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# OPTIONAL SAMPLING OF SUBMARTINGALES INDEXED BY PARTIALLY ORDERED SETS

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The optional sampling theorem for martingales indexed by a partially ordered set is true if the index set is directed. However, the corresponding result for submartingales indexed by a partially ordered set is not true in general. In this paper we completely characterize the class of stopping times for which the optional sampling theorem is true for all uniformly bounded submartingales indexed by countable partially ordered sets. By assuming a conditional independence property, we show that when the index set is $R^2$ the optional sampling theorem is true for all uniformly bounded, right continuous submartingales and all stopping times. This conditional independence property is satisfied in cases where the submartingales and stopping times are measurable with respect to the two-parameter Wiener process. A counterexample shows that the optional sampling result is false for $R^n$ when $n > 2$ even if the conditional independence property is satisfied.

**1. Introduction.** Bochner (1955) formulated the martingale theory of Doob (1953) for random functions on a directed[1] index set with the intention of clarifying and simplifying several probabilistic concepts in terms of martingale concepts. With this motivation he defined martingales, submartingales and stopping times in the general context of a directed index set and he stated general versions of the martingale convergence theorem and the optional sampling theorem. Since that time several authors have studied these conjectures and found that the general case of directed indices requires additional hypotheses to obtain generalized versions of the results for linearly ordered index sets. On the question of martingale convergence, Krickeberg (1956), Helms (1958) and Chow (1960) have obtained generalized versions of Doob's (1953) results for linearly ordered index sets. See also the monograph of Hayes and Pauc (1970). More recently, Gut (1976) and Gabriel (1977) have studied convergence of martingales indexed by directed sets and applied these results to investigate the law of large numbers for multiparameter stochastic processes.

Using a restricted definition of stopping time, Chow (1960) proved that the optional sampling theorem was true for martingales in the general case of directed index sets. Kurtz (1977) removed Chow's restrictions on the stopping time and extended the results to the case when the index set is a topological lattice. In addition to proving an optional sampling theorem for martingales with a directed index set, Chow (1960) also showed that the analogous result for submartingales was false even for very simple examples. Nevertheless, the optional sampling theorem is true for submartingales with a partially ordered index set if suitable assumptions are made. Haggstrom (1966), extending the work of Snell (1952) on martingale systems theorems and optimal stopping problems, defined submartingales indexed by a *tree*,[2] a special type of partially ordered set which is not directed. In addition, he proved a version of the optional sampling theorem for a special class of stopping times called *control variables*.[2]

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[1] A directed set is a partially ordered set with the additional property that every two elements in the set have a common upper bound.

[2] See Section 2 for these definitions.

In the first part of this paper (Section 2) we consider the optional sampling theorem for submartingales indexed by countable (but otherwise general) partially ordered sets. We define the concept of *reachability* for pairs of stopping times and we reformulate Haggstrom's problem for general partially ordered time sets. The concept of reachable stopping times generalizes Haggstrom's notion of control variable. We show that if a stopping time $\tau$ is reachable from a stopping time $\sigma$, then the optional sampling theorem is true for all submartingales satisfying a uniform bound. Conversely, we show that if the optional sampling theorem is true for the pair $\tau$, $\sigma$ of stopping times and for any uniformly bounded submartingale, then $\tau$ must be reachable from $\sigma$. Thus, we obtain a complete characterization of the case in which the optional sampling theorem is true for general submartingales.

In the second part (Section 3) of this paper we assume that the increasing family of $\sigma$-fields satisfies a special conditional independence property that can be defined when the index set is such that any two elements have a greatest lower bound. This is a straightforward generalization of the conditional independence property that Cairoli and Walsh (1975) define for the index set $R_+^2$.[3] Assuming this conditional independence hypothesis, we can show that if the index set $T$ is a tree with respect to its order relation $\le$ and if $\sigma$, $\tau$ are stopping times with $\sigma \le \tau$, then $\tau$ is always reachable from $\sigma$. Likewise, assuming the conditional independence property, we can show that if $\sigma$ and $\tau$ are stopping times on $Z^2$ and $\sigma \le \tau$, then $\tau$ is reachable from $\sigma$. Consequently, the optional sampling theorem is true for all stopping times and all uniformly bounded submartingales defined on either a tree or on $Z^2$, when the conditional independence property is satisfied. It is a simple matter to extend the optional sampling theorem for the case of the index set $Z^2$ to the case of right continuous martingales defined on $R^2$. A counterexample reveals that the optional sampling theorem is not true for $Z^n$ or $R^n$ when $n > 2$.

## 2. Optional sampling for submartingales indexed by partially ordered sets.

**2.1 Notation, conventions and basic definitions.** We will let $T$ denote the partially ordered index set in this paper and except for one case in Section 3, we will always assume that $T$ is countable. We will use $\le$ to denote both the partial order relation on $T$ and the usual linear order relation on the set $R$ of real numbers, but there should be no confusion as to which case is meant. Let $(\Omega, F, P)$ denote the underlying probability space and $\{F(t): t \in T\}$ a family of sub $\sigma$-fields of $F$ indexed by $T$. We will always assume that $F$ and each $F(t)$ are complete with respect to the probability measure $P$. Following convention, we omit "a.s." from all equalities and inequalities between random functions, although we implicitly assume that these relationships only hold almost surely.

It is straightforward to extend the usual definitions of increasing family, submartingale and stopping time to the case of a partially ordered index set. Nevertheless, we assemble these definitions here for the sake of completeness. The family $\{F(t): t \in T\}$ is said to be *increasing* with respect to $\le$ if $s \le t$ implies that $F(s) \subset F(t)$. A mapping $X: T \times \Omega \to R$ is *adapted* to the family $\{F(t): t \in T\}$ if $\omega \to X(t, \omega)$ is $F(t)$-measurable for each $t$ in $T$. To be concise, let us denote the random variable $\omega \to X(t, \omega)$ by $X(t)$. A mapping $X$ is *uniformly bounded* if there exists a nonnegative random variable $X_+$ with finite expectation $E(X_+)$ such that $|X(t)| \le X_+$ for all $t$. A *submartingale* $X$ with respect to the increasing family $\{F(t): t \in T\}$ is a map $X: T \times \Omega \to R$ such that $X$ is adapted to $\{F(t): t \in T\}$, such that the expectation $E(|X(t)|)$ is finite for each $t$ and such that the conditional expectation satisfies

$$X(s) \le E(X(t) \mid F(s)) \tag{2.1}$$

whenever $s \le t$. Similarly, a *martingale* is a submartingale for which equality holds in (2.1). Note that in this paper we will always assume that submartingales are uniformly bounded.

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[3] The reals and nonnegative reals are denoted by $R$ and $R_+$, respectively, and the integers are denoted by $Z$. The Cartesian products are denoted $R^n$, $R_+^n$ and $Z^n$, respectively.

