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ANOTHER VERSION OF STRASSEN'S LOG LOG LAW WITH AN APPLICATION TO APPROXIMATE UPPER FUNCTIONS OF A GAUSSIAN PROCESS WITH A POSITIVE INDEX

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Let $\{Y(t, \omega) = (X_1(t, \omega), \cdots, X_d(t, \omega)); 0 \le t \le 1\}$ be a d-dimensional Gaussian process whose components are independent copies of a Gaussian process with index α; that is, $E[X(t, \omega)] = 0$, $X(0, \omega) = 0$, and $E[(X(t, \omega) - X(s, \omega))^2] = \sigma^2(|t - s|)$, where $\sigma(t) = t^\alpha$, $0 < \alpha < 1$. Let $h(t)$ be a positive, non-increasing, continuous function and set

$$q = \sup\left\{r \ge 0; \int_{+0} e^{-rh^2(t)/2}\, dt/t = +\infty\right\}.$$

Then, as an application of a version of Strassen's log log law, we have

$$\limsup_{t \downarrow 0} t^{-1} m(\{0 \le s \le t; \| Y(s, \omega) \| > \sigma(s) h(s)\}) = \sup_{x \in B} m(\{0 \le s \le 1; \| x(s) \| \ge \sigma(s)/\sqrt{q}\}), \quad \text{a.s.},$$

where $\| \ \|$ denotes the usual Euclidean norm, $m(\Gamma)$ denotes the Lebesgue measure of a linear set Γ, and B is the unit ball of the direct sum of d copies of the reproducing kernel Hilbert space with the kernel $R(s, t) = (\sigma^2(t) + \sigma^2(s) - \sigma^2(|t - s|))/2$. In case of the d-dimensional Brownian motion, Strassen [7] had proved that the right-hand side of the above formula is equal to $1 - \exp\{-4(q - 1)\}$ if $q \ge 1$, and 0 if $q \le 1$.

As a corollary, $\sigma(t)h(t)$ is an approximate upper function as introduced by D. Geman [2] if and only if $q \le 1$. Especially, if $\lim_{t \downarrow 0} h(t)/\sqrt{2 \log \log 1/t} = c$, $\sigma(t)h(t)$ is an approximate upper function if and only if $c \ge 1$.

1. Introduction. In [2], D. Geman introduced the notion of approximate upper and lower functions of a stochastic process; locally, the former can be thought of as a modulus of approximate continuity of the sample paths, whereas the latter provides a lower bound on the growth of the path.

Let $\{X^d(t, \omega); 0 \le t \le 1\}$ be a measurable stochastic process defined on a probability space taking the value in R^d, d-dimensional Euclidean space with the Euclidean norm $\| \ \|$. A continuous function $\varphi(t)$ is called an approximate upper function (at $t = 0$) if, with probability 1,

$$\lim_{t \downarrow 0} t^{-1} m(\{0 \le s \le t; \| X^d(s, \omega) - X^d(0, \omega) \| > \varphi(s)\}) = 0,$$

where $m(\Gamma)$ denotes the Lebesgue measure of a linear set Γ.

Among many results in [2] and [3], Geman has shown that for any real-valued, centered Gaussian process, if $h \equiv \varphi/\sigma \uparrow +\infty$ as $t \downarrow 0$, then φ is an approximate upper function whenever

$$\int_{+0} h(t)^{-1} e^{-h^2(t)/2}\, dt/t < +\infty$$

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holds. (In particular, $\varphi(t) = \sigma(t)\sqrt{2 \log_{(2)} 1/t + \gamma \log_{(3)} 1/t}$, $\gamma > 1$.)

In this paper, we will focus our concern on a d-dimensional Gaussian process with positive index, which means that each component is an independent copy of a centered, path continuous Gaussian process with $X(0) = 0$ and $E[(X(t) - X(s))^2] = \sigma^2(|t - s|) = |t - s|^{2\alpha}$, $0 < \alpha < 1$.

For the 1-dimensional case of this class, Geman [2] showed that $\sigma(t)\sqrt{2 \log_{(2)} 1/t - \gamma \log_{(3)} 1/t}$, $\gamma < -1 + 1/\alpha$, is an approximate upper function. Even in the d-dimensional case, however, we will prove a much better result than this as a corollary of our theorem; namely, for $h \uparrow +\infty$ as $t \downarrow 0$, σh is an approximate upper function if and only if $q \leq 1$, where

$$q = \sup\left\{ r \geq 0;\ \int_{+0} e^{-rh^2(t)/2}\, dt/t = +\infty \right\}.$$

Especially, if $\lim_{t\downarrow 0} h(t)/\sqrt{2 \log \log 1/t} = c$, σh is an approximate upper function if and only if $c \geq 1$.

2. Main results. To describe our results, we need two function spaces: the space C of all d-dimensional continuous functions defined on $[0, 1]$ with the sup norm $\| x \|_C = \sup_{0\leq t\leq 1} \| x(t) \|$, and a Hilbert space $K = H \oplus \cdots \oplus H$ (d copies), the direct sum of the reproducing kernel Hilbert space H with the reproducing kernel

$$R(s, t) = E[X(s)X(t)] = (\sigma^2(t) + \sigma^2(s) - \sigma^2(|t - s|))/2,$$

where $\sigma(t) = t^\alpha$, $0 < \alpha < 1$. It is well known that for $x \in K$, $x(t) = (x_1(t), \cdots, x_d(t))$ is an element of C and

$$\| x \|_C^2 = \sup_{0\leq t\leq 1} \sum_{i=1}^d (x_i(\cdot), R(t, \cdot))_H^2 \leq \sup_{0\leq t\leq 1} R(t, t) \sum_{i=1}^d \| x_i \|_H^2 = \| x \|_K^2, \tag{2.1}$$

where we denote by $\| \; \|_H$ and $\| \; \|_K$ the norms in H and K respectively.

