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ON THE CONVERGENCE OF THE EMPIRIC AGE DISTRIBUTION FOR ONE DIMENSIONAL SUPERCRITICAL AGE DEPENDENT BRANCHING PROCESSES

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The age distribution for the supercritical age dependent branching processes is shown to converge on the set of nonextinction to a particular distribution function if the offspring distribution $\{p_j\}$ satisfies $1 < \Sigma j p_j < \infty$.

1. Introduction. Consider an age-dependent branching process (see Harris [3] for definitions) governed by $\{p_j\}$, the common probability distribution of the number of progeny having been born to an individual by time of death, and $G(\cdot)$, the common distribution function (d.f.) of the length of life of an individual. For a realization ω of the process, let $Z(t, \omega)$ denote the total number of individuals alive at time t, $Z(x, t, \omega)$, the number among these that have ages no more than x, and $A(\cdot, t, \omega)$ defined by $A(x, t, \omega) = Z(x, t, \omega)/Z(t, \omega)$ denote the empiric age d.f. of those alive at time t.

There has been considerable interest shown in the past in the limiting behavior of the age distribution $A(\cdot, t, \omega)$, as $t \to \infty$. In [3], Harris showed the almost sure (a.s.) convergence of $A(\cdot, t, \omega)$ if $\{p_j\}$ has a second moment and $G(\cdot)$ is sufficiently regular. Later, Jagers [4] obtained the same result assuming only that $\{p_j\}$ has a second moment. More recently, Athreya and Kaplan [1] showed the validity of the above result assuming that $\Sigma(j \cdot \log j) \cdot p_j$ is finite. After the present results were obtained, a later paper of Athreya and Kaplan [2] was brought to the attention of the author, where they have shown that the result holds if $1 < m = \Sigma jp < \infty$ and $G(\cdot)$ satisfies a certain tail condition. In contrast to these, the present paper assumes only $1 < m < \infty$, with no conditions imposed on $G(\cdot)$. Recently, independent but related work of O. Nerman has become known to the author. See Section 5.

The approach adopted here follows in part the basic steps of [1], namely that we decompose $A(\cdot, t, \omega)$ into three terms, and then tackle each term separately. In so doing, we use a rather interesting embedding technique leading to the final proof.

In Section 2, we give notation from [1] and our basic assumptions. In Section 3, we state the main theorem and the three lemmas necessary to prove it. Finally, to complete the proof of the basic theorem in Section 3, a theorem is given in Section 4 which provides certain lower bounds achieved through a special embedding. Of the three lemmas of Section 3, one is already proved in [1], while the other two are proven in Section 4.

2. Notation and basic assumptions. We always assume, whether stated or not, that (a) $p_0 = 0$ (rather than conditioning on the set of nonextinction), (b) $1 < m = \Sigma jp_j < \infty$, and (c) $G(0+) = 0$. For any realization ω and $x \geq 0$ we define,

$$Z(t, \omega) = \text{number of particles living at time } t,$$

$$Z(x, t, \omega) = \text{number of particles of age} \leq x \text{ living at time } t,$$

$$A(x, t, \omega) = Z(x, t, \omega)/Z(t, \omega),$$

$$\{x_i(t, \omega);\ i = 1, 2, \cdots, Z(t, \omega)\} = \text{the age chart at time } t,$$

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$Z_{x_i(t,\omega)}(x, s, \omega)$ = number of particles alive at time $t + s$ with ages $\leq x$, in a line of descent initiated by a particle of age $x_i(t, \omega)$ living at time t,

and $$Z_{x_i(t,\omega)}(s, \omega) = \lim_{x\to\infty} Z_{x_i(t,\omega)}(x, s, \omega).$$