A *stopping time* $\tau$ with respect to an increasing family $\{F(t):t \in T\}$ is a mapping $\tau:\Omega \to T$ which satisfies the measurability property $\{\tau = t\} \in F(t)$ for all $t$. Corresponding to each stopping time $\tau$ there is a $\sigma$-field denoted by $F(\tau)$ and defined to be the $\sigma$-field of sets $A$ in $F$ such that $A \cap \{\tau = t\}$ lies in $F(t)$ for each $t$. If $\sigma$ is a stopping time, let $ST(\sigma)$ denote the collection of all stopping times $\tau$ such that $\sigma \leq \tau$. The optional sampling theorem gives conditions under which

$$X(\sigma) \leq E(X(\tau)\,|\,F(\sigma)) \tag{2.2}$$

for a given $\tau$ in $ST(\sigma)$ and for a given submartingale $X$. Let $OS(\sigma)$ denote the collection of all $\tau$ in $ST(\sigma)$ such that (2.2) is true for all uniformly bounded submartingales $X$. If the index set is the set of integers ordered as usual, then standard theorems (see Neveu (1975)) imply that $ST(\sigma) = OS(\sigma)$. As Chow (1960) showed with a simple counterexample, this is not true for general partially ordered index sets, and in general one only has $OS(\sigma) \subset ST(\sigma)$. In the remainder of this section we are going to characterize $OS(\sigma)$ in terms of the concept of reachability which we discuss next. In Section 3 we will show that in certain special cases in which $T$ is not linearly ordered we can still have $OS(\sigma) = ST(\sigma)$.

**2.2 Reachability.** Throughout the following definitions and discussions let us assume that the partially ordered index set, the underlying probability space and the increasing family of $\sigma$-fields are fixed. Thus, for example, "stopping time" will mean "stopping time on $T$ with respect to the increasing family $\{F(t):t \in T\}$."

DEFINITION 1. A *decision function* $\phi$ is a mapping $\phi: T \times \Omega \to T$ satisfying

$$t \leq \phi(t, \omega) \tag{2.3}$$

and

$$\{\omega:\phi(t, \omega) = s\} \in F(t) \text{ for all } t, s \in T. \tag{2.4}$$

Let $D$ denote the collection of all decision functions. Note that $D$ depends on $T$, $\leq$ and $\{F(t):t \in T\}$. For any positive integer $k$ let $\phi^k$ denote $k$ applications of the random function $\phi$. That is, $\phi^k$ is defined recursively by

$$\phi^{k+1}(t, \omega) = \phi(\phi^k(t, \omega), \omega) \tag{2.5}$$

where we define $\phi^0(t, \omega) = t$ for all $t$ and $\omega$. Also, for a random function $\sigma:\Omega \to T$ let $\phi(\sigma)$ denote the random function $\omega \to \phi(\sigma(\omega), \omega)$. Thus, $\phi^{k+1} = \phi(\phi^k(t)) = \phi^k(\phi(t))$. The concept of a decision function is central to our development of the notion of reachability. Having defined decision functions, we can define several types of reachability as follows. In each of the following definitions assume that $\sigma$ is a stopping time and that $\tau$ is a mapping $\tau:\Omega \to T$ such that $\{\tau = t\}$ is $F$-measurable for each $t$ in $T$.

DEFINITION 2. We say that $\tau$ is *finitely reachable* from $\sigma$ if there is a decision function $\phi$ in $D$ and an integer $k$ such that

$$\phi^k(\sigma) = \tau \tag{2.6}$$

Let $FR(\sigma)$ denote the collection of all $\tau$ which are finitely reachable from $\sigma$.

DEFINITION 3. We say that $\tau$ is *strongly reachable* from $\sigma$ if there is a decision function $\phi$ in $D$ such that the limit

$$\lim_{k\to\infty}\phi^k(\sigma) \tag{2.7}$$

exists almost surely and is equal to $\tau$. The limit (2.7) is interpreted in terms of the discrete topology on $T$. That is, for almost all $\omega$ we have $\lim_{k\to\infty}\phi^k(\sigma(\omega), \omega) = t$ if and only if $\phi^n(\sigma(\omega),$

$\omega) = t$ for some integer $n$. Let $SR(\sigma)$ denote the collection of all $\tau$ which are strongly reachable from $\sigma$.

DEFINITION 4. We say that $\tau$ is *reachable* from $\sigma$ if there is a sequence $\{\tau_k\}$ of $\tau_k$ in $FR(\sigma)$ such that

$$\lim_{k\to\infty} P(\tau_k \neq \tau) = 0. \tag{2.8}$$

Let $R(\sigma)$ denote the collection of all $\tau$ which are reachable from $\sigma$.

In general one has the following relationships between the collections of random functions we have just defined:

$$FR(\sigma) \subset SR(\sigma) \subset R(\sigma) = OS(\sigma) \subset ST(\sigma). \tag{2.9}$$

In particular, note that the relationship $R(\sigma) = OS(\sigma)$ characterizes those pairs of stopping times for which the optional sampling theorem is true for uniformly bounded submartingales. In the present subsection we will show that $FR(\sigma) \subset SR(\sigma) \subset R(\sigma) \subset OS(\sigma) \subset ST(\sigma)$, and in Subsection 2.2 we will show the converse relation $OS(\sigma) \subset R(\sigma)$. First we prove the following simple theorem.

THEOREM 1. *Suppose that $\sigma$ is a stopping time. Then the following relationships are true*:

$$FR(\sigma) \subset SR(\sigma) \subset R(\sigma) \subset ST(\sigma). \tag{2.10}$$

PROOF. Note that for $\phi \in D$ and $\tau \in ST(\sigma)$ we always have $\phi(\tau) \in ST(\sigma)$. From this one easily deduces that $FR(\sigma) \subset ST(\sigma)$. The remaining inclusion relations can be deduced easily from the definitions given above. □

The inclusion relations in (2.10) generally cannot be replaced by equalities. The following simple examples illustrate this fact.

EXAMPLE 1. $FR(\sigma) \neq SR(\sigma)$

Let $T$ denote the set of positive integers ordered in the usual way. Let $\tau$ be any random function taking values in $T$ such that $P(\tau = t) > 0$ for all $t$. Define $F(t)$ as the $\sigma$-field generated by $\{\tau = s\}$ for $s \leq t$. Let $\sigma = 1$. Then with respect to $\{F(t): t \in T\}$, $\sigma$ is a stopping time and $\tau$ is strongly reachable from $\sigma$ but not finitely reachable from $\sigma$.

EXAMPLE 2. $SR(\sigma) \neq R(\sigma)$

Let $T = \{0\} \cup \{1/n: n \geq 1\}$ and order $T$ in the usual way. Let $\tau$ be any random function taking values in $T - \{0\}$ such that $P(\tau = t) > 0$ for all $t \neq 0$. Define $F(t)$ as the $\sigma$-field generated by $\{\tau = s\}$ for $s \leq t$, and let $\sigma = 0$. In this case $F(0)$ is the trivial $\sigma$-field and any decision function $\phi(t)$ is almost surely constant at $t = 0$. Thus, we must have $\phi(\sigma) = 1/n$ for some $n$ and there is always a non-zero probability $P(\tau \leq 1/n)$ that $\phi^k(\sigma) \neq \tau$ for all $k$. Therefore, $\tau$ is not strongly reachable from $\sigma$. However, by choosing a decision function with $\phi(0) = 1/n$ for sufficiently large $n$ we can make the probability that $\phi^k(\sigma) \neq \tau$ for some $k$ arbitrarily small, and thus, $\tau$ is reachable from $\sigma$.

