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L_∞-BOUND FOR ASYMPTOTIC NORMALITY OF WEAKLY DEPENDENT SUMMANDS USING STEIN'S RESULT

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Let $\{X_n\}$ be a strictly stationary process satisfying some mixing conditions, including ϕ-mixing condition. It is the aim of the present paper to give, using a slight modification of Stein's result, a rate $O(n^{-1/2} \log n)$ of the normal approximation for a sum $S_n = X_1 + \cdots + X_n$.

1. Introduction. Let $\{X_n, -\infty < n < \infty\}$ be a strictly stationary process with $EX_0 = 0$ and $EX_0^2 < \infty$ and let $\rho(k)$ $k = 1, 2, \cdots$ be a sequence of nonnegative numbers such that, if A and B are any two finite sets of natural numbers for which

$$inf_{i \in A, j \in B} |i - j| \geqq k \tag{1.1}$$

and Y and Z are random variables with finite variance depending only on the $\{X_i\}_{i \in A}$ and $\{X_j\}_{j \in B}$ respectively, then

$$|\mathrm{Corr}(Y, Z)| \leqq \rho(k). \tag{1.2}$$

In [6], C. Stein studied a method to prove the rate of the normal approximation to the distribution of a sum $S_n = X_1 + X_2 + \cdots + X_n$. He applied his results to the m-dependent case which implied $\rho(k) = 0$ for all $k \geqq m + 1$, and obtained the rate $O(n^{-1/2})$. In another case, he obtained the rate $O(n^{-1/2} \log^2 n)$ assuming that $\rho(k)$ decreases to zero exponentially and that EX_0^8 is finite. Up to now, however, the relation (1.2) does not seem to have been proved under some usual mixing conditions, for example, *-mixing, ϕ-mixing [4] [1], absolute regularity [7] [8] and strong mixing [5]. The purpose of this paper is to prove exactly the rate $O(n^{-1/2}$ log $n)$ using a slight modification of Stein's result in [6]. Our central job is to give inequalities (Corollary 3.1 and 3.2) corresponding to (1.2) under the mixing conditions above. Regrettably we could not obtain any result for strong mixing sequences.

2. Preliminaries. Let $\{X_n\}$ be a strictly stationary process. Denote by $\mathscr{M}_a^b$ the sigma field of events generated by $X_a, X_{a+1}, \cdots, X_b$. We shall consider the following four mixing conditions

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(A) *-mixing condition [4]

$$\psi(n) = \sup_{A \in \mathscr{M}_{-\infty}^0, B \in \mathscr{M}_n^\infty} |P(AB) - P(A)P(B)|/P(A)P(B) \downarrow 0 \qquad (n \to \infty)$$

(B) ϕ-mixing condition [4] [1]

$$\phi(n) = \sup_{A \in \mathscr{M}_{-\infty}^0, B \in \mathscr{M}_n^\infty} |P(AB) - P(A)P(B)|/P(A) \downarrow 0 \qquad (n \to \infty)$$

(C) absolutely regular condition [7] [8]

$$\beta(n) = E \sup_{B \in \mathscr{M}_n^\infty} |P(B \mid M_{-\infty}^0) - P(B)| \downarrow 0 \qquad (n \to \infty)$$

(D) strong mixing condition [2] [5]

$$\alpha(n) = \sup_{A \in \mathscr{M}_{-\infty}^0, B \in \mathscr{M}_n^\infty} |P(AB) - P(A)\, P(B)| \downarrow 0 \qquad (n \to \infty).$$

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Then we have $\psi(n) \geqq \phi(n) \geqq \beta(n) \geqq \alpha(n)$. The following lemmas are well known.

