{2.2}$$

Since K is finite and A_n^c decreases to ϕ as $n \to \infty$, the right-hand side of (2.2) converges to 0 as $n \to \infty$. Therefore,

LEMMA 2.1. *$X_n(t)$ converges to $X(t)$ in $L^2(\Omega)$ uniformly in t-compact sets as $n \to \infty$.*

Let n be fixed and $|u| < 1/n$. Then $uX_n(t)$ is a semimartingale with jumps bounded by $|u|n < 1$. The integral equation

$$Z(t) = 1 + u\int_0^t Z(s-)\, dX_n(s) \tag{2.3}$$

has a unique solution. According to a theorem of Doléans-Dade (see [7], Theorem 25, page 304 or [8], Theorem 2-1, page 453), this solution is, when expressed in terms of $\nu(t, dx)$, $\mu(dx)$,

$$\begin{aligned} Z_n(t) &= \exp\left\{uX_n(t) - \tfrac{1}{2}t\sigma^2u^2 + \int_{A_n} \log(1 + ux)\nu(t, dx) - u\int_{A_n} x\nu(t, dx)\right\} \\ &= \exp\left\{uW(t) - \tfrac{1}{2}t\sigma^2u^2 + \int_{A_n} \log(1 + ux)\nu^*(t, dx) - t\int_{A_n} [ux - \log(1 + ux)]\mu(dx)\right\}. \end{aligned} \tag{2.4}$$

Note that $Z_n(t)$ is $L^2(\Omega)$-continuous in t. Hence the integral of $Z_n(s-)$ w.r.t. $X_n(t)$ which appeared in (2.3) will also exist in the $L^2(\Omega)$-sense.

Let $I_n^0(t) \equiv 1$ and define $I_n^k(t) = \int_0^t I_n^{k-1}(s-)\, dX_n(s)$, $k \geq 1$. There is no problem with the existence of $I_n^k(t)$, $k \geq 1$, since it is not hard to see, inductively, that $I_n^{k-1}(t)$, $k \geq 1$, is $L^2(\Omega)$-continuous in t. By using (1.2), we can derive

$$E\{I_n^k(t)I_n^j(t)\} = (K_n + \sigma^2)\int_0^t E\{I_n^{k-1}(s)I_n^{j-1}(s)\}\, ds$$

for $k, j \geq 1$, where $K_n = \int_{A_n} x^2\mu(dx)$. Repeating the above procedure, we have

$$\begin{aligned} E\{I_n^k(t)I_n^j(t)\} &= 0, & k \neq j, \\ &= t^k(K_n + \sigma^2)^k/k!, & k = j. \end{aligned} \tag{2.5}$$

Therefore, $Y_n(t) = \sum_{k=0}^{\infty} u^k I_n^k(t)$ converges in $L^2(\Omega)$. It is also trivial that $Y_n(t)$ satisfies equation (2.3) in $L^2(\Omega)$-sense. Hence, $Y_n(t) = Z_n(t)$ for $|u| < 1/n$. From this equality and (2.4), we obtain

THEOREM 2.2. *For each $k \geqslant 0$,*

$$I_n^k(t) = \left\{\frac{\partial^k}{\partial u^k} \exp\left\{uW(t) - \frac{t\sigma^2 u^2}{2} + \int_{A_n} \log(1+ux)\nu^*(t, dx) - t\int_{A_n} [ux - \log(1+ux)]\mu\,(dx)\right\}\right\}_{u=0} \Big/ k!. \tag{2.6}$$

Now let $I^0(t) \equiv 1$ and $I^k(t) = \int_0^t I^{k-1}(s-)\, dX(s)$, $k \geqslant 1$. The reason for $I^k(t)$, $k \geqslant 1$, to exist is similar to that for $I_n^k(t)$, $k \geqslant 1$.