Let $\{Y(t, \omega);\ 0 \leq t \leq 1\}$ be a d-dimensional Gaussian process whose components are independent copies of the Gaussian process with index α, $0 < \alpha < 1$, mentioned in Section 1, and let $h(t)$ be a non-increasing, positive, continuous function defined on the positive half-line.

Our first theorem is analogous to that of Strassen [7] and Oodaira [6].

THEOREM 1. *Assume that for any* $\varepsilon \neq 0$,

$$\int_{+0} e^{-(1+\varepsilon)h^2(t)/2}\, dt/t < +\infty, \qquad \text{or} \quad = +\infty$$

depending on whether $\varepsilon > 0$, *or* $\varepsilon < 0$. *Then the random set in C given by* $\{f_n(t, \omega) = Y(t/n, \omega)/(\sigma(1/n)h(1/n));\ n = 1, 2, \cdots\}$ *is, with probability* 1, *relatively compact in C and the set of all limit points coincides with the unit ball of K.*

Applying Theorem 1 to obtain approximate upper functions for Gaussian sample paths, we have the following. Following Uchiyama (who, in a private communication, has applied this criterion to obtain the same result in case of the Brownian motion, although his proof is completely different from ours), set

$$q = \sup\left\{ r \geq 0,\ \int_{+0} e^{-rh^2(t)/2}\, dt/t = +\infty \right\}. \tag{2.2}$$

THEOREM 2. (i) *With probability* 1,

$$\begin{aligned}
&\lim_{t\downarrow 0}\sup t^{-1}m(\{0 \le s \le t; \| Y(s,\omega)\| > \sigma(s)h(s)\}) && \\
(2.3)\qquad &= \sup_{x\in B} m(\{0 \le s \le 1; \|x(s)\| \ge \sigma(s)/\sqrt{q}\}) && \text{if } 0 < q < +\infty, \\
&= 1 && \text{if } q = +\infty, \\
&= 0 && \text{if } q = 0,
\end{aligned}$$

where B is the unit ball of the Hilbert space K.

(ii) *The function*

$$F(q) = \sup_{x\in B} m(\{0 \le s \le 1; \|x(s)\| \ge \sigma(s)/\sqrt{q}\})$$

is continuous for $q > 0$; in particular, positive strictly increasing for $q > 1$ with $\lim_{q\uparrow+\infty} F(q) = 1$ *and $F(q) = 0$ for $q \le 1$.*

COROLLARY. *As a corollary of Theorem* 2, *it follows that σh is an approximate upper function if and only if $q \le 1$, where q is defined by* (2.2).

REMARK. In case of one-dimensional Brownian motion, Strassen [7] had shown that

$$\begin{aligned}
\sup_{x\in B} m(\{0 \le s \le 1; \|x(s)\| \ge \sqrt{s}/\sqrt{q}\}) &= 1 - e^{-4(q-1)} && \text{if } q \ge 1, \\
&= 0 && \text{if } 0 < q \le 1.
\end{aligned}$$

Nothing changes in the higher dimensional case: Uchiyama [8] has proved that the left-hand side of (2.3) in Theorem 2 is equal to $1 - e^{-4(q-1)}$ if $q \ge 1$ and 0 of $q \le 1$ in case of d-dimensional Brownian motion applying the methods of diffusion processes.

3. Stochastic version: Proof of Theorem 1. The proof of Theorem 1 splits into the following two lemmas.

LEMMA 1. *Assume that for any* $1 > \varepsilon > 0$

$$(3.1)\qquad \int_{+0} e^{-(1+\varepsilon)h^2(t)/2}\, dt/t < +\infty.$$

Then, with probability 1, $\{f_n(t,\omega) = Y(t/n,\omega)/(\sigma(1/n)h(1/n));\ n = 1, 2, \cdots\}$ *is relatively compact in C and the set of all limit points is included in the unit ball B of K.*

LEMMA 2. *In addition to* (3.1), *assume that for any* $1 > \varepsilon > 0$

$$(3.2)\qquad \int_{+0} e^{-(1-\varepsilon)h^2(t)/2}\, dt/t = +\infty.$$

Then, with probability 1, *the set of all limit points of* $\{f_n(t,\omega);\ n = 1, 2, \cdots\}$ *coincides with B.*

The main difference between our Lemma 1 and Theorem 1 of [5], or Theorems 1 and 2 of [6], is that our function h is not necessarily comparable with the function $\sqrt{\log\log 1/t}$; therefore, we need to modify their proof.

Before starting the proof of Lemmas 1 and 2, we notice that (3.1) and (3.2) are equivalent to the following statements respectively: for any $1 > \varepsilon > 0$ and any j

$$(3.1)'\qquad \int_{+0} h^j(t)e^{-(1+\varepsilon)^2h^2(t)/2}\, dt/t < +\infty,$$

and

$$(3.2)' \qquad \int_{+0} h^j(t)e^{-(1-\varepsilon)^2h^2(t)/2}\,dt/t = +\infty.$$

PROOF OF LEMMA 1. For any $\varepsilon > 0$, we can choose $\delta > 1$ and a positive integer q such that the following inequalities are fulfilled;

$$\sigma(n_{r+1}/n_r) \le 1+\varepsilon, \qquad \sigma(n_{r+1}/n_r) - (1+\varepsilon)^{-1} \le 2\varepsilon, \qquad \sigma(1-n_r/n_{r+1}) \le \varepsilon,$$

and

$$(3.3) \qquad \sigma(n_q n_r/n_{r+1}) \ge 2(1+\varepsilon)\varepsilon^{-1},$$

where $n_r = [\delta^r]$, the largest integer not exceeding δ^r.