LEMMA 2.1[1]. *Suppose that X and Y are $\mathscr{M}_{-\infty}^{0}$- and $\mathscr{M}_{n}^{-\infty}$-measurable, respectively, with $E|X|^p < \infty$, $E|Y|^q < \infty$ $(p^{-1} + q^{-1} = 1, p, q > 0)$. Then*

$$|E(XY) - E(X)E(Y)| \leqq 2\phi^{1/p}(n)(E|X|^p)^{1/p}(E|Y|^q)^{1/q}. \tag{2.1}$$

LEMMA 2.2[2] [5]. *Suppose that X and Y are $\mathscr{M}_{-\infty}^{0}$- and $\mathscr{M}_{n}^{-\infty}$-measurable, respectively, with $E|X|^p < \infty$, $E|Y|^q < \infty$ $(p^{-1} + q^{-1} < 1, p, q > 0)$. Then*

$$|E(XY) - E(X)E(Y)| \leqq 10\,\alpha^{1/s}(n)(E|X|^p)^{1/p}(E|X|^q)^{1/q} \tag{2.2}$$

where $s^{-1} = 1 - (p^{-1} + q^{-1})$. In above, $\alpha(n)$ can be replaced by $\beta(n)$.

3. Some properties of mixing sequences. Let $\{X_n\}$ be as defined in Section 2. For brevity, for $m \geqq n > 0$, introduce the notations

$$\mathscr{F}_1 = \mathscr{M}_{-\infty}^{0},\ \mathscr{F}_2 = \mathscr{M}_{n}^{m},\ \mathscr{F}_3 = \mathscr{M}_{m+n}^{\infty}$$

and

$$\mathscr{B} = \{\cup_{i=1}^{k} (B_i \cap C_i);\ B_i \in \mathscr{F}_1,\ C_i \in \mathscr{F}_3\ 1 \leqq i \leqq k,\ k \text{ any positive integer}\}.$$

Let $\mathscr{F}_4 = \sigma(\mathscr{F}_1 \cup \mathscr{F}_3)$ be the sigma field generated by $\mathscr{F}_1$ and $\mathscr{F}_3$. Then it is easy to show $\sigma(\mathscr{B}) = \mathscr{F}_4$. In this section, we shall prove the following theorems, from which we have two inequalities corresponding to (1.2) (see Corollary 3.1 and 3.2).

THEOREM 3.1. *For any $A \in \mathscr{F}_2$ and $D \in \mathscr{F}_4$,*

$$|P(AD) - P(A)P(D)| \leqq 3\beta(n). \tag{3.1}$$

THEOREM 3.2. *For any $A \in \mathscr{F}_2$ and $D \in \mathscr{F}_4$,*

$$|P(AD) - P(A)P(D)| \leqq 3P(A)\psi(n). \tag{3.2}$$

LEMMA 3.1. *$\mathscr{B}$ is a field.*

PROOF. An arbitrary finite union of elements in $\mathscr{B}$ belongs to $\mathscr{B}$. Hence it is sufficient to prove that, for any $D \in \mathscr{B}$, D^c belongs to $\mathscr{B}$. Now write $D = \cup_{i=1}^{k}(B_i \cap C_i)$, $B_i \in \mathscr{F}_1$, $C_i \in \mathscr{F}_3$ $1 \leqq i \leqq k$. Then $D^c = \cap_{i=1}^{k}(B_i^c \cup C_i^c) = \cap_{i=1}^{k}(B_i^c \cup (B_i \cap C_i^c))$. This equality is equivalent to the indicator form equality $1_{D^c} = \prod_{i=1}^{k} (1_{B_i^c} + 1_{(B_i \cap C_i^c)})$. Expanding this, we have 2^k terms. And rewrite this as $\sum_{i=1}^{2^k} 1_{D_i}$. Then, since it is impossible that $1_{D_i} = 1$ and $1_{D_j} = 1$ for $i \neq j$ simultaneously, we have $D_i \cap D_j = \phi$ for $i \neq j$. And it is easy to see that we can write $D_i = F_i \cap G_i$, $F_i \in \mathscr{F}_1$, $G_i \in \mathscr{F}_3$ $1 \leqq i \leqq 2^k$ and that $F_i \cap F_j = \phi$ for $i \neq j$. Thus $D^c = \cup_{i=1}^{2^k} (F_i \cap G_i)$, $F_i \in \mathscr{F}_1$, $G_i \in \mathscr{F}_3$ $1 \leqq i \leqq 2^k$, i.e., $D^c \in \mathscr{B}$. Hence $\mathscr{B}$ is a field.