LEMMA 2.3. *For each $k \geqslant 1$, $I_n^k(t)$ converges to $I^k(t)$ in $L^2(\Omega)$ uniformly in t-compact sets as $n \to \infty$.*

PROOF. We shall prove it by induction. Lemma 2.3 is trivial for $k = 0, 1$. Suppose that it is true for $k = j$. By using the property of independent increments, (2.5) and (2.2), we can derive

$$E\{I^{j+1}(t) - I_n^{j+1}(t)\}^2 = E\left\{\int_0^t I^j(s-)\, dX(s) - \int_0^t I_n^j(s-)\, dX_n(s)\right\}^2$$

$$\leqslant 2E\left\{\int_0^t [I^j(s-) - I_n^j(s-)]\, dX(s)\right\}^2 + 2E\left\{\int_0^t I_n^j(s-) d[X(s) - X_n(s)]\right\}^2$$

$$\leqslant 2(\sigma^2 + K) \int_0^t E\{I^j(s) - I_n^j(s)\}^2\, ds + 2\int_0^t E\{I_n^j(s)\}^2\, ds \int_{A_n^c} x^2\mu(dx)$$

$$\leqslant 2(\sigma^2 + K) \int_0^t E\{I^j(s) - I_n^j(s)\}^2\, ds + 2t^{j+1}\,(K_n + \sigma^2)^j \int_{A_n^c} x^2\mu(dx)/(j+1)!.$$

Consider the right-hand side of the last inequality as $n \to \infty$. The first term tends to 0 by the induction hypothesis. The second term converges to 0 since A_n^c decreases to ϕ. Clearly, these two convergences are uniform in t-compact sets. Therefore, Lemma 2.3 holds also for $k = j + 1$. This proves Lemma 4.3.

For each n, let

$$p_{n,j} = \int_{A_n} x^j \nu^*(t, dx), \quad j \geqslant 1, \qquad q_{n,j} = \int_{A_n} x^j \mu(dx), \quad j \geqslant 2.$$

From (2.6), it is clear that I_n^k is a polynomial of $W(t), p_{n,j}$, $1 \leqslant j \leqslant k$, $q_{n,j}$, $2 \leqslant j \leqslant k$. Let f_k denote this polynomial, that is,

$$I_n^k = f_k(W(t), p_{n,1}, \cdots, p_{n,k}, q_{n,2}, \cdots, q_{n,k}). \tag{2.7}$$

Identity (2.1) and the assumption that $E|X^m(t)| < \infty$ imply the existence of $\int_{R'} |x|^m \mu\,(dx)$ which in turn implies the convergence of

$$\lim_{n\to\infty} \int_{A_n} x^k \mu\,(dx) = \int_{R'} x^k \mu\,(dx), \qquad 2 \leqslant k \leqslant m,$$

$$\lim_{n\to\infty} \int_{A_n} x^k \nu^*(t, dx) = \int_{R'} x^k \nu^*(t, dx), \qquad 1 \leqslant k \leqslant m.$$

The last convergence is in the $L^1(\Omega)$-sense and is uniform in t-compact set. Therefore, $f_k(W(t), p_{n,1}, \cdots, p_{n,k}, q_{n,2}, \cdots, q_{n,k})$ converges at least in probability to $f_k(W(t), p_1, \cdots, p_k, q_2, \cdots, q_k)$ for each $k \leq m$, where

$$p_k = \int_{R'} x^k \nu^*(t, dx), \quad 1 \leq k \leq m, \qquad q_k = \int_{R'} x^k \mu\,(dx), \quad 2 \leq k \leq m. \tag{2.8}$$

This fact, together with Lemma 2.3, implies that both sides of (2.7) converge. Although they might converge in different senses, their limits must be equal almost everywhere. Therefore, we obtain

THEOREM 2.4. *The k-multiple integrals of $X(t)$, $1 \leq k \leq m$, are*

$$I^k(t) = f_k(W(t), p_1, \cdots, p_k, q_2, \cdots, q_k)$$

where p_j's, q_j's are defined in (2.8). *Or, formally,* $I^k(t) = \{(\partial^k/\partial u^k)\exp\{uW(t) - t\sigma^2 u^2/2 + \int_{R'} \log(1 + ux)\nu^*(t, dx) - t \int_{R'} [ux - \log(1 + ux)]\mu(dx)\}\}_{u=0}/k!$.

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