For $x \geq 0$, $y \geq 0$, let $M(x, t) = E\{Z(x, t)\}$, $M(t) = E\{Z(t)\}$, $M_y(t) = E\{Z_y(t)\}$, $M_y(x, t) = E\{Z_y(x, t)\}$, and $m = \Sigma j p_j$. Also for $x \geq 0$, $y \geq 0$, let $G_y(x) = (G(x + y) - G(y))/(1 - G(y))$, $V(x) = m \int_0^\infty e^{-\alpha u} G_x(du)$,

$$n_1 = \left[\int_0^\infty e^{-\alpha t}(1 - G(t))\, dt\right] \Big/ \left[m \int_0^\infty t e^{-\alpha t}\, G(dt)\right],$$

$$A(x) = \left[\int_0^x e^{-\alpha t}(1 - G(t))\, dt\right] \Big/ \left[\int_0^\infty e^{-\alpha t}(1 - G(t))\, dt\right],$$

$$V_t = \int_0^\infty V(x) Z(dx, t, \omega) = \sum_{i=1}^{Z(t,\omega)} V(x_i(t, \omega)),$$

where α is the Malthusian parameter defined as the root of the equation $m \int_0^\infty e^{-\alpha t}\, G(dt) = 1$.

3. The theorem and three lemmas. The proof of the following theorem is based on a natural decomposition of $A(x, t)$ into three parts as in [1], and a separate lemma is proven for each part, the difference being that two of the three lemmas given below are stronger than those of [1] or [2].

THEOREM 3.1. *If $1 < m < \infty$, then*

$$\lim_{t\to\infty} \sup_{x\geq 0} |A(x, t, \omega) - A(x)| = 0 \quad \text{a.s.} \tag{3.1}$$

Before indicating the proof of Theorem 3.1, we will first define the decomposition, and then give the three corresponding lemmas.

Clearly one may write (suppressing subscripts)

$$Z(x, t + s) = \sum_{i=1}^{Z(t)} Z_{x_i}(x, s). \tag{3.2}$$

Also, as in [1], by defining

$$a_t(x, s) = \frac{1}{Z(t)} \sum_{i=1}^{Z(t)} [Z_{x_i}(x, s) - M_{x_i}(x, s)] e^{-\alpha s}, \tag{3.3}$$

$$b_t(x, s) = \frac{1}{Z(t)} \sum_{i=1}^{Z(t)} [M_{x_i}(x, s) e^{-\alpha s} - n_1 V(x_i) A(x)], \tag{3.4}$$

and

$$c_t = V_t / Z(t) \tag{3.5}$$

we have

$$A(x, t + s) = \frac{a_t(x, s) + b_t(x, s) + c_t A(x)}{a_t(\infty, s) + b_t(\infty, s) + c_t},$$

where $a_t(\infty, s)$ and $b_t(\infty, s)$ are the respective limits of $a_t(x, s)$ and $b_t(x, s)$ as $x \to \infty$. The following lemma and corollary are from [1].

LEMMA 3.1. *As $s \to \infty$, for fixed x,*

$$\sup_{y\geq 0}\{|M_y(x, s) e^{-\alpha s} - n_1 V(y) A(x)|, |M_y(\infty, s) e^{-\alpha s} - n_1 V(y)|\} \to 0. \tag{3.6}$$

COROLLARY 3.1. *As $s \to \infty$, for fixed x,*

(3.7) $$\sup_{t,\omega}\{|b_t(x, s)|, |b_t(\infty, s)|\} \to 0.$$

The next two lemmas are proved at the end of Section 4 as the corollaries of results in Section 4.

LEMMA 3.2. *If $1 < m < \infty$ and $\delta > 0$, then,*

(3.8) $$Y_{n\delta} = \frac{1}{Z(n\delta)} \sum_{i=1}^{Z(n\delta)} [Z_{x_i}(x, s) - M_{x_i}(x, s)] \to 0 \quad \text{a.s.}$$

In [1] it was shown that $Y_t \to 0$ in probability, if $1 < m < \infty$, and that $Y_{n\delta} \to 0$ a.s., if $\Sigma(j \log j)p_j < \infty$. In [2] it was shown that $Y_t \to 0$ a.s., if $1 < m < \infty$ and a certain tail condition on $G(\cdot)$ holds.

LEMMA 3.3. *If $1 < m < \infty$, then for some $\eta > 0$*

(3.9) $$\liminf_{t\to\infty} [V_t/Z(t)] > \eta \quad \text{a.s.}$$

It was shown in [1] that (3.9) holds if $\inf_{x\in \operatorname{supp} G} V(x) > 0$ or instead if $\Sigma(j \log j)p_j < \infty$.