PROOF OF THEOREM 3.1. We consider the following set of events

$$\mathscr{A} = \{D \in \mathscr{F}_4;\ |P(AD) - P(A)P(D)| \leqq 3\beta(n) \text{ for any } A \in \mathscr{F}_2\}.$$

As the first step of proof, we shall show $\mathscr{B} \subset \mathscr{A}$. For this, firstly we remark that if $D \in \mathscr{A}$, then $D^c \in \mathscr{A}$, because, for any $A \in \mathscr{F}_2$, $|P(AD^c) - P(A)P(D^c)| = |P(A) - P(AD) - P(A)(1 - P(D))| = |P(AD) - P(A)P(D)| \leqq 3\beta(n)$.

Now we prove $\mathscr{B} \subset \mathscr{A}$. For $D \in \mathscr{B}$, set $D = \cup_{i=1}^{k} (B_i \cap C_i)$. In order to show $D \in \mathscr{A}$, from the above remark, it is sufficient to show $D^c \in \mathscr{A}$. From the proof of Lemma 3.1, it is possible to write

$$D^c = \cup_{i=1}^{2^k} (F_i \cap G_i),\ F_i \in \mathscr{F}_1,\ G_i \in \mathscr{F}_3,\ 1 \leqq i \leqq 2^k,\ F_i \cap F_j = \phi\ (i \neq j).$$

Below, summations are always taken from 1 to 2^k. Let A be in $\mathscr{F}_2$.

$$|P(A \cap D^c) - P(A)P(D^c)| = |\sum P(A \cap F_i \cap G_i) - P(A) \sum P(F_i \cap G_i)|$$

$$\leqq |\sum P(A \cap F_i \cap G_i) - \sum P(F_i)P(A \cap G_i)|$$

$$+ \sum P(F_i)|P(A \cap G_i) - P(A)P(G_i)|$$

$$+ P(A) \sum |P(F_i \cap G_i) - P(F_i)P(G_i)| = I_1 + I_2 + I_3, \qquad \text{(say)}.$$

$$I_1 = |\sum E\{1_{F_i}(P(A \cap G_i | F_1) - P(A \cap G_i))\}|$$

$$\leqq \sum E\{1_{F_i} \sup_{H \in \mathscr{M}_n^\infty} |P(H|\mathscr{F}_1) - P(H)|\} \leqq E \sup_{H \in \mathscr{M}_n^\infty} |P(H|\mathscr{F}_1) - P(H)|$$

$$\leqq \beta(n)$$

since F_i are disjoint.

$$I_2 = \sum P(F_i)|E\{1_A(P(G_i|\mathscr{M}_{-\infty}^m) - P(G_i))\}| \leqq \beta(n) \sum P(F_i) \leqq \beta(n)$$

and, by the argument used to bound I_1, we have

$$I_3 \leqq \beta(m+n) \leqq \beta(n)$$

Therefore we have $D^c \in \mathscr{A}$, hence $\mathscr{B} \subset \mathscr{A}$.

As the second step of the proof, we must show $\mathscr{A} = \mathscr{F}_4$. It is trivial that $\mathscr{A}$ forms a monotone class (for the definition, see [3]) and $\mathscr{A}$ contains $\mathscr{B}$. Hence the monotone class generated by $\mathscr{B}$ is contained in $\mathscr{A}$. On the other hand, $\mathscr{B}$ generates $\mathscr{F}_4$. Therefore, by Proposition I.4.2[3], $\mathscr{F}_4 \subset \mathscr{A}$. By the definition, $\mathscr{A} \supset \mathscr{F}_4$. Thus we have $\mathscr{A} = \mathscr{F}_4$, i.e., for any $A \in \mathscr{F}_2$ and $D \in \mathscr{F}_4$, $|P(AD) - P(A)P(D)| \leqq 3\beta(n)$.