EXAMPLE 3. $R(\sigma) \neq ST(\sigma)$

We present an example from Chow (1960) to show that $OS(\sigma) \neq ST(\sigma)$ by constructing a stopping time $\tau$ in $ST(\sigma)$ which does not belong to $OS(\sigma)$. In the next theorem we will show that $R(\sigma) \subset OS(\sigma)$ and thus, this stopping time $\tau$ cannot belong to $R(\sigma)$. Unlike the previous examples which used linearly ordered index sets, to show $R(\sigma) \neq ST(\sigma)$ we must use a partially ordered index set. Let $T$ consist of three points $a$, $b$, $c$ with the order relations $a \leq b$ and $a \leq c$. Let $\tau$ be a random function taking only the values $b$ and $c$ each with probability one-half. Let $F(t)$ be the $\sigma$-field generated by $\tau = t$ for each $t$ in $T$. Then $F(a)$ is the trivial $\sigma$-field, and $F(b) = F(c)$. If $t = b$ or $t = c$, define $X(t) = 1$ if $t \neq \tau$ and $X(t) = -1$ if $t = \tau$. Let $X(a) = 0$. Then $E(X(b) \mid F(a)) = E(X(c) \mid F(a)) = X(a)$ and $X$ is a uniformly bounded martingale on $T$. However, $E(X(\tau)\,|\,F(\sigma)) = -1 < X(\sigma)$ and hence $\tau \notin OS(\sigma)$.

In special cases we can have some equalities in (2.10). For example, the argument of example 2 is easily extended to show that $R(\sigma) = ST(\sigma)$ in case $T, \leq$ is linearly ordered. In Theorem 3 we will show that $SR(\sigma) = R(\sigma)$ if the index set is a special type of partially ordered set called a tree. In subsection 2.3 we will give a more general condition for the equality $SR(\sigma) = R(\sigma)$. Note that $FR(\sigma) = SR(\sigma) = R(\sigma)$ whenever the index set is finite.

Intuitively, $\tau$ is reachable from $\sigma$ if there is a finite sequence of decisions which reaches $\tau$ from $\sigma$ with arbitrarily large probability. The sequence is $\{\phi^j(\sigma): 0 \leq j \leq k\}$ and the decisions $\phi^j(\sigma)$ must be nondecreasing with respect to the partial order (2.3) and each decision $\phi^{j+1}(\sigma)$ must be measurable with respect to the previous decision $\phi^j(\sigma)$ as required in (2.4). Given this definition of reachability, it is not surprising that the optional sampling theorem is true for reachable pairs of stopping times. Indeed, one merely applies the result for submartingales indexed by integers as we now show.

THEOREM 2. *Suppose that $\sigma$ is a stopping time. Then the following relationship is true*:

$$R(\sigma) \subset OS(\sigma). \tag{2.11}$$

PROOF. Suppose that $X$ is a submartingale uniformly bounded by $X^+$, and let $\phi$ be a decision function. It is easy to show that $\phi(\sigma) \in OS(\sigma)$. Applying the principle of mathematical induction, it follows that $\phi^k(\sigma) \in OS(\sigma)$ for any $k$. Consequently, we have $FR(\sigma) \subset OS(\sigma)$. Applying this result to prove the theorem is straightforward. Let $\{\tau_k\}$ be a sequence in $FR(\sigma)$ which converges to $\tau$ in the sense of (2.8). Rewrite $X(\tau_k)$ as

$$X(\tau_k) = X(\tau) + (X(\tau_k) - X(\tau))\,1_{\tau\neq\tau_k} \tag{2.12}$$

and note that we have already shown

$$E(X(\tau_k)\,|\,F(\sigma)) \geq X(\sigma) \tag{2.13}$$

for each $k$. By assumption, $|X(\tau_k) - X(\tau)|$ is uniformly bounded by $2X^+$. Since $P(\tau \neq \tau_k) \to 0$ as $k \to \infty$, it follows that

$$\lim\nolimits_{k\to\infty} E((X(\tau_k) - X(\tau))\,1_{\tau\neq\tau_k}\,|\,F(\sigma)) = 0. \tag{2.14}$$

The desired result $E(X(\tau)\,|\,F(\sigma)) \geq X(\sigma)$ follows immediately from (2.12), (2.13) and (2.14). □

Before proceeding, let us discuss the relationship of reachability to Haggstrom's control variables. Haggstrom (1966) considered a special type of partially ordered index set, called a tree, and a special type of stopping time which he called a control variable. A *tree* $T, \leq$ is a partially ordered set which consists of finite sequences $(t_1, t_2, \cdots, t_n)$ where $t_i$ are elements of some abstract set which we leave unspecified. The set $T$ must have the property that if $(t_1, t_2, \cdots, t_n)$ lies in $T$, then so does $(t_1, t_2, \cdots, t_k)$ for each $k$, $1 \leq k \leq n$. The partial order $\leq$ on $T$ is defined so that $s \leq t$ for two sequences $t = (t_1, t_2, \cdots, t_n)$ and $s = (s_1, s_2, \cdots, s_m)$ if and only if $m \leq n$ and $s_i = t_i$ for each $i$, $1 \leq i \leq m$. Associated with $T$ is an increasing family which we denote by $\{F(t): t \in T\}$ as before. A *control variable* $\tau$ is a random function $\tau: \Omega \to T$ with the property that for each $n \geq 1$, the events $\{t = \tau\}$ and $\{(t_1, t_2, \cdots, t_n, t_{n+1}) \leq \tau\}$ are $F(t)$-measurable for $t = (t_1, t_2, \cdots, t_n)$. In addition, include the empty sequence $o$ in $T$ and assume that the events $\{o = \tau\}$ and $\{(t_1) \leq \tau\}$ are $F(o)$-measurable for each sequence of the form $(t_1)$ in $T$. The following proposition shows that Haggstrom's control variables are equivalent to random functions reachable from the constant stopping time $\sigma = o$ in our formulation.

THEOREM 3. *Suppose that $T, \leq$ is a countable tree and let $\tau: \Omega \to T$ be a random function. Then $\tau$ is a control variable if and only if it is reachable from the constant stopping time $\sigma = o$. Moreover, in this case $SR(\sigma) = R(\sigma)$.*

PROOF. Suppose first that $\tau$ is a control variable. Then for each $t = (t_1, t_2, \cdots, t_n)$ in $T$ define $\phi(t)$ as