PROOF OF THEOREM 3.1. The above lemmas show that if $\delta > 0$, then $A_{n\delta}(x) \to A(x)$ a.s. This fact and the continuity of $A(x)$, along with technical arguments in [1], give us Theorem 3.1.

4. A theorem based on an embedding and proofs of Lemmas 3.2 and 3.3. By considering a certain type of process embedded within the Bellman-Harris process, the following theorem is proved, which in turn implies Lemma 3.3.

THEOREM 4.1. *If $1 < m < \infty$, then for some C_1, C_2, both positive and finite,*

(4.1) $$\liminf_{t\to\infty}[Z(C_1, t)/Z(t)] > C_2 \quad \text{a.s.}$$

Before proving the above theorem, we shall need the concept of what we call a short term branching process. Without loss of generality, assume $G(t) < 1$ for $t < \infty$. (The theorem is trivially true if not.) As usual, $p_0 = 0$. Fix $K > 0$ such that

(4.2) $$G(K)\cdot m > 1, \quad \text{and}$$

(4.3) $$G(2K) - G(K) > 0.$$

With this particular K, for a particle born at time 0, define

(4.4) $\bar{Z}(\tau) =$ {number of particles alive at time τ, descended from the original particle, such that (1) each has a life-span $\le K$ in length, and (2) each of its ancestors, up to and including the original particle, had lifespan $\le K$}.

The above definition implies that $\bar{Z}(\tau)$ is itself a Bellman-Harris process (on set ancestor lives $\le K$) with lifetime distribution

(4.5) $$\bar{G}(x) = G(x)/G(K), \quad 0 \le x \le K,$$

and offspring distribution

(4.6) $$\bar{p}_n = \sum_{m=n}^{\infty} \binom{m}{m} (G(K))^n (1 - G(K))^{m-n} p_m, \qquad n = 0, 1, \cdots.$$

Evidently

(4.7) $$\sum n\bar{p}_n = G(K) \sum n p_n = G(K)m > 1,$$

so that $0 < \beta < 1$, where

$$\beta \equiv p(\bar{Z}(\tau) > 0, \quad \text{for all} \quad \tau > 0). \tag{4.8}$$

Another concept we need is that of particles of order n at time t. Recursively, define them as follows.

The *particles of order one* at time t are those particles ever born up to time t such that (1) their life-length is $>K$, (2) no ancestor (born at or after $t = 0$) has life-length $>K$.

We also define

$$Z_1(t) = \{\text{number of particles of order one born by time } t\}, \tag{4.9}$$

$$S_{1i}(\tau) = 2 \textstyle\sum_{j=1}^{J_i} \bar{Z}_{1i}^j(\tau) \tag{4.10}$$

where $\bar{Z}_{1i}^j(\tau)$ is the short term branching process at time τ after the birth of the jth of J_i progeny of the ith of $Z_1(t)$ particles of order one born by t, as well as

$$Y_{1i} = \text{life-length of } i\text{th particle of order one.} \tag{4.11}$$

We add that t is suppressed in some expressions for notational convenience, and $i = 1, 2, \cdots, Z_1(t)$.

Assume that *particles of order* n at time t have been defined and that this set is not null. Define *particles of order* $n + 1$ at time t to be those particles born up to time t such that (1) each has lifespan $>K$, and (2) each has an ancestor with lifespan $>K$ and the nearest such ancestor is a particle of order n.

Clearly one may define $Z_{n+1}(t)$, $S_{n+1,i}(\tau)$, $\bar{Z}_{n+1,i}^j(\tau)$, and $Y_{n+1,i}$ for particles of order $n + 1$ just as they were defined for particles of order one. Also note that if there are no particles of order n at time t there are none of higher order at time t.

Also define

$$X_{ni} = I_{[Y_{ni} \in (K, 2K]]} \cdot I_{[\liminf S_{ni}(\tau) > 0]} \tag{4.12}$$

for $i = 1, 2, \cdots, Z_n(t)$, $n = 1, 2, \cdots, [t/K] + 1$. We note that $Z_n(t)$ is void if $n > [t/K] + 1$. For notational simplicity let