Proof of Theorem 3.2. In the notation of the proof of Theorem 3.1, clearly

$$I_1 \leqq \sum P(AF_i)P(G_i)\psi(n) \leqq P(A)\psi(n).$$

Similarly for I_2 and I_3, we have $I_2 \leqq P(A)\psi(n)$ and $I_3 \leqq P(A)\psi(n)$.

From this, by the analogous way to the proof above, we can prove Theorem 3.2.

From Theorem 3.1 and 3.2, we have

Corollary 3.1. *Suppose that X and Y are $\mathscr{F}_2$- and $\mathscr{F}_4$-measurable, respectively, with $E|X|^p < \infty$, $E|Y|^q < \infty$ $(p^{-1} + q^{-1} < 1, p, q > 0)$. Then*

$$|E(XY) - E(x)E(Y)| \leqq 30\, \beta^{1/s}(n)(E|X|^p)^{1/p}(E|Y|^q)^{1/q} \tag{3.3}$$

where $s^{-1} = 1 - (p^{-1} + q^{-1})$. In above, $\beta(n)$ can be replaced by $\phi(n)$.

Corollary 3.2. *Suppose that X and Y are $\mathscr{F}_2$- and $\mathscr{F}_4$-measurable, respectively, with $E|X|^p < \infty$, $E|Y|^q < \infty$ $(p^{-1} + q^{-1} = 1, p, q > 0)$. Then*

$$|E(XY) - E(X)\, E(Y)| \leqq 6\psi^{1/p}(n)(E|X|^p)^{1/p}(E|Y|^q)^{1/q}. \tag{3.4}$$

Remark. In the inequality (3.4), if we put $p = q = 2$, then we have a special form of (1.2).

4. The summands of some mixing stationary sequence. Throughout this section, we borrow the notations from [6], and so the details should be referred to [6]. Let $\{X_n\}$ be a strictly stationary process which satisfies the following conditions (as in [6]).

$$E(X_1) = 0,\ E(X_1^8) < \infty \tag{4.1}$$

and

$$0 < \sigma^2 = \lim_{n\to\infty} \frac{1}{n} E(S_n^2) \tag{4.2}$$

where $S_n = \sum_1^n X_\iota$.

In this section, we shall prove the following theorems using a slight modification of Stein's result [6].

THEOREM 4.1. *Suppose that* $\phi(n) = O(e^{-\lambda n})$ *for some* $\lambda > 0$. *Then*

$$\Delta_n = \sup_x | P(S_n \leqq x\sqrt{n}\,\sigma) - \Phi(x) | = O(n^{-1/2} \log n) \tag{4.3}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$

THEOREM 4.2. *Suppose that* $\beta(n) = O(e^{-\lambda n})$ *for some* $\lambda > 0$ *and that* $E|X_1|^{8+\delta} < \infty$ *for some* $\delta > 0$. *Then the conclusion of the theorem above remains valid* [cf. Yoshihara [8]].

In what follows, we shall prove only Theorem 4.2 and omit the proof of the other since it can be proved by the analogous argument to the one of Theorem 4.2.