$$\phi(t) = (t_1, t_2, \cdots, t_n, t_{n+1}) \tag{2.15}$$

if $(t_1, t_2, \cdots, t_n, t_{n+1}) \le \tau$, and

$$\phi(t) = t \tag{2.16}$$

if $(t_1, t_2, \cdots, t_n, t_{n+1}) \not\le \tau$ for any $t_{n+1}$. Note that since $T, \le$ is a tree, the events $\{(t_1, t_2, \cdots, t_n, t_{n+1}) \le \tau\}$ are disjoint for different $t_{n+1}$ and hence, $\phi$ is well-defined. To see that $\phi$ is a decision function as defined in Section 2, we must check that (2.3) and (2.4) are satisfied. Property (2.3) is clear from the definition of $\phi$. Since $\tau$ is a control variable, the events $\{(t_1, t_2, \cdots, t_n, t_{n+1}) \le \tau\}$ are $F(t)$-measurable for $t = (t_1, t_2, \cdots, t_n)$ and hence, property (2.4) is satisfied. Now let us show that $\lim_{k\to\infty}\phi^k(\sigma)$ exists and is equal to $\tau$, so that $\tau \in SR(o)$. Consider the events $\{t \not< \tau\}$ and $\{t < \tau\}$ as follows. If $t \not< \tau(\omega)$, then by construction we have $\phi(t, \omega) = t$ and hence, $\lim_{k\to\infty}\phi^k(t, \omega) = t$. If $t < \tau(\omega)$, then by construction we have $t < \phi(t, \omega) \le \tau(\omega)$. If $t = (t_1, t_2, \cdots, t_n)$ and $\tau(\omega) = (\tau_1(\omega), \tau_2(\omega), \cdots, \tau_{n+k}(\omega))$, then $(\tau_1(\omega), \tau_2(\omega), \cdots, \tau_{n+j}(\omega)) \le \phi^j(t, \omega)$ for $1 \le j \le k$. Hence $\phi^k(t, \omega) = \tau(\omega)$ and $\lim_{k\to\infty}\phi^k(t, \omega) = \tau(\omega)$. Thus, we see that $\lim_{k\to\infty}\phi^k(t, \omega) = \tau(\omega)$ whenever $t \le \tau(\omega)$. In particular, we must have that $\lim_{k\to\infty}\phi^k(o) = \tau$. It follows that $\tau$ is strongly reachable from $o$.

To show the converse, assume that $\tau$ is reachable from $o$. We must show that $\{t' \le \tau\}$ is $F(t)$-measurable for $t = (t_1, t_2, \cdots, t_n)$ and $t' = (t_1, t_2, \cdots, t_n, t_{n+1})$ in $T$. First suppose that $\tau$ is finitely reachable from $o$ so that $\tau = \phi^k(o)$ for some decision function $\phi$ and some integer $k \ge 0$. Thus, we have

$$\{t' \le \tau\} = \cup_{j=0}^{k} \{\phi^j(o) \le t, t' \le \phi^{j+1}(o)\}. \tag{2.17}$$

As we proved in Theorem 1, $\phi^j(o)$ is a stopping time for each $j$ and hence, $\{\phi^j(o) = s\}$ is $F(s)$-measurable and hence $F(t)$-measurable for all $s \le t$. Since $\phi$ is a decision function, the event $\{t' \le \phi(s)\}$ is $F(s)$-measurable, and hence $F(t)$-measurable for $s \le t$. We can write

$$\{\phi^j(o) \le t, t' \le \phi^{j+1}(o)\} = \cup_{s\le t} (\{\phi^j(o) = s\} \cap \{t' \le \phi(s)\}) \tag{2.18}$$

and thus, the event $\{\phi^j(o) \le t, t' \le \phi^{j+1}(o)\}$ must be $F(t)$-measurable. From (2.17) it follows that $\{t' \le \tau\}$ is also $F(t)$-measurable, and consequently, each $\tau$ in $FR(o)$ is a control variable.

To see that $\tau$ in $R(o)$ are also control variables, let $\{\tau_k\}$ be a sequence in $FR(o)$ which converges to $\tau$ in the sense of (2.8). We have just showed that $\{t' \le \tau_k\}$ is $F(t)$-measurable for each $k$. It is not hard to deduce that $\{t' \le \tau\}$ is also $F(t)$-measurable. Hence, each $\tau$ in $R(o)$ is a control variable.

Thus, we have shown that all control variables are strongly reachable from $o$ and that all $\tau$ which are reachable from $o$ are control variables. Using the result $SR(o) \subset R(o)$ from Theorem 1, we see that in fact $SR(o) = R(o)$ in this case and the notions of strongly reachable, reachable and control variable are equivalent. □

**2.3 Optimal Stopping Problem and Converse Optional Sampling.** We now turn to proving the converse optional sampling theorem, namely that $OS(\sigma) \subset R(\sigma)$. To do this we first consider an optimal stopping problem, defined on partially ordered index sets, which is a generalization of Haggstrom's (1966) stopping problem on trees.

THEOREM 4. Suppose that the mapping $c: T \times \Omega \to \mathbb{R}$ is uniformly bounded and adapted to $\{F(t): t \in T\}$. For any random function $\tau: \Omega \to T$ define $\pi(t, \tau)$ as

$$\pi(t, \tau) = E(c(\tau) \mid F(t)). \tag{2.19}$$

Define $\pi(t)$ as

(2.20) $$\pi(t) = \text{ess inf}\{\pi(t, \tau) : \tau \in R(t)\}$$

and let $\pi$ denote the mapping from $T \times \Omega$ to $\mathbb{R}$ defined by (2.20). Then $\pi$ satisfies the equation

(2.21) $$\pi(t) = \inf\{E(\pi(s) \mid F(t)), c(t) : t < s\}$$

for all $t$ in $T$. Furthermore, for any stopping time $\sigma$ and $\epsilon > 0$ there exists $\tau$ in $SR(\sigma)$ such that

(2.22) $$\pi(\sigma) + \epsilon \geq \pi(\sigma, \tau).$$

Proof. Note first that

(2.23) $$\text{ess inf}\{\pi(t, \tau) : \tau \in R(t)\} = \text{ess inf}\{\pi(t, \tau) : \tau \in FR(t)\}.$$

Define $\tilde{\pi}(t)$ as

(2.24) $$\tilde{\pi}(t) = \inf\{E(\pi(s) \mid F(t)), c(t) : t < s\}.$$

It is easy to show that $\pi(t) \geq \tilde{\pi}(t)$. In order to do this, let $\tau$ be an element of $FR(t)$ and let $\tau = \phi^k(t)$ for the decision function $\phi$. Then it is a straightforward computation to show

(2.25) $$E(c(\tau) \mid F(t)) \geq \tilde{\pi}(t).$$

Since $\tau \in FR(t)$ in (2.25) was chosen arbitrarily, the relations (2.20) and (2.23) imply that $\pi(t) \geq \tilde{\pi}(t)$.

The opposite inequality, $\pi(t) \leq \tilde{\pi}(t)$, is slightly harder to prove, but it follows easily once we show that (2.22) is true for constant stopping times. As shown in Chow, Siegmond and Robbins (1971), the essential infimum (ess inf) has the property that it is almost surely equal to an infimum over a countable collection of random variables. Using (2.20) and (2.23), we see that there is a countable set $\{\tau_k\}$ of random functions in $FR(t)$ such that

(2.26) $$\pi(t) = \inf\{\pi(t, \tau_k) : k \geq 1\}.$$

In Theorem 1 we proved that $FR(t) \subset SR(t)$ and therefore, there exist decision functions $\phi_k$ such that for each $k$

(2.27) $$\lim_{n\to\infty} \phi_k^n(t) = \tau_k.$$

Define the integer-valued random function $k^*$ to be the least integer $k \geq 1$ such that $\pi(t) + \epsilon \geq \pi(t, \tau_k)$. The random function $k^*$ is thus defined almost everywhere and it is $F(t)$-measurable. Let $\phi$ be defined so that $\phi(r) = r$ for all $r$ such that $t \not\leq r$, and such that $\phi(r) = \phi_{k^*}(r)$ for $t \leq r$. Then the mapping $\phi : T \times \Omega \to T$ is defined for almost all $\omega$ and by defining $\phi(r, \omega) = r$ where $k^*(\omega)$ is not defined, one easily sees that $\phi \in \text{D}$. Moreover,

(2.28) $$\lim_{n\to\infty} \phi^n(t) = \tau_{k^*}.$$

Hence, $\tau_{k^*}$ must be in $SR(t)$ and by definition of $k^*$ it must be true that $\pi(t) + \epsilon \geq \pi(t, \tau_{k^*})$.