Step 1. With probability 1, $\{f_n(t,\omega); n = 1, 2, \cdots\}$ is equicontinuous in C. In fact, set

$$A_r = \{\omega;\ \sup_{n_r\le m\le n_{r+1}}\sup_{|t-s|\le n_q^{-1}}\|\, Y(t/m,\omega) - Y(s/m,\omega)\,\| \ge \varepsilon\sigma(1/n_{r+1})h(1/n_r)\}.$$

Then, we have

$$\begin{aligned} A_r &\subset \{\omega;\ \sup_{0\le t\le t+h\le n_r^{-1}, 0\le h\le n_r^{-1}n_q^{-1}}\|\, Y(t+h,\omega) - Y(t,\omega)\,\| \\ &\qquad \ge \varepsilon\sigma(1/n_{r+1})h(1/n_r)\} \\ &\subset \bigcup_{k=0}^{n_q-1} \{\omega;\ \sup_{kn_r^{-1}n_q^{-1}\le t\le (k+1)n_r^{-1}n_q^{-1}, 0\le h\le n_r^{-1}n_q^{-1}}\|\, Y(t+h,\omega) - Y(t,\omega)\,\| \\ &\qquad \ge \varepsilon\sigma(1/n_{r+1})h(1/n_r)\} \\ &\equiv \bigcup_{k=0}^{n_q-1} A_{r,k}. \end{aligned}$$

Now, in order to apply Lemma 3 of [4] which is an extension of Fernique's inequality, set

$$S = \{(u,v);\ 0\le v\le u\le n_r^{-1}n_q^{-1},\ 0\le u-v\le n_r^{-1}n_q^{-1}\},$$

and

$$X^d(u,v,\omega) = Y(u,\omega) - Y(v,\omega).$$

Since we have

$$\begin{aligned} &E[(X_i(u,\omega) - X_i(v,\omega) - X_i(u',\omega) + X_i(v',\omega))^2] \\ &\le 2E[(X_i(u,\omega) - X_i(u',\omega))^2 + (X_i(v,\omega) - X_i(v',\omega))^2] \\ &= 2(|u-u'|^{2\alpha} + |v-v'|^{2\alpha}) \\ &\le 4(\sqrt{|u-u'|^2 + |v-v'|^2})^{2\alpha}, \end{aligned}$$

and

$$E[(X_i(u,\omega) - X_i(v,\omega))^2] = |u-v|^{2\alpha} \le (n_r n_q)^{-2\alpha},$$

it follows from Lemma 3 of [4] that

$$P(\sup_{(u,v)\in S}\|\, X^d(u,v,\omega)\,\| \ge 2(n_r n_q)^{-\alpha}x) \le c_1 x^{d-2}e^{-x^2/2}$$

for sufficiently large x, where c_1 is a constant independent of x, r and q. Applying this inequality to have an upper bound of $P(A_{r,k})$ with the definition of q, we have

$$P(A_r) \le \sum_{k=0}^{n_q-1} P(A_{r,k}) \le n_q c_1(1+\varepsilon)^{d-2}h^{d-2}(1/n_r)e^{-(1+\varepsilon)^2h^2(1/n_r)/2},$$

and

$$\begin{aligned} \sum_r^\infty P(A_r) &\le n_q c_1(1+\varepsilon)^{d-2}\sum_r^\infty h^{d-2}(1/n_r)e^{-(1+\varepsilon)^2h^2(1/n_r)/2} \\ &= n_q c_1(1+\varepsilon)^{d-2}\sum_r^\infty h^{d-2}(1/n_r)e^{-(1+\varepsilon)^2h^2(1/n_r)/2}\, n_{r-1}(1-n_{r-1}/n_r)^{-1}(n_{r-1}^{-1} - n_r^{-1}) \\ &\le c_2\int_{+0} h^{d-2}(t)e^{-(1+\varepsilon)^2h^2(t)/2}\,dt/t < +\infty, \qquad \text{by (3.1)}'. \end{aligned}$$

Therefore, by the Borel-Cantelli lemma, with probability 1, there exists $r_1 = r_1(\varepsilon, \omega)$ such that

$$\| f_m(t, \omega) - f_m(s, \omega) \| < \varepsilon \tag{3.4}$$

holds for any $m \geq n_{r_1}$ and $|t - s| < n_q^{-1}$.

Before entering the next step, we need some notation. Let $\{e_j(t);\ j = 1, 2, \cdots\}$ be a complete orthonormal system of H and let $\psi_n^{(i)}$ be the isometric isomorphism defined by $\psi_n^{(i)} R(t, \cdot) = X_i(t/n)/\sigma(1/n)$ between H and the closed linear subspace $L_2^{(n,i)}$ spanned by $\{X_i(t/n)/\sigma(1/n);\ 0 \leq t \leq 1\}$, recall that $X_i(t)$ is the ith component of $Y(t)$. Then, $\xi_n^{(k,i)}(\omega) \equiv \psi_n^{(i)}(e_k)$, $k = 1, 2, \cdots, i = 1, \cdots, d$, are independent standard normal random variables. Set

$$Z_r(t, \omega) = (Z_r^{(1)}(t, \omega), \cdots, Z_r^{(d)}(t, \omega)),$$

where

$$Z_r^{(i)}(t, \omega) = \textstyle\sum_{k=1}^{j} \xi_{n_r}^{(k,i)}(\omega) e_k(t),$$

and j is defined in the next step depending on ε.