As in [6] we consider only $X_1, X_2, \cdots, X_n$ and define I as a random variable uniformly distributed on $J = \{1, 2, \cdots, n\}$ which is independent of $X_1, X_2, \cdots, X_n$. Remark that if $i = 5$ and $m = 8$, then $\sum_{|j-i|>m} X_j$ denotes the sum $\sum_{14}^n X_j$, hence that $\sum_{|j-\iota|>m} X_j$ $i = 1, 2, \cdots, n$ are not identically distributed in general. Let $\mathscr{F}$ be the sigma field generated by X_1, $X_2, \cdots, X_n$ and $\mathscr{C}$ the sigma field generated by all events of the form $\{I = i$ and for all j such that $|j - i| > m, X_j \leqq a_j\}$ where the a_j are real numbers. Let $G = n/(\sqrt{n}\sigma)\, X_I$, $W = E^{\mathscr{F}}G = 1/(\sqrt{n}\alpha) \sum_1^n X_i$ and $W^* = 1/(\sqrt{n}\sigma) \sum_{|j-I|>m} X_j$. Then W^* is $\mathscr{C}$-measurable and $W - W^* = 1/(\sqrt{n}\sigma) \sum_{|j-I|\leqq m} X_j$. Here we remark that in [6] Stein asserts that $EG(W - W^*)$ equals to one using $\sqrt{\delta_n}$ [6] in place of $\sqrt{n}\,\sigma$, but it is, in general, false. The fundamental equality (2.13) in [6] can be rewritten as follows

$$\begin{aligned} Eh(W) = Nh + E\Big\{\Big(h(W) - E^{\mathscr{F}}\Big[G \int_{W^*}^{W} h(z)\, dz\Big]\Big) + \Big(Wf(W) \\ - E^{\mathscr{F}}\Big[G \int_{W^*}^{W} zf(z)\, dz\Big]\Big) - (E^{\mathscr{C}}G)f(W^*)\Big\} - Nh(1 - E\{G(W - W^*)\}) \end{aligned} \tag{4.4}$$

where h is a bounded measurable function,

$$Nh = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x) e^{-x^2/2}\, dx$$

and

$$f(w) = e^{w^2/2} \int_{-\infty}^{w} [h(x) - Nh] e^{-x^2/2}\, dx.$$

From (4.4) we have as well as [6]

$$\begin{aligned} EE^{\mathscr{F}}\Big[G \int_{W^*}^{W} h(z)\, dz\Big] = Nh + E\Big\{Wf(W) - E^{\mathscr{F}}\Big[G \int_{W^*}^{W} zf(z)\, dz\Big] \\ - (E^{\mathscr{C}}G)f(W^*)\Big\} - Nh(1 - E\{G(W - W^*)\}) \end{aligned} \tag{4.5}$$

(cf. Lemma 2.1 in [6]).

Remark that, by the Hölder inequality, we may use as λ of (2.86) in [6] $\max\{12[E|G(W - W^*)^2|^{8/3}]^{3/8}, 5[E|G(W - W^*)^3|^{2+\alpha}]^{1/(4+2\alpha)}\}$ instead of $\max\{12E|G(W - W^*)^2|, 5[E|G(W - W^*)^3|]^{1/2}\}$. By this remark, if, keeping (4.5) in mind, readers review the paper

[6] carefully, they shall find that if $E\{G(W-W^*)\}$ is sufficiently close to one, then the following estimate is valid.

REMARK. If n and m are sufficiently large, then $E\{G(W-W^*)\}$ is very close to one. See (E_{10}) below.

$$|P(W \leqq a) - \Phi(a)| \leqq R \tag{4.6}$$

for all a, where

$$\begin{aligned} R = {} & 6\{\operatorname{Var} E^{\mathscr{F}}[G(W-W^*)]\}^{1/2} + 3E|G(W-W^*)^3| \\ & + 3\left\{E\left(|W| + \frac{1}{2}\right)^2 E[E^{\mathscr{F}}|G(W-W^*)^2|]^2\right\}^{1/2} \\ & + 15(E[G(W-W^*)]^4)^{1/4}\left\{\frac{\operatorname{Var} E^{\mathscr{F}}|G(W-W^*)^2|}{(E|G(W-W^*)^2|^{8/3})^{3/4}} \right. \\ & \left. + \frac{\operatorname{Var} E^{\mathscr{F}}|G(W-W^*)^3|}{(E|G(W-W^*)^3|^{2+\alpha})^{2/(2+\alpha)}}\right\}^{3/4} \\ & + \max\{54[E|G(W-W^*)^2|^{8/3}]^{3/8}, 23[E|G(W-W^*)^3|^{2+\alpha}]^{1/(4+2\alpha)}\} \\ & + 3|1 - E\{G(W-W^*)\}| + 3E|E^{\mathscr{G}}G| = I_1 + \cdots + I_7, \qquad \text{(say)}. \end{aligned} \tag{4.7}$$

Here α is a positive number such that $0 < (2+\alpha)/(2-3\alpha) \leqq 1 + \delta/8$ and $\alpha \leqq \delta/4$.