We can now show $\pi(t) \leq \tilde{\pi}(t)$ as follows. For $\epsilon > 0$ and for $s$ such that $t < s$, choose $\tau$ in $SR(s)$ such that

(2.29) $$\pi(s) + \epsilon \geq \pi(s, \tau).$$

Let $\tau = \lim_{k\to\infty} \phi^k(s)$ for $\phi \in D$. Define a new decision function $\psi$ as $\psi(\text{r}) = \phi(\text{r})$ for $s \leq r$, $\psi(t) = s$ and $\psi(r) = r$ for all other $r$. Then $\lim_{k\to\infty} \psi^k(t) = \lim_{k\to\infty} \phi^k(s)$ and hence, $\tau$ is an element of $SR(t)$ and also an element of $R(t)$. Conditioning (2.29) with respect to $F(t)$ we obtain $E(\pi(s) \mid F(t)) + \epsilon \geq \pi(t, \tau)$ and consequently, $E(\pi(s) \mid F(t)) + \epsilon \geq \pi(t)$. Since $\epsilon$ was arbitrary, we obtain $E(\pi(s) \mid F(t)) \geq \pi(t)$. It is clear that $c(t) \geq \pi(t)$, and thus, we have $\tilde{\pi}(t) \geq \pi(t)$.

To finish the proof we must demonstrate the inequality (2.22) for arbitrary stopping times $\sigma$. From above we know that for each $t$ there exists $\tau_t \in SR(t)$ such that

$$\pi(t) + \epsilon \geq \pi(t, \tau_t) \tag{2.30}$$

for a given $\epsilon > 0$. Let $\phi_t \in D$ be such that

$$\lim_{k\to\infty} \phi_t^k(t) = \tau_t. \tag{2.31}$$

Define a new decision function $\phi$ such that $\phi(t) = \phi_r(t)$ if $r \leq t$ and $\sigma = r$, and $\phi(t) = t$ if $\sigma \not\leq t$. In this case the limit $\tau = \lim_{k\to\infty}\phi^k(\sigma)$ exists and is equal to $\lim_{k\to\infty}\phi_r^k(r)$ whenever $\sigma = r$. Thus, $\tau \in SR(\sigma)$ and from (2.30) it follows that $\pi(\sigma) + \epsilon \geq \pi(\sigma, \tau)$. □

THEOREM 5. *If $\sigma$ is a stopping time on $T, \leq$ with respect to $\{F(t) : t \in T\}$ then $OS(\sigma) = R(\sigma)$.*

PROOF. We have already shown in Theorem 2 that $R(\sigma) \subset OS(\sigma)$. Thus, it suffices to show $OS(\sigma) \subset R(\sigma)$. Suppose that $\tau \in OS(\sigma)$. We apply Theorem 4 to the optimal stopping problem with cost function $c(t) = 1_{\tau\neq t}$. It follows that $\pi$ in (2.20) is a submartingale uniformly bounded by 1. Since we assume that $\tau \in OS(\sigma)$, the optional sampling inequality (2.2) is true for $X = \pi$ and thus $E(\pi(\tau) \mid F(\sigma)) \geq \pi(\sigma)$. Since $\pi(\tau)$ is clearly 0 by definition of $c$ and since $\pi(\sigma) \geq 0$, we have $\pi(\sigma) = 0$. From (2.22) there exist $\tau_k$ in $SR(\sigma)$ such that for each positive integer $k$

$$\pi(\sigma) + \frac{1}{k} \geq \pi(\sigma, \tau_k). \tag{2.32}$$

Noting that $\pi(\sigma) = 0$ and $\pi(\sigma, \tau_k) = P(\tau_k \neq \tau \mid F(\sigma))$ in (2.32), we obtain the following result:

$$P(\tau_k \neq \tau \mid F(\sigma)) \leq \frac{1}{k}. \tag{2.33}$$

Taking the expectation of (2.33) gives $P(\tau_k \neq \tau) \leq 1/k$. Since each $\tau_k$ is strongly reachable from $\sigma$, according to Theorem 1 it is also reachable from $\sigma$, and hence there are $\tau_k' \in FR(\sigma)$ such that $P(\tau_k' \neq \tau_k) \leq 1/k$. It follows that $P(\tau_k' \neq \tau) \leq 2/k$ for each $k$ and consequently $\tau \in R(\sigma)$. □

The following corollary, which follows from (2.33), improves the approximation (2.8).

COROLLARY 5. *For each $\tau \in R(\sigma)$ there exist $\tau_k \in SR(\sigma)$ such that*

$$\lim_{k\to\infty} P(\tau_k \neq \tau \mid F(\sigma)) = 0 \tag{2.34}$$

*where the convergence in* (2.34) *is uniform on a set of probability one.*

When the infimum in (2.21) is a minimum, we may refine the results of Theorem 5 and in some cases prove that all reachable random functions are in fact strongly reachable. We present these results below in Theorem 7 and its corollaries. The following simple theorem shows that the infimum is actually a minimum for a large class of index sets. If $T, \leq$ is a partially ordered set and $t \in T$, then we say that $s$ is an immediate successor of $t$ and write $t <\cdot s$ if $t \leq s$ and if $t \leq r \leq s$ for no $r$ other than $t$ or $s$.

THEOREM 6. *Suppose that the partially ordered index set $T, \leq$ in Theorem 4 is such that each $t$ in $T$ has at most a finite number of immediate successors. Then the infimum in* (2.21) *is a minimum.* □

PROOF. Note that since $\pi$ is a submartingale we have that $E(\pi(s) \mid F(t)) \leq E(\pi(r) \mid F(t))$ for each $t \leq s \leq r$. Since for each $r$ such that $t < r$ there is an immediate successor $s$ of $t$ such that $s \leq r$, we have

$$\inf\{E(\pi(s)\,|\,F(t)),\, c(t): t < s\} = \inf\{E(\pi(s)\,|\,F(t)),\, c(t): t \lessdot s\}. \tag{2.35}$$

The infimum on the right hand side of (2.35) is taken over a finite set of $s$ by assumption, and consequently it is a minimum.