$$n_t = [t/K] + 1 \quad \text{and} \tag{4.13}$$

$$N_t = \textstyle\sum_{n=1}^{n_t} Z_n(t). \tag{4.14}$$

The purpose of the previous definitions was to define the random variables $\{X_{ni}\}$ and $\{Z_n(t)\}$. The following lemmas concern distributional aspects of $\Sigma\Sigma X_{ni}$. These results will be crucial to the proof of Theorem 4.1. Before proceeding to the lemmas, more notation is needed. Let

$$\begin{aligned} p(x_{11}, \cdots, x_{n_t m_{n_t}}, m_1, \cdots, m_{n_t}) \\ = P(X_{11} = x_{11}, \cdots, X_{n_t m_{n_t}} = x_{n_t m_{n_t}}, Z_1(t) = m_1, \cdots, Z_{n_t}(t) = m_{n_t}). \end{aligned} \tag{4.15}$$

We note that some x are necessarily 0 if $m_i = 0$ for some $i \le n_t$. We also use $p(,\cdots,)$ for marginals. The following is trivial.

$$\begin{aligned} p(x_{11}, \cdots, x_{n_t m_{n_t}}, m_1, \cdots, m_{n_t}) &= p(x_{11}, \cdots, x_{1m_1} \mid m_1) \\ &\cdot p(x_{21}, \cdots, x_{2m_2} \mid m_2, x_{11}, \cdots, x_{1m_1}, m_1) \cdots \\ &\cdot p(x_{n_t 1}, \cdots, x_{n_t m_{n_t}} \mid m_{n_t}, x_{n_t - 1,1}, \cdots, x_{1m1}, m_1) \\ &\cdot p(m_1, \cdots, m_{n_t}). \end{aligned} \tag{4.16}$$

LEMMA 4.1. *On* $\{Z_i = m_i\}$ *for* $m_i \ge 1$,

$$p(x_{i1}, \cdots, x_{im_i} \mid m_i, x_{i-1,1}, \cdots, x_{11}, m_1) = p^{\sum_{j=1}^{m_i} x_{ij}} (1 - p)^{m_i - \sum_{j=1}^{m_i} x_{ij}}. \tag{4.17}$$

In (4.17), p has the value

$$p = [(G(2K) - G(K))/(1 - G(K))]\cdot[1 - E(1 - \beta)^J] \tag{4.18}$$

where J is the random number of offspring at split and β is as in (4.8).

PROOF (Lemma 4.1). Clearly $X_{i1}, \cdots, X_{im_i}$ are Bernoulli (p) random variables for p as in (4.18). Using definition 2.3 from [3], one may define particles by sequences; $\langle i_1, \cdots, i_k\rangle$, for example, is a particle of generation $k + 1$. Now if we condition on event that $\{\alpha_1, \cdots, \alpha_{m_i}$ (α's represent sequences) are all and only particles of order i at time t, and $x_{i-1,1}, \cdots, x_{11}, m_1$ also occur$\}$ then conditioned on this set

$$p(x_{i1}, \cdots, x_{im_i}) = p^{\sum_{j=1}^{m_i} x_{ij}}(1 - p)^{m_i - \sum_{j=1}^{m_i} x_{ij}},$$

since the future is conditionally independent of the past. Unconditioning on the particular α's gives (4.17). □

LEMMA 4.2.

$$p(x_{11}, \cdots, x_{n_t}, m_{n_t} \mid m_1, \cdots, m_{n_t}) = p^{\sum_{n=1}^{n_t}\sum_{i=1}^{m_n} x_{ni}}(1 - p)^{N_t - \sum_{n=1}^{n_t}\sum_{i=1}^{m_n} x_{ni}}. \tag{4.19}$$

PROOF. Use Lemma 4.1 and equation (4.16). By Chebyshev, the following holds:

$$p(|[(\textstyle\sum_{n=1}^{n_t}\sum_{i=1}^{m_n} X_{ni})/N_t] - p| > \varepsilon \mid m_1, m_2, \cdots, m_{n_t}) \le p(1 - p)/\varepsilon^2 N_t. \tag{4.20}$$

Since N_t is really an overestimate of the number of particles of age $>K$ at time $t + K$, one may argue, by means of results of Jagers [4], that for any $\delta > 0$, there exists M_δ finite and $C > 1$ such that

$$N_{m\cdot\delta} > C^m \quad \text{a.s.} \tag{4.21}$$

if $m \ge M_\delta$. Consequently, we obtain

LEMMA 4.3. *For* $\delta > 0$,

$$P([\textstyle\sum_{n=1}^{n_t}\sum_{i=1}^{m_n} X_{ni}]/N_t < p/2 \quad \text{i.o.}, \qquad t = m\delta) = 0. \tag{4.22}$$