Step 2. With probability 1, there exist integers $j = j(\varepsilon)$, independent of ω, and $r_2 = r_2(\varepsilon, \omega)$ such that

$$\| f_{n_r}(t, \omega) - Z_r(t, \omega)/h(1/n_r) \|_C < \varepsilon \tag{3.5}$$

holds for all $r \geq r_2$.

In fact, it is well known that there exists $j_0 = j_0(\varepsilon)$ such that

$$\sup_{0 \leq t \leq 1} | R(t, t) - \textstyle\sum_{k=1}^{j} e_k^2(t) | < \varepsilon$$

holds for all $j \geq j_0$. Now, set

$$B_r = \{\omega;\ \| W(t, \omega) \|_C \geq \varepsilon\, h(1/n_r)\},$$

where $W(t, \omega) = (W_1(t, \omega), \cdots, W_d(t, \omega))$ is defined by

$$W_i(t, \omega) = X_i(t/n_r, \omega)/\sigma(1/n_r) - Z_r^{(i)}(t, \omega).$$

Then, $\{W_i(t, \omega);\ i = 1, \cdots, d\}$ are independent centered Gaussian random processes such that

$$E[W_i^2(t, \omega)] = R(t, t) - \textstyle\sum_{k=1}^{j} e_k^2(t) \leq \sup_{0 \leq t \leq 1} \sum_{k=j+1}^{\infty} e_k^2(t) \equiv \Gamma_j^2$$

and

$$E[(W_i(t, \omega) - W_i(s, \omega))^2] = \textstyle\sum_{k=j+1}^{\infty} (e_k(t) - e_k(s))^2$$
$$\leq \sup_{|t-s|=h, 0 \leq t,s \leq 1} \textstyle\sum_{k=j+1}^{\infty} (e_k(t) - e_k(s))^2 \equiv \sigma_j^2(h).$$

Therefore, again by Lemma 3 of [4], we have

$$P(\| W(t, \omega) \|_C \geq x(\Gamma_j + 4 \int_0^\infty \sigma_j(e^{-u^2})\, du))$$
$$\leq c_3 x^{d-2} e^{-x^2/2},$$

where c_3 is a constant independent of x and j. In this inequality, choosing sufficiently large j such that $(1 + \varepsilon)(\Gamma_j + 4 \int_0^\infty \sigma_j(e^{-u^2})\, du) \leq \varepsilon$, and setting $x = (1 + \varepsilon) h(1/n_r)$, we have

$$P(B_r) \leq c_3 (1 + \varepsilon)^{d-2} h^{d-2}(1/n_r) e^{-(1+\varepsilon)^2 h^2(1/n_r)/2},$$

and

$$\sum_r^\infty P(B_r) \le c_3(1+\varepsilon)^{d-2} \sum_r^\infty h^{d-2}(1/n_r) e^{-(1+\varepsilon)^2 h(1/n_r)/2} \times n_{r-1}(1 - n_{r-1}/n_r)^{-1}(n_{r-1}^{-1} - n_r^{-1})$$
$$\le c_4 \int_{+0} h^{d-2}(t) e^{-(1+\varepsilon)^2 h^2(t)/2} dt/t < +\infty.$$

Applying the Borel-Cantelli lemma, we have (3.5).

Step 3. With probability 1, $\{Z_r(t, \omega)/h(1/n_r); r = 1, 2, \cdots\}$ is pre-compact in C and all limit points are contained in the unit ball B of K.

In fact, set

$$C_r = \{\omega : \| Z_r(t, \omega) \|_K > (1+\varepsilon) h(1/n_r)\}.$$

Since we have

$$\| Z_r(t, \omega) \|_K^2 = \sum_{i=1}^d \| Z_r^{(i)}(t, \omega) \|_H^2 = \sum_{i=1}^d \sum_{k=1}^j | \xi_{n_r}^{(k,i)}(\omega) |^2$$

and $\{\xi_{n_r}^{(k,i)}(\omega); k = 1, \cdots, j, i = 1, \cdots, d\}$ are independent standard normal random variables, it follows that

$$\sum_r^\infty P(C_r) \le c_5 \sum_r^\infty (1+\varepsilon)^{dj} h^{dj}(1/n_r) e^{-(1+\varepsilon)^2 h^2(1/n_r)/2}$$
$$\le c_6 \int_{+0} h^{dj}(t) e^{-(1+\varepsilon)^2 h^2(t)/2} \, dt/t < +\infty.$$

By the Borel-Centelli lemma, with probability 1, there exists an $r_3 = r_3(\varepsilon, \omega)$ such that

$$\| Z_r(t, \omega) \|_K \le (1+\varepsilon) h(1/n_r) \tag{3.6}$$

holds for any $r \ge r_3$.