THEOREM 7. *Let $\sigma$ be a stopping time with respect to $\{F(t): t \in T\}$ and let $\tau \in R(\sigma)$. Let $\pi$ be the uniformly bounded submartingale defined by* (2.20) *in Theorem* 4 *with the cost function $c(t) = 1_{\tau \neq t}$. Suppose that there exists $\phi$ such that*

$$\pi(t) = E\{\pi(\phi(t))\,|\,F(t)\} \tag{2.36}$$

*for each $t$ and $\phi(t) = t$ if and only if $\pi(t) = c(t)$. Then $\tau \in R(\phi^k(\sigma))$ for each $k$, and in particular, $\phi^k(\sigma) \le \tau$ for all $k$. Furthermore, whenever the limit $\lim_{k\to\infty}\phi^k(\sigma)$ exists, it is equal to $\tau$.*

PROOF. If $\rho$ is a stopping time, then from (2.36) it follows that

$$\pi(\rho) = E\{\pi(\phi(\rho))\,|\,F(\rho)\}. \tag{2.37}$$

Letting $\rho = \phi^k(\sigma)$ successively for $k \ge 0$ we see that $\{\pi(\phi^k(\sigma))\}$ is a one parameter martingale with respect to $\{F(\phi^k(\sigma))\}$. Since $\tau \in R(\sigma)$ implies that $\pi(\sigma) = 0$ and since $\pi \ge 0$, we see that $\pi(\phi^k(\sigma)) = 0$ also for each $k$. Using the same argument as in Theorem 5, we deduce that $\tau \in R(\phi^k(\sigma))$. It follows that $\phi^k(\sigma) \le \tau$ a.s. from the definition of $R(\phi^k(\sigma))$. Finally, to prove the last assertion of the theorem suppose that $\lim_{k\to\infty}\phi^k(\sigma)$ exists so that $\phi^k(\sigma) = \phi^{k+1}(\sigma)$ for some $k$. The condition that $\phi(t) = t$ if and only if $\pi(t) = c(t)$ implies that $\pi(\phi^k(\sigma)) = c(\phi^k(\sigma))$. Since $\pi(\phi^k(\sigma)) = 0$, it follows that $c(\phi^k(\sigma)) = 0$ and hence, $\phi^k(\sigma) = \tau$. □

The following corollaries are immediate consequences of Theorems 6 and 7.

COROLLARY 7.1. *Assume the same conditions as in Theorem 7. Suppose that the index set $T$ has the property that for any $s$, $t$ in $T$ with $s < t$, there is no infinite sequence $\{r_n\}$ in $T$ such that $s < r_n < r_{n+1} < t$ for all $n$. Then for the decision function $\phi$ satisfying* (2.36) *we have*

$$\lim_{k\to\infty}\phi^k(\sigma) = \tau. \tag{2.38}$$

COROLLARY 7.2. *If the partially ordered index set $T, \le$ is the set of integer n-tuples $Z^n$ with the coordinate-wise partial ordering, then $SR(\sigma) = R(\sigma)$.*

COROLLARY 7.3. *If the partially ordered index set $T, \le$ is finite, then $FR(\sigma) = SR(\sigma) = R(\sigma)$.*

**3. Conditional Independence and Optional Sampling.** For particular types of index sets $T, \le$ and increasing families $\{F(t): t \in T\}$ it may be true that for any pair $\tau$, $\sigma$ of stopping times with $\sigma \le \tau$ that $\tau$ is reachable from $\sigma$. For example, this is true if $T, \le$ is countable and linearly ordered. In this section we present two other general cases where this is also true and where the index set is not linearly ordered.

To begin we make two assumptions, one concerning the index set $T, \le$ and the other concerning the collection $\{F(t): t \in T\}$ of $\sigma$-fields. Namely, assume that for any two elements $t$, $s$ of $T$ there is a greatest lower bound $t \wedge s$ of $t$ and $s$ with respect to the partial ordering of $T$. This is true, for example, if $T, \le$ is a tree, as defined in Section 2, or if $T, \le$ is a lattice such as $Z^n$ or $R^n$ with the coordinate-wise partial ordering. In the second case, the $i$th coordinate of $t \wedge s$ is $\min\{t_i, s_i\}$ where $t_i$ and $s_i$ are the $i$th coordinates of $t$ and $s$ respectively. The second assumption we make is that $\{F(t): t \in T\}$ satisfies the following conditional independence property.

DEFINITION 5. The increasing family $\{F(t):t \in T\}$ satisfies the *conditional independence property* if for each $s$ and $t$ in $T$, the $\sigma$-fields $F(s)$ and $F(t)$ are conditionally independent given $F(s \wedge t)$.

This conditional independence property was defined for the case of $T = R^2_+$ by Cairoli and Walsh (1975) in their study of stochastic integrals on the plane. The multiparameter Wiener process on $R^n_+$ defined by Park (1970) generates $\sigma$-fields which satisfy the conditional independence property. If $T, \leq$ is an index set with the property that any two elements of $T$ have a greatest lower bound, then we can construct a simple example of a collection of $\sigma$-fields with the conditional independence property as follows. Let $\{x(t):t \in T\}$ be a collection of independent random variables and let $F(t)$ be the $\sigma$-field generated by the collection $\{x(s):s \leq t\}$ of random variables. It is not hard to see that the collection $\{F(t):t \in T\}$ so defined satisfies Definition 5.

We will show that if $T, \leq$ is either a tree or $Z^2$ with the coordinate-wise ordering and if $\{F(t):t \in T\}$ satisfies the conditional independence property, then $SR(\sigma) = ST(\sigma)$ for all stopping times $\sigma$. The first case we consider is that for which $T, \leq$ is a countable tree as defined in Section 2.

THEOREM 8. *Suppose that $T, \leq$ is a countable tree and that the increasing family $\{F(t):t \in T\}$ has the conditional independence property. If $\sigma$ is any stopping time, then $SR(\sigma) = ST(\sigma)$.*

PROOF. Having proved that $SR(\sigma) \subset ST(\sigma)$ in Theorem 1, we need only prove $ST(\sigma) \subset SR(\sigma)$. For a given stopping time $\tau$ we will construct a decision function $\phi$ such that for any stopping time $\sigma$ with $\sigma \leq \tau$, the limit $\lim_{k\to\infty} \phi^k(\sigma)$ exists and is equal to $\tau$.

Fix $t$ in $T$ and define $\phi(t):\Omega \to T$ as follows. For each immediate successor $s$ of $t$, let $A_s$ denote the $F(t)$-measurable event

$$A_s = \{\omega:P(s \leq \tau \mid F(t)) > 0\}. \tag{3.1}$$

Note that for each $s$ the definition (3.1) implies that

$$P(A_s \cap \{s \leq \tau\}) = P(\{s \leq \tau\}). \tag{3.2}$$

Suppose that $s$ and $s'$ are immediate successors of $t$ with $s \neq s'$, and let $s \leq r$ and $s' \leq r'$. Because $T, \leq$ is a tree, we must have $r \neq r'$ and $r \wedge r' = s \wedge s' = t$. Thus, the conditional independence property implies that

$$P(\tau = r \mid F(t))P(\tau = r' \mid F(t)) = 0. \tag{3.3}$$

Summing (3.3) over all $r, r'$ such that $s \leq r$ and $s' \leq r'$ gives

$$P(s \leq \tau \mid F(t))P(s' \leq \tau \mid F(t)) = 0. \tag{3.4}$$

From (3.4) and (3.1) it follows that for each $s \neq s'$

$$P(A_s \cap A_{s'}) = 0. \tag{3.5}$$

Using the assumed completeness of $F(t)$ and redefining the $A_s$ on sets of measure zero if necessary, we deduce from (3.2) and (3.5) that

$$\{s \leq \tau\} \subset A_s \tag{3.6}$$

$$A_s \cap A_{s'} = \varnothing \tag{3.7}$$

for all $s, s'$ such that $t \lessdot s$, $t \lessdot s'$ and $s \neq s'$. Let $A_t$ denote the event in $F(t)$ defined by

$$A_t = (\Omega - \{A_s:t \lessdot s\}) \cup \{\tau = t\}. \tag{3.8}$$

We can define $\phi(t)$ in terms of the sets $A_t$ and $A_s$ for $t \lessdot s$ as follows.