Now to prove the theorem, we must estimate each term in (4.7). As a main tool, we shall use the following lemma repeatedly.

LEMMA 4.1. *For $1 \leqq p \leqq 8$, there exists a constant C such that*

$$E|\textstyle\sum_{i=1}^n X_i|^p \leqq Cn^{p/2} \tag{4.8}$$

for all n.

This is an immediate corollary to Theorem 3 in [9].

Now we list the inequalities which we make use of in estimating I_i's. In what follows, we shall agree that the large letter K denotes some absolute positive constant, not necessarily identical at different occurrences.

(E_1) $\operatorname{Var} E^{\mathscr{F}}[G(W-W^*)] \leqq Kn^{-1}(2m+1)^2$.
(E_2) $E|G(W-W^*)^3| \leqq Kn^{-1}(2m+1)^{3/2}$.
(E_3) $E[E^{\mathscr{F}}|G(W-W^*)^2|]^2 \leqq Kn^{-1}(2m+1)^2$.
(E_4) $E(|W|+1/2)^2 \leqq K$.
(E_5) $(E[G(W-W^*)]^4)^{1/4} \leqq K(2m+1)^{1/2}$.
(E_6) $\operatorname{Var} E^{\mathscr{F}}|G(W-W^*)^2| \leqq Kn^{-1}[(2m+1) + \sum_1^\infty \beta^{1/4}(i)][E|G(W-W^*)^2|^{8/3}]^{3/4}$.
(E_7) $\operatorname{Var} E^{\mathscr{F}}|G(W-W^*)^3| \leqq Kn^{-1}[(2m+1)$
$+ \sum_1^\infty \beta^{1/4}(i)][E|G(W-W^*)^3|^{2+\alpha}]^{2/(2+\alpha)}$.
(E_8) $E|G(W-W^*)^2|^{8/3} \leqq Kn^{-4/3}(2m+1)^{8/3}$.
(E_9) $E|G(W-W^*)^3|^{2+\alpha} \leqq Kn^{-(2+\alpha)}(2m+1)^{(6+3\alpha)/2}$.
(E_{10}) $|1 - E\{G(W-W^*)\}| \leqq K(n^{-1}[(n-2m)\sum_{j>m} \beta^{\alpha/(2+\alpha)}(j) + m])$.
(E_{11}) $E|E^{\mathscr{G}}G| \leqq K\beta^{1/4}(m)n^{1/2}$.

PROOF of (E_1).

$$\begin{aligned} \operatorname{Var} E^{\mathscr{F}}[G(W-W^*)] & \leqq Kn^{-2} \operatorname{Var}[\textstyle\sum_1^n X_i(\sum_{|j-i|\leqq m} X_j)] \\ & \leqq Kn^{-2} \textstyle\sum_{i,i'} |\operatorname{Cov}(X_i \sum_{|j-i|\leqq m} X_j, X_{i'} \sum_{|j'-i'|\leqq m} X_{j'})| \\ & \leqq Kn^{-2} \{\textstyle\sum_{|i-i'|>2m} \beta^{\alpha/(2+\alpha)}(|i-i'|-2m) + \sum_{|i-i'|\leqq 2m}\} A_i A_{i'} \end{aligned}$$

where $A_i = \{E|X_i \sum_{|j-i|\leqq m} X_j|^{2+\alpha}\}^{1/(2+\alpha)}$. Here we used Lemma 2.2 and the Hölder inequality. On the other hand, by the Schwarz inequality and Lemma 4.1, we have $A_i \leqq K(2m+1)^{1/2}$ for all i. Hence we have

$$\operatorname{Var} E^{\mathscr{F}}[G(W-W^*)] \leqq Kn^{-2}\{n(2m+1)^2 \textstyle\sum_i \beta^{\alpha/(2+\alpha)}(i) + n(2m+1)^2\}$$
$$\leqq Kn^{-1}(2m+1)^2.$$