Step 4. Conclusion. Since $f_n(0, \omega) = 0$ for all $n = 1, 2, \cdots$ and for any ω, $\{f_n(t, \omega); n = 1, 2, \cdots\}$ is pre-compact in C by Step 1. Now, consider the following triangular inequality for $r \ge r_4 = \max(r_1, r_2, r_3)$ and $n_r \le m < n_{r+1}$;

$$\left\| f_m(t, \omega) - \frac{Z_r(t, \omega)}{h(1/m)(1+\varepsilon)} \right\|_C \le \frac{\sigma(1/n_r) h(1/n_r)}{\sigma(1/m) h(1/m)} \left\| f_{n_r}\left(\frac{n_r}{m} t, \omega\right) - Z_r\left(\frac{n_r}{m} t, \omega\right) / h(1/n_r) \right\|_C$$
$$+ \frac{\sigma(1/n_r)}{\sigma(1/m) h(1/m)} \left\| Z_r\left(\frac{n_r}{m} t, \omega\right) - Z_r(t, \omega) \right\|_C$$
$$+ \left(\frac{\sigma(1/n_r)}{\sigma(1/m)} - \frac{1}{1+\varepsilon} \right) \| Z_r(t, \omega) \|_C / h(1/m)$$
$$\equiv I_1 + I_2 + I_3.$$

From (3.3), (3.5), and monotonicity of h, we have $I_1 \le \varepsilon(1+\varepsilon)$. Using the reproducing property of the kernel Hilbert space, monotonicity of h, (3.3) and (3.6), we have

$$I_2 \le \sigma(n_{r+1}/n_r) \sigma(1 - n_r/n_{r+1}) \| Z_r(t, \omega) \|_K / h(1/n_r) \le (1+\varepsilon)^2 \varepsilon.$$

Finally, from (2.1), (3.3), (3.6) and monotonicity of h, we have $I_3 \le 2\varepsilon(1+\varepsilon)$, and $I_1 + I_2 + I_3 \le 10\varepsilon$ because of $\varepsilon < 1$. Since, with probability 1, we have $(1+\varepsilon)^{-1} Z_r(t, \omega)/h(1/m) \in B$ by (3.6), it follows that $f_m(t, \omega) \in B_{10\varepsilon}$, the 10ε-neighborhood of B in C. This completes the proof of Lemma 1.

PROOF OF LEMMA 2. The proof of Lemma 2 is much more complicated than that of Lemma 1; still, it is routine work for Gaussian processes. It seems that Oodaira's or Lai's proof for the corresponding result to Lemma 2 does not work for our case, but the ideas are similar. Because of separability of K, it is sufficient to prove that for any $x \in B$ such that $0 < \|x\|_K < 1$, with probability 1, x is a limit point of $\{f_n(t, \omega) = Y(t/n, \omega)/(\sigma(1/n)h(1/n)); n = 1, 2, \cdots\}$. Now, fix any $\varepsilon > 0$ such that $1 - \varepsilon > \|x\|_K > 4\varepsilon$. For $0 < a < 1$, we denote by H_a, K_a, and C_a the restriction to $[a, 1]$ of H, K, and C respectively, we choose a so small that

$$\|x\|_K - \|x\|_{K_a} < \varepsilon, \tag{3.7}$$

and

$$\sup_{0\le t\le a}\|y(t)\| \le \sigma(a)\|y\|_K \le \varepsilon\|y\|_K \tag{3.8}$$

for any $y \in K$. Then, it follows by (3.8) that

$$\sup_{0\le t\le a}\|y(t)\| \le 11\varepsilon \tag{3.9}$$

holds for any $y \in B_{10\varepsilon}$, the 10ε-neigoborhood of B in C. Consider the compact operator $\varphi \to \int_a^1 R(t, s)\varphi(s)\, ds$ on $L^2([a, 1], ds)$ and let $\{\varphi_k : k = 1, 2, \cdots\}$ be normalized eigenfunctions corresponding to the eigenvalues $\{\lambda_1 \ge \lambda_2 \ge \cdots > 0\}$. That is,

$$\int_a^1 R(t, s)\varphi_k(s)\, ds = \lambda_k\varphi_k(t), \qquad a \le t \le 1, \tag{3.10}$$

and

$$\begin{aligned} \int_a^1 \varphi_k(s)\varphi_{k'}(s)\, ds &= 1, \qquad \text{if } k = k', \\ &= 0, \qquad \text{if } k \ne k'. \end{aligned} \tag{3.11}$$

We notice that not only is φ_k an element of H_a but also $\{\sqrt{\lambda_k}\ \varphi_k;\ k = 1, 2, \cdots\}$ forms a C.O.N.S of H_a.

Next, in order that the following Step 1 and Step 4 go well, we have to choose a positive integer j sufficiently large such that

$$(1 + \varepsilon)(\bar{\Gamma}_j + 4\int_0^\infty \bar{\sigma}_j(e^{-u^2})\, du) \le \varepsilon, \tag{3.12}$$

where

$$\bar{\Gamma}_j = \sup_{a\le t\le 1}(R(t, t) - \textstyle\sum_{k=1}^{j} \lambda_k\varphi_k^2(t)) = \sup_{a\le t\le 1}(\sum_{k=j+1}^{\infty} \lambda_k\varphi_k^2(t)),$$

and

$$\bar{\sigma}_j(h) = \sup_{|t-s|=h, a\le t,s\le 1} \textstyle\sum_{k=j+1}^{\infty} \lambda_k(\varphi_k(t) - \varphi_k(s))^2,$$

and such that

$$\|x - \bar{x}\|_{K_a} < \varepsilon, \tag{3.13}$$

where $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_d)$ is the jth partial sum of the expansion of $x = (x_1, \cdots, x_d)$ in K_a; that is,

$$x_i = \textstyle\sum_{k=1}^{\infty} x_i^{(k)}\ \sqrt{\lambda_k}\ \varphi_k \text{ in } H_a,$$

and

$$\bar{x}_i = \textstyle\sum_{k=1}^{j} x_i^{(k)}\ \sqrt{\lambda_k}\ \varphi_k.$$

Finally, we need to choose a positive integer Δ such that

$$\Delta \ge 2/a, \quad 3\lambda_j^{-1}a^{-\beta}j(1 + \varepsilon)\Delta^{-\beta} \le \varepsilon, \tag{3.14}$$

and

$$27(1 + 2\varepsilon)^2\lambda_j^{-2}a^{-2\beta}j(\|x\|_K - 3\varepsilon)^{-2}\Delta^{-2\beta} \le (\|x\|_K - 4\varepsilon)^2/4 \tag{3.15}$$

hold, where $\beta = \min(\alpha, 1-\alpha)$.