$$\phi(t, \omega) = t \quad \text{if } \omega \in A_t \tag{3.9}$$

$$\phi(t, \omega) = s \quad \text{if } \omega \in A_s - A_t. \tag{3.10}$$

Property (3.7) and the definition (3.8) ensure that $\phi(t)$ is a well-defined function. Property (3.6) ensures that if $t \le \tau$, then $\phi(t) \le \tau$. Furthermore, if $t \le \tau$ and $\phi(t) = t$, then $t = \tau$. If $\sigma$ is any stopping time such that $\sigma \le \tau$, then $\phi^k(\sigma) \le \tau$ for each $k$. Since $T, \le$ is a tree and has the property mentioned in Corollary 7.1, the limit $\lim_{k\to\infty} \phi^k(\sigma)$ exists almost surely. By construction this limit $\rho$ is such that $\rho \le \tau$ and $\phi(\rho) = \rho$. Consequently, $\rho = \tau$. □

The property of a tree that makes Theorem 8 possible is that for each $s, s'$ such that $t \lessdot s, s'$ and $s \ne s'$ we have $\{r : s \le r\} \cap \{r' : s' \le r'\} = \varnothing$. This property will not hold for more general partially ordered index sets such as $T = Z^n$. Nevertheless, we can adapt the proof of Theorem 8 to the more general case of $T = Z^2$.

THEOREM 9. *Suppose that $T = Z^2$ and $\le$ is the coordinate-wise partial ordering of $Z^2$. Furthermore, suppose that the increasing family $\{F(t): t \in T\}$ satisfies the conditional independence property. If $\sigma$ is any stopping time, then $SR(\sigma) = ST(\sigma)$.*

PROOF. As in Theorem 8, for a given stopping time $\tau$ we will construct a decision function $\phi$ such that $\lim_{k\to\infty} \phi^k(\sigma) = \tau$ for any stopping time $\sigma$ with $\sigma \le \tau$.

Fix $t = (t_1, t_2)$ in $Z^2$ and define $\phi(t): \Omega \to Z^2$ as follows. Define the events $K_1$ and $K_2$ as

$$K_1 = \cup_{n>0} \{\tau = (t_1, t_2 + n)\} \tag{3.11}$$

$$K_2 = \cap_{m>0} \{\tau = (t_1 + m, t_2)\}. \tag{3.12}$$

The conditional independence property implies that $F((t_1, t_2 + n))$ and $F((t_1 + m, t_2))$ are conditionally independent given $F((t_1, t_2))$ for any $m, n \ge 0$. Thus, we have

$$P(\tau = (t_1, t_2 + n) \mid F(t))P(\tau = (t_1 + m, t_2) \mid F(t)) = 0 \tag{3.13}$$

for all $n, m > 0$. From (3.13), (3.11) and (3.12) it follows that

$$P(K_1 \mid F(t))P(K_2 \mid F(t)) = 0. \tag{3.14}$$

Define the $F(t)$-measurable events $A_{(t_1,t_2+1)}$ and $A_{(t_1+1,t_2)}$ as

$$A_{(t_1,t_2+1)} = \{\omega : P(K_1 \mid F(t)) > 0\} \tag{3.15}$$

$$A_{(t_1+1,t_2)} = \Omega - A_{(t_1,t_2+1)}. \tag{3.16}$$

It is not difficult to see that

$$P(A_{(t_1,t_2+1)} \cap K_1) = P(K_1) \tag{3.17}$$

and that

$$P(A_{(t_1,t_2+1)} \cap K_2) = 0. \tag{3.18}$$

From this point, the proof follows Theorem 8. Using the assumed completeness of $F(t)$ and redefining $A_{(t_1,t_2+1)}$ and $A_{(t_1+1,t_2)}$ on sets of measure zero if necessary, we deduce from (3.17) and (3.18) that

$$K_1 \subset A_{(t_1,t_2+1)} \tag{3.19}$$

$$K_2 \subset A_{(t_1+1,t_2)} \tag{3.20}$$

where $A_{(t_1,t_2+1)} \cap A_{(t_1+1,t_2)} = \varnothing$ and $A_{(t_1,t_2+1)} \cup A_{(t_1+1,t_2)} = \Omega$. Let $A_{(t_1,t_2)} = \{\tau = (t_1, t_2)\}$ and define $\phi((t_1, t_2))$ as:

$$\phi((t_1, t_2), \omega) = (t_1, t_2) \quad \text{if } \omega \in A_{(t_1,t_2)} \tag{3.21}$$

$$\phi((t_1, t_2), \omega) = (t_1 + 1, t_2) \quad \text{if } \omega \in A_{(t_1+1,t_2)} - A_{(t_1,t_2)} \tag{3.22}$$

$$\phi((t_1, t_2), \omega) = (t_1, t_2 + 1) \quad \text{if } \omega \in A_{(t_1,t_2+1)} - A_{(t_1,t_2)}. \tag{3.23}$$

For this decision function, $\phi^k(\sigma) \le \tau$ for any stopping time $\sigma$ such that $\sigma \le \tau$. Since $Z^2$ also has the property of Corollary 7.1, the limit $\lim_{k\to\infty} \phi^k(\sigma)$ exists almost surely and is equal to $\tau$. □

We cannot extend the proof of Theorem 9 to the case of $T = Z^n$ for $n > 2$. The following example shows that in fact the result is not true in general for $n > 2$.

EXAMPLE 4. $R(\sigma) \ne ST(\sigma)$ when $T = Z^3$ and when the conditional independence property holds true.

Without loss of generality, we construct the example for the index set $T$ defined as the Cartesian product $\{0, 1\} \times \{0, 1\} \times \{0, 1\}$, a subset of $Z^3$, namely the vertices of the unit cube. One can easily extend this example to one on all of $Z^3$ or to $Z^n$ for some $n > 3$. Let $\Omega = \{\omega_i : 1 \le i \le 8\}$ with $P(\{\omega_i\}) = 1/8$ for each $i$, and let $F$ be the collection of all subsets of $\Omega$. Define three random functions $\alpha$, $\beta$, $\gamma$ from $\Omega$ into $\{0, 1\}$ as follows.

$$\alpha(\omega_1) = \alpha(\omega_2) = \alpha(\omega_3) = \alpha(\omega_4) = 0$$

$$\alpha(\omega_5) = \alpha(\omega_6) = \alpha(\omega_7) = \alpha(\omega_8) = 1$$

$$\beta(\omega_1) = \beta(\omega_2) = \beta(\omega_5) = \beta(\omega_6) = 0$$

$$\beta(\omega_3) = \beta(\omega_4) = \beta(\omega_7) = \beta(\omega_8) = 1$$

$$\gamma(\omega_1) = \gamma(\omega_3) = \gamma(\omega_5) = \gamma(\omega_7) = 0$$

$$\gamma(\omega_2) = \gamma(\omega_4) = \gamma(\omega_6) = \gamma(\omega_8) = 1$$

It is not difficult to check that $\alpha$, $\beta$, $\gamma$ are independent random functions. We can now define the $\sigma$-fields $\{F(t) : t \in T\}$ in terms of these random variables. Let $F((0, 0, 0))$ be the trivial $\sigma$-field $\{\Omega, \varnothing\}$. The $\sigma$-field $F((1, 0, 0))$ is generated by $\alpha$, $F((0, 1, 0))$ is generated by $\beta$, $F((0, 0, 1))$ is generated by $\gamma$, $F((1, 1, 0))$ is generated by $\alpha$ and $\beta$, $F((1, 0, 1))$ is generated by $\alpha$ and $\gamma$, $F((0, 1, 1))$ is generated by $\beta$ and $\gamma$, and $F((1, 1, 1))$ is generated by all three random variables—hence, $F((1, 1, 1)) = F$. Since $\alpha$, $\beta$, and $\gamma$ are independent, it is easy to check that $\{F(t) : t \in T\}$ satisfies the conditional independence property.