PROOF OF $(E_2)-(E_5)$, (E_8) AND (E_9). By Lemma 4.1, we have

$$E(W-W^*)^8 = \frac{1}{n^4\sigma^8} E(\textstyle\sum_{|j-I|\leqq m} X_j)^8 \leqq Kn^{-4}(2m+1)^4.$$

From this, instead of (3.20) [6], we have, for $0 \leqq k, l, k+l \leqq 8$,

$$E\{|G|^k|W-W^*|^l\} \leqq K(2m+1)^{l/2} n^{(k-l)/2}. \tag{4.9}$$

As special ones of (4.9), we have the following inequalities.

$$\begin{aligned} &E|G(W-W^*)^3| \leqq Kn^{-1}(2m+1)^{3/2},\\ &E[E^{\mathscr{F}}|G(W-W^*)^2|]^2 \leqq Kn^{-1}(2m+1)^2\\ &E|G(W-W^*)^2| \leqq Kn^{-1/2}(2m+1),\\ &E|G(W-W^*)^2|^{8/3} \leqq Kn^{-4/3}(2m+1)^{8/3} \end{aligned} \tag{4.10}$$

and

$$E[G(W-W^*)]^4 \leqq K(2m+1)^2.$$

Thus we proved (E_2), (E_3), (E_5) and (E_8). (E_4) is, by Lemma 4.1, trivial. We show (E_9). The choice of α and the Hölder inequality permit us to have

$$\begin{aligned} E|G(W-W^*)^3|^{2+\alpha} &\leqq (E(|G|^{8+\delta}))^{(2+\alpha)/(8+\delta)}(E|W-W^*|^8)^{(6+3\alpha)/8}\\ &\leqq K(n^{(8+\delta)/2})^{(2+\alpha)/(8+\delta)}(n^{-4}(2m+1)^4)^{(6+3\alpha)/8}\\ &\leqq Kn^{-(2+\alpha)}(2m+1)^{(6+3\alpha)/2} \end{aligned}$$

proving (E_9).

PROOF OF (E_6) AND (E_7). For brevity, we shall denote $X_i\,(\sum_{|j-i|\leqq m} X_j)^2$ by Z_i.

$$\begin{aligned} \operatorname{Var} E^{\mathscr{F}}|G(W-W^*)^2| &= \frac{1}{n^3\sigma^6}\sum|\operatorname{Cov}(|Z_i|,|Z_j|)|\\ &\leqq Kn^{-3}\,(\textstyle\sum_{|i-j|\geqq 2m} \beta^{1/4}(|i-j|-2m) + \sum_{|i-j|<2m})\\ &\quad (E|Z_i|^{8/3})^{3/8}(E|Z_j|^{8/3})^{3/8}\\ &\leqq Kn^{-3}\,(\textstyle\sum_{|i-j|\geqq 2m} \beta^{1/4}(|i-j|-2m) + \sum_{|i-j|<2m})\\ &\quad [(E|Z_i|^{8/3})^{3/4} + (E|Z_j|^{8/3})^{3/4}]\\ &\leqq Kn^{-2}[(2m+1) + \textstyle\sum_1^\infty \beta^{1/4}(i)] \quad (E|X_I\,(\sum_{|j-I|\leqq m} X_j)^2|^{8/3})^{3/4}\\ &\leqq Kn^{-1}[(2m+1) + \textstyle\sum_1^\infty \beta^{1/4}(i)][E|G(W-W^*)s2|^{8/3}]^{3/4}. \end{aligned}$$