In the sequel, we use abbreviations $n_r = \Delta^r$, $r = 1, 2, \cdots$, and $h_r = h(1/n_r)$.

Step 1. Set

$$\eta_r^{(k,i)}(\omega) = \frac{\int_a^1 X_i(s/n_r, \omega)\varphi_k(s)\, ds}{\sqrt{\lambda_k}\,\sigma(1/n_r)},$$

$k = 1, 2, \cdots, j$, $i = 1, \cdots, d$, and

$$\bar{Z}_r(t, \omega) = (\bar{Z}_r^{(1)}(t, \omega), \cdots, \bar{Z}_r^{(d)}(t, \omega)),$$

where

$$\bar{Z}_r^{(i)}(t, \omega) = \textstyle\sum_{k=1}^{j} \sqrt{\lambda_k}\,\varphi_k(t)\eta_r^{(k,i)}(\omega).$$

Then, with probability 1, there exists $r_1 = r_1(\varepsilon, \omega)$ such that

$$\| f_{n_r}(t, \omega) - \bar{Z}_r(t, \omega)/h_r \|_{C_a} < \varepsilon \tag{3.16}$$

holds for all $r \geq r_1$.

In fact, $\{\eta_r^{(k,i)}(\omega);\, k = 1, 2, \cdots, j, i = 1, \cdots, d\}$ are independent standard normal random variables such that

$$\begin{aligned} E[X_i(t/n_r)\eta_r^{(k,i)}] &= \int_a^1 R(t/n_r, s/n_r)\varphi_k(s)\, ds/(\sqrt{\lambda_k}\,\sigma(1/n_r)) \\ &= \sqrt{\lambda_k}\,\varphi_k(t)\sigma(1/n_r). \end{aligned}$$

On the other hand, by Mercer's theorem, we have

$$\textstyle\sum_{k=1}^{\infty} \lambda_k\varphi_k(t)\varphi_k(s) = R(t, s),$$

where the convergence is uniform in s and t. Therefore, by an analogous way with that of Step 2 of Lemma 1 using (3.12), we have (3.16).

Step 2. Set

$$D_r = \{\omega;\, \|\bar{Z}_r(\cdot, \omega)/h_r - \bar{x}(\cdot)\|_{K_a} \leq \varepsilon\}.$$

Then, we have

$$\textstyle\sum_{r=1}^{\infty} P(D_r) = +\infty. \tag{3.17}$$

Since we have $\sum_{i=1}^{d}\sum_{k=1}^{j} |x_i^{(k)}|^2 \leq \|x\|_{K_a}^2 \leq \|x\|_K^2$, it follows that

$$\begin{aligned} P(D_r) &= P(\{\omega;\, \textstyle\sum_{i=1}^{d}\sum_{k=1}^{j} (\eta_r^{(k,i)}(\omega)/h_r - x_i^{(k)})^2 \leq \varepsilon^2\}) \\ &\geq \textstyle\prod_{i=1}^{d}\prod_{k=1}^{j} P(\{\omega;\, |\eta_r^{(k,i)}(\omega)/h_r - x_i^{(k)}| \leq \varepsilon\sqrt{dj}\}) \\ &= \textstyle\prod_{i=1}^{d}\prod_{k=1}^{j} \int_{(|x_i^{(k)}|-\varepsilon/\sqrt{dj})h_r}^{(|x_i^{(k)}|+\varepsilon/\sqrt{dj})h_r} (2\pi)^{-1/2}e^{-u^2/2}\, du \\ &\geq \left(\frac{\varepsilon h_r}{\sqrt{2\pi dj}}\right)^{dj} \textstyle\prod_{i=1}^{d}\prod_{k=1}^{j} e^{-(|x_i^{(k)}|^2+\varepsilon^2(dj)^{-1})h_r^2/2}. \\ &\geq \left(\frac{\varepsilon h_r}{\sqrt{2\pi dj}}\right)^{dj} e^{-(\|x\|_K^2+\varepsilon^2)h_r^2/2}. \end{aligned}$$

Hence, (3.17) follows from $\|x\|_K^2 + \varepsilon^2 < 1 - 2\varepsilon + 2\varepsilon^2 < 1$ and (3.2)$'$.

Step 3. There exists a constant c_8, independent of r, such that

$$h_r^2 \geq \log\log n_r - \log c_8 \tag{3.18}$$

holds for all r. In fact, from (3.1) we have

$$+\infty > c_8 \geq \int_{n_r^{-1}}^1 e^{-h^2(t)}\, dt/t \geq e^{-h_r^2} \int_{n_r^{-1}}^1 dt/t = e^{-h_r^2} \log n_r \qquad \text{for all } r.$$

Step 4. Set

$$\Lambda = \{r;\, h_r \leq (\|x\|_K - 3\varepsilon)^{-1} \sqrt{3 \log\log n_r}\}.$$