Define a submartingale $X$ on $T$ as follows. Let $X((1, 1, 1))(\omega_3) = X((1, 1, 1))(\omega_6) = -1$ and let $X((1, 1, 1))(\omega_i) = 1$ for $i \ne 3, 6$. Define $X(t) = 0$ for $t \ne (1, 1, 1)$. To check that $X$ is a submartingale, it suffices to show that $E(X((1, 1, 1)) \mid F(t)) \ge 0$, for $t = (1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$. A simple calculation shows that $E(X((1, 1, 1)) \mid F((1, 1, 0)))$ is equal to 1 if $\omega = \omega_1$, $\omega_2$, $\omega_7$, $\omega_8$ and it is equal to 0 if $\omega = \omega_3$, $\omega_4$, $\omega_5$, $\omega_6$. The other conditional expectations are similar.

Finally, define $\tau(\omega) = (1, 1, 0)$ if $\omega = \omega_1, \omega_2$; let $\tau(\omega) = (1, 0, 1)$ if $\omega = \omega_5, \omega_7$; let $\tau(\omega) = (0, 1, 1)$ if $\omega = \omega_4, \omega_8$; and let $\tau(\omega) = (1, 1, 1)$ if $\omega = \omega_3, \omega_6$. It is easy to check that $\tau$ is a stopping time. Let $\sigma = (0, 0, 0)$. Then a simple calculation shows that

$$E(X(\tau) \mid F(\sigma)) = E(X(\tau)) = (-1)\,\tfrac{1}{4} + (0)\,\tfrac{3}{4} = -\tfrac{1}{4} \not\ge 0 = X(\sigma).$$

Consequently, $\tau \notin OS(\sigma)$ and from Theorem 2 it follows that $\tau \notin R(\sigma)$. Thus, $R(\sigma) \ne ST(\sigma)$.

To conclude the results of this paper we state an optional sampling theorem which follows easily from Theorems 8 and 9 and Theorem 2.

THEOREM 10. *Suppose that $T, \le$ is either a countable tree, $Z^2$ or $R^2$. Let the increasing family $\{F(t) : t \in T\}$ satisfy the conditional independence property, and let $X$ be a uniformly bounded submartingale with respect to this increasing family. If $T = R^2$ then also let $X$ be right continuous in the sense that*

$$\lim_{s\to t, t\le s} X(s) = X(t) \text{ for all } t$$

If $\sigma$ and $\tau$ are stopping times with $\sigma \le \tau$, then $E(X(\tau) \mid F(\sigma)) \ge X(\sigma)$.

PROOF. For $T = Z^2$ or $T$ a tree the proof follows immediately from Theorems 8, 9 and 2. For $T = R^2$ note that a stopping time $\tau$ is a random function $\tau:\Omega \to R^2$ such that $\{\tau \leq t\}$ is $F(t)$-measurable for each $t$ in $T$. This is equivalent to the definition of stopping time in Section 2.1 if $\tau$ takes only countably many values. To prove the theorem for $T = R^2$, one proceeds as for $R^1$ by taking limits of stopping times taking only countable many values in $R^2$ and using the result for $T = Z^2$. Since there are very few changes from the one-parameter case, for example as given in Neveu (1965), we omit the proof. □

**4. Conclusions.** We have shown that for a given pair $\sigma$, $\tau$ of stopping times such that $\sigma \leq \tau$ on a countable partially ordered set, the optional sampling inequality

$$E(X(\tau) \mid F(\sigma)) \geq X(\sigma) \tag{4.1}$$

is true for all uniformly bounded *submartingales* $X$ if and only if $\tau$ is reachable from $\sigma$. This result stands in sharp contrast to the *martingale* versions of Chow (1960) and Kurtz (1977). If $X$ is a uniformly bounded martingale and if the index set is directed, then (4.1) is true for any stopping times $\sigma$, $\tau$ such that $\sigma \leq \tau$. Note that our result does not require that the index set be directed.

Our characterization of the collection $OS(\sigma)$ of stopping times $\tau$ for which (4.1) is true shows that optional sampling is intimately associated with *sequential* sampling problems—namely, reachable stopping times are defined in terms of sequential decision functions as described in Section 2. Thus, even for the case of a partially ordered index set, the optional sampling theorem is necessarily a one-parameter result. As we have shown in Section 2, the optional sampling theorem is true for general uniformly bounded submartingales if and only if the stopping times are reachable and hence if and only if the theorem can be reduced to its one-parameter version.

In certain special cases discussed in Section 3 it is possible to show that the optional sampling inequality (4.1) is true for all stopping times and all uniformly bounded submartingales. To obtain such results we assumed that the increasing family of $\sigma$-fields satisfied a conditional independence property originally given by Cairoli and Walsh (1975). This property is possessed by $\sigma$-fields generated by multiparameter Wiener processes, for example. Thus, we proved the optional sampling theorem for the special cases when $T$ is a tree or when $T = Z^2$ or $R^2$. The counter-example of Section 3 shows that the theorem is not true for $T = Z^3$.

The preceding results completely characterize the situations in which the optional sampling theorem is true with no restriction on the submartingales other then uniform boundedness. If further restrictions are placed on the submartingales, it may be possible to obtain different optional sampling results. For example, if $X$ has a Doob decomposition[4], it is easy to apply Kurtz's (1977) result to show that (4.1) is satisfied for all stopping times. Thus, suppose $X(t) = M(t) + A(t)$ where $M$ is a martingale and $A$ is an *increasing process* in the sense that $A(s) \leq A(t)$ if $s \leq t$ in the partially ordered index set. Then if the index set is directed, Kurtz's result implies that $E(M(\tau) \mid F(\sigma)) = M(\sigma)$ for all stopping times $\sigma$, $\tau$ such that $\sigma \leq \tau$. It is clear that $A(\sigma) \leq A(\tau)$ and hence $E(A(\tau) \mid F(\sigma)) \geq A(\sigma)$. Thus, $E(X(\tau) \mid F(\sigma)) \geq X(\sigma)$ is also true.

Of course, the fact that the optional sampling theorem is not generally true in the case of partially ordered index sets means that not all submartingales have such Doob decompositions. Indeed, if $\tau$ is not reachable from some stopping time $\sigma$ where $\sigma \leq \tau$, then the submartingale $\pi$ of Theorem 5 has no Doob decomposition. For otherwise (4.1) would be true for $X = \pi$ and the argument of Theorem 5 would imply that $\tau$ is reachable from $\sigma$, a contradiction.

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[4] See Doob (1953) for the discrete index version or Meyer (1966) for the continuous index version.

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