Similarly we have

$$\operatorname{Var} E^{\mathscr{F}}|G(W-W^*)^3|$$
$$\leqq Kn^{-1}[(2m+1) + \textstyle\sum_1^\infty \beta^{1/4}(i)][E|G(W-W^*)^3|^{2+\alpha}]^{2/(2+\alpha)}.$$

PROOF OF (E_{10}).

$$|1 - EG(W - W^*)| = \frac{1}{n\sigma^2} |nEX_0^2 + 2n \textstyle\sum_1^\infty E(X_0X_j) - \sum_{|j-i|\leqq m} E(X_iX_j)|$$

$$= \frac{2}{n\sigma^2} |(n - 2m) \textstyle\sum_1^\infty E(X_0X_j) + 2m \sum_1^\infty E(X_0X_j)$$

$$- \sum_{\substack{m\leqq i \leqq n-m \\ 0<|j-i|\leqq m}} E(X_iX_j) - \sum_{\substack{i<m \\ i>n-m \\ 0<|j-i|\leqq m}} E(X_iX_j)|$$

$$\leqq Kn^{-1}[(n - 2m) \textstyle\sum_{j>m} \beta^{\alpha/(2+\alpha)}(j)(E|X_0|^{2+\alpha})^{2/(2+\alpha)} + 2Km]$$

by Lemma 2.2

PROOF OF (E_{11}). Under the condition $\{I = i\}$, the random variable $E^{\mathscr{C}}G$ is measurable with respect to the sigma field generated by $\{X_j;\ |j - i| > m\}$. Thus, by Corollary 3.1, we have

$$\begin{aligned} E|E^{\mathscr{C}}G|^2 &= E\{GE^{\mathscr{C}}G\} = E\{E^I[G(E^{\mathscr{C}}G)]\} \\ &\leqq 30\beta^{1/4}(m)E[\{E^IG^4\}^{1/4}\{E^I(E^{\mathscr{C}}G)^2\}^{1/2}] \\ &\leqq 30\beta^{1/4}(m)\{E(E^IG^4)\}^{1/4}\{E[E^I(E^{\mathscr{C}}G)^2]\}^{1/2} \\ &= 30\beta^{1/4}(m)\{EG^4\}^{1/4}\{E(E^{\mathscr{C}}G)^2\}^{1/2}. \end{aligned}$$

Therefore, as well as [6], we have

$$E|E^{\mathscr{C}}G| \leqq K\beta^{1/4}(m)n^{1/2}.$$

REMARK. The results in Section 3 are used only to prove (E_{11}).

Now we shall examine each term I_i of (4.7) making use of (E_1), (E_2), ..., (E_{11}). At first, setting $m = [c \log n]c > 0$, we can choose $c > 0$ satisfying the conditions

(4.11) $$\textstyle\sum_{j>m} \beta^{\alpha/(2+\alpha)}(j) = O(n^{-1}) \quad \text{and} \quad \beta^{1/4}(m)n^{1/2} = O(n^{-1/2}).$$

By (E_1), $I_1 = O(n^{-1/2}\log n)$. By (E_2), $I_2 = O(n^{-1} \log^{3/2} n)$. By ($E_3$) and ($E_4$), $I_3 = O(n^{-1/2} \log n)$. By (E_5), (E_6) and (E_7), $I_4 = O(n^{-3/4} \log^{5/4} n)$. By ($E_8$) and ($E_9$), $I_5 = O(n^{-1/2} \log n)$. By (E_{10}) and (4.11), $I_6 = O(n^{-1} \log n)$. By (E_{11}) and (4.11), $I_7 = O(n^{-1/2})$. Thus we proved $\Delta_n = O(n^{-1/2} \log n)$.

Added in proof. The author has been informed by a referee that A.N. Tihomirov obtained a better result than Theorem 4.1, in his paper entitled "On the speed of convergence in the central limit theorem for weakly dependent random variables" (to appear in *Teorya. Veroyat. i ee Primen.*).

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