Then, we have

$$\textstyle\sum_{r\in\Lambda} P(D_r) = +\infty. \tag{3.19}$$

Since by (3.7) and (3.13) we have

$$\begin{aligned} P(D_r) &\leq P(\{\omega;\, \|\bar{Z}_r(\cdot,\omega)\|_{K_a} \geq (\|\bar{x}\|_{K_a} - \varepsilon)h_r\}) \\ &\leq P(\{\omega;\, \|\bar{Z}_r(\cdot,\omega)\|_{K_a} \geq (\|x\|_{K_a} - 2\varepsilon)h_r\}) \\ &\leq P(\{\omega;\, \textstyle\sum_{i=1}^d \sum_{k=1}^j |\eta_r^{(k,i)}(\omega)|^2 \geq (\|x\|_K - 3\varepsilon)^2 h_r^2\}) \\ &\leq c_9(\|x\|_K - 3\varepsilon)^{dj-2} h_r^{dj-2} e^{-(\|x\|_K - 3\varepsilon)^2 h_r^2/2}, \end{aligned}$$

it follows that

$$\textstyle\sum_{r\notin\Lambda} P(D_r) < +\infty.$$

Combining this with (3.17), we have (3.19).

Step 5. Set $g_r = 2(\beta \log \Delta)^{-1} \log\log r$. Then, for $r \in \Lambda$ and $r' \in \Lambda$ with $r' \geq r + g_r$, we have

$$P(D_r \cap D_{r'}) \leq c_r P(D_r) P(D_{r'}),$$

where c_r is a constant independent of r' such that $\lim_{r\to\infty} c_r = 1$.

In order to show Step 5, we need the following inequalities; for $0 < s < t$,

$$\begin{aligned} \frac{R(s,t)}{\sigma(s)\sigma(t)} &= \frac{\sigma(s)}{2\sigma(t)} + \frac{(\sigma(t) - \sigma(t-s))(\sigma(t) + \sigma(t-s))}{2\sigma(s)\sigma(t)} \\ &\leq \frac{\sigma(s)}{2\sigma(t)} + \frac{\sigma(t) - \sigma(t-s)}{\sigma(s)} \leq \frac{\sigma(s)}{2\sigma(t)} + \frac{\sigma(t-s)}{\sigma(s)} \frac{s}{t-s}, \end{aligned}$$

where the last inequality follows from concavity of $\sigma(t)$. Hence, for $2s \leq t$ we have

$$\frac{R(s,t)}{\sigma(s)\sigma(t)} \leq 2^{-1}(s/t)^\alpha + 2^{1-\alpha}(s/t)^{1-\alpha} < 3(s/t)^\beta, \tag{3.20}$$

where $\beta = \min(\alpha, 1-\alpha)$.

Checking by (3.14) that

$$2s/n_{r'} \leq 2s/n_{r+1} \leq 2/n_{r+1} \leq a/n_r \leq t/n_r$$

holds for all $a \leq s, t \leq 1$, from (3.20) we have

$$\begin{aligned} |E[\eta_{r'}^{(k',i)}\eta_r^{(k,i)}]| &= (\sqrt{\lambda_{k'}\lambda_k}\, \sigma(1/n_{r'})\sigma(1/n_r))^{-1} \left| \int_a^1 \int_a^1 R(s/n_{r'}, t/n_r)\varphi_{k'}(s)\varphi_k(t)\, ds\, dt \right| \\ &\leq (\lambda_k\lambda_{k'})^{-1/2} 3a^{-\beta}(n_r/n_{r'})^\beta \int_a^1 \int_a^1 |\varphi_{k'}(s)|\, |\varphi_k(t)|\, ds\, dt \end{aligned} \tag{3.21}$$

$$\leq 3\lambda_j^{-1}a^{-\beta}\Delta^{-(r'-r)\beta}\left(\int_a^1 |\varphi_{k'}(s)|^2\,ds \int_a^1 |\varphi_k(t)|^2\,dt\right)^{1/2}$$

$$= 3\lambda_j^{-1}a^{-\beta}\Delta^{-(r'-r)\beta}.$$

By definition, $\{\eta_r^{(k,i)}, \eta_{r'}^{(k',i)}; k = 1, \cdots, j, \text{ and } k' = 1, \cdots, j\}$ forms a Gaussian system, so $\eta_{r'}^{(k',i)}$ can be represented as follows:

$$\eta_{r'}^{(k',i)}(\omega) = \bar{\eta}_{r'}^{(k',i)}(\omega) + \zeta_{r'}^{(k',i)}(\omega), \tag{3.22}$$
$$\zeta_{r'}^{(k',i)}(\omega) = \textstyle\sum_{k=1}^{j} a_{r',k'}^{(k,i)}\, \eta_r^{(k,i)}(\omega),$$

where $\{\eta_r^{(k,i)}; k = 1, \cdots, j\}$ and $\{\bar{\eta}_{r'}^{(k,i)}; k = 1, \cdots, j\}$ are independent but we notice that the random variables of the latter system are not necessarily independent. From (3.21) we have

$$|a_{r',k'}^{(k,i)}| = |E[\zeta_{r'}^{(k',i)}\eta_r^{(k,i)}]| = |E[\eta_{r'}^{(k',i)}\eta_r^{(k,i)}]| \leq 3\lambda_j^{-1}\, a^{-\beta}\Delta^{-(r'-r)\beta}. \tag{3.23}$$