PROOF. Let $\varepsilon = p/2$. Since $M_\delta < \infty$ a.s. using (4.20)

$$\textstyle\sum_{m=M_\delta}^{\infty} P([\sum_{n=1}^{n_t}\sum_{i=1}^{m_n} X_{ni}]/N_t < p/2,\ t = m\delta) \le \sum_{m=1}^{\infty} [p(1 - p)]/[(\varepsilon/2)^2 C^m] < \infty. \tag{4.23}$$

Borel-Cantelli gives the rest. □

Now the tools are available to prove Theorem 4.1.

PROOF OF THEOREM 4.1. It is easy to see that equation (4.1) is equivalent to showing, for some C_1', C_2', that

$$\liminf_{t\to\infty}[Z(C_1', t)/[Z(t) - Z(C_i', t)]] > C_2' > 0 \quad \text{a.s.} \tag{4.24}$$

Now note that by definition of X_{ni} (equation (4.12)),

$$\textstyle\sum_{n=1}^{n_t}\sum_{i=1}^{Z_n(t)} X_{ni} \le Z(2K, t + 2K) \tag{4.25}$$

and

$$N_t \ge Z(t + 2K) - Z(2K, t + 2K). \tag{4.26}$$

This yields the lower bound

(4.27) $[\sum_{n=1}^{n_t} \sum_{i=1}^{Z_n(t)} X_{ni}]/N_t \leq Z(2K, t+2K)/[Z(t+2K) - Z(2K, t+2K)]$.

Directly from Lemma 4.3 we may infer that, if $\delta > 0$

(4.28) $P(Z(2K, t+2K)/[Z(t+2K) - Z(2K, t+2K)] < p/2 \quad \text{i.o.} \quad t = m\cdot\delta) = 0$.

It is not difficult to show that

(4.29) $$\liminf_{t\to\infty} Z(4K, t)/[Z(t) - Z(4K, t)] \geq \liminf_{m\to\infty} Z(2K, m\cdot\delta)/[Z(m\delta) - Z(2K, m\delta)] \quad \text{a.s.}$$

Therefore with $C_1' = 4K$, $C_2' = p/2$ we have (4.24). □

Proof of Lemma 3.3. If we let

(4.30) $$a = \inf_{x\in[0,C_1]} V(x) > 0$$

then

(4.31) $$\liminf_{t\to\infty} V_t/Z(t) \geq a \liminf_{t\to\infty} Z(C_1, t)/Z(t)$$

(4.32) $$\geq a\cdot C_2 \quad \text{a.s.}$$ □

Corollary 4.1. *For some $K' > 0$, there is a constant $C' > 1$, such that*

(4.33) $$\liminf_{t\to\infty} Z(t+K')/Z(t) > C' \quad \text{a.s.}$$

Proof. See Lemma 4.5 of [2] for details. □

Proof (Lemma 3.2). In view of corollary 4.1, this follows by the proof of Lemma 3.2 in [2]. □

5. Remarks. Independently of the preceeding work, O. Nerman (Nerman [5]) has obtained similar results for the Crump-Mode-Jagers or generalized branching process. Let

(5.1) $\xi(t)$ = number of offspring born to a particle by time t, which itself was born at time 0,

and α be such that

(5.2) $$E\int_0^\infty e^{-\alpha t}\,\xi(dt) = 1.$$

If in addition, one assumes that for some $\beta < \alpha$

(5.3) $$E\int_0^\infty e^{-\beta t}\,\xi(dt) < \infty,$$

then the corresponding age result holds. In particular, for B-H processes this implies Theorem 3.1.

On the other hand, Theorem 4.1 evidently holds for the generalized process without the assumption (5.3), needed in Nerman. In fact, choosing K, K′ satisfying the analogous equations to (4.2) and (4.3), one may bound the tail of the age distribution (asymptotically) for the generalized process, again without (5.3). However, this does not imply the ages result for the generalized process without (5.3), which is still open (and probably true).

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