Now, we have an upper bound of the joint probability of D_r and $D_{r'}$. For simplicity, we use the following abbreviation;

$$\eta_r = (\eta_r^{(k,i)}), \qquad \bar{\eta}_{r'} = (\bar{\eta}_{r'}^{(k',i)}), \qquad \zeta_{r'} = (\zeta_{r'}^{(k',i)})$$

and $\bar{x} = (x_i^{(k)})$ are regarded as elements of dj-dimensional Euclidean space with the Euclidean norm $\|\ \|$. Since by (3.23) we have

$$\|\zeta_{r'}\| \leq 3\lambda_j^{-1}a^{-\beta}j\Delta^{-(r'-r)\beta}\|\eta_r\|$$

and we have $\|\bar{x}\| = \|\bar{x}\|_{K_\alpha} < 1$ and $h_r/h_{r'} \leq 1$, it follows that

$$\begin{aligned} P(D_r \cap D_{r'}) &= P(\|\eta_r/h_r - \bar{x}\| \leq \varepsilon,\ \|(\bar{\eta}_{r'} + \zeta_{r'})/h_{r'} - \bar{x}\| \leq \varepsilon) \\ &\leq P(\|\eta_r/h_r - \bar{x}\| \leq \varepsilon,\ \|\bar{\eta}_{r'}/h_{r'} - \bar{x}\| \leq \varepsilon + \|\zeta_{r'}/h_{r'}\|) \\ &\leq P(\|\eta_r/h_r - \bar{x}\| \leq \varepsilon,\ \|\bar{\eta}_{r'}/h_{r'} - \bar{x}\| \leq \varepsilon + c_{10}\Delta^{-(r'-r)\beta}) \\ &\qquad (c_{10} = 3\lambda_j^{-1}a^{-\beta}j(1+\varepsilon)) \\ &= P(\|\eta_r/h_r - \bar{x}\| \leq \varepsilon)P(\|\bar{\eta}_{r'}/h_{r'} - \bar{x}\| \leq \varepsilon') \\ &\qquad (\varepsilon' = \varepsilon + c_{10}\Delta^{-(r'-r)\beta}) \\ &\equiv P(D_r)P(\bar{D}_r), \end{aligned} \tag{3.24}$$

where

$$\bar{D}_{r'} = \{\omega : \|\bar{\eta}_{r'}/h_{r'} - \bar{x}\| \leq \varepsilon'\}$$

and

$$\begin{aligned} \varepsilon' &= \varepsilon + c_{10}\Delta^{-(r'-r)\beta} \\ &= \varepsilon + 3\lambda_j^{-1}\, a^{-\beta}j(1+\varepsilon)\Delta^{-(r'-r)\beta} \leq 2\varepsilon, \qquad \text{by (3.15)}. \end{aligned}$$

From (3.23) we have

$$\begin{aligned} E[|\bar{\eta}_{r'}^{(p,i)}|^2] &\equiv r_{p,p}^{(i)} = 1 - E[|\xi_{r'}^{(p,i)}|^2] \\ &\geq 1 - 9\lambda_j^{-2}a^{-2\beta}j\Delta^{-2(r'-r)\beta} \equiv 1 - \theta_{r,r'}, \end{aligned}$$

and

$$\begin{aligned} |E[\bar{\eta}_{r'}^{(p,i)}\bar{\eta}_{r'}^{(q,i)}] &\equiv |r_{p,q}^{(i)}| = |\textstyle\sum_{k=1}^{j} a_{r',p}^{(k,i)}\, a_{r',q}^{(k,i)}| \\ &\leq 9\lambda_j^{-2}a^{-2\beta}j\Delta^{-2(r'-r)\beta} = \theta_{r,r'}. \end{aligned} \tag{3.25}$$

We denote by R_i and R_i^{-1} a positive definite matrix $(r_{p,q}^{(i)})_{p,q=1}^{j}$ and its inverse matrix, respectively. Then, under the condition (3.25), there exists a constant c_{11}, independent of $\theta_{r,r'}$ and any vector $\bar{y} = (y_1, \cdots, y_j)$, such that

$$|(R_i^{-1}\bar{y}, \bar{y}) - (\bar{y}, \bar{y})| \leq c_{11}\,\theta_{r,r'}(\bar{y}, \bar{y}) \tag{3.26}$$

and

$$\begin{aligned} \det R_i &\geq (1-\theta_{rr'})^j - (j!-1)\theta_{r,r'}^{j-1} \\ &\geq (1-\theta_{r,r+g_r})^j - (j!-1)\theta_{r,r+g_r}^{j-1} \\ &\equiv b_r \uparrow 1 \quad \text{as} \quad r \to +\infty. \end{aligned} \tag{3.27}$$

Therefore, by (3.26) and (3.27), we have

$$P(\bar{D}_{r'}) \leq (2\pi)^{-dj/2} b_r^{-d/2} \int_{\|\bar{u}/h_{r'} - \bar{x}\| \leq \varepsilon'} e^{-(R^{-1}\bar{u},\bar{u})/2}\, d\bar{u}$$

(where R^{-1} is the inverse matrix of $R = \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_d \end{pmatrix}$)

$$\leq (2\pi)^{-dj/2} b_r^{-d/2} \int_{\|\bar{u}/h_{r'} - \bar{x}\| \leq \varepsilon'} e^{-(\bar{u},\bar{u})/2 + c_{11}\theta_{r,r'}(\bar{u},\bar{u})}\, d\bar{u} \tag{3.28}$$

(setting $(\bar{u}/h_{r'} - \bar{x})\varepsilon = (\bar{v}/h_{r'} - \bar{x})\varepsilon'$)

$$\begin{aligned} &\leq (2\pi)^{-dj/2} b_r^{-d/2} (1 + c_{10}\varepsilon^{-1}\Delta^{-g_r\beta})^{dj} \\ &\quad \times \int_{\|\bar{v}/h_{r'} - \bar{x}\| \leq \varepsilon} e^{-(1/2 - c_{11}\theta_{r,r'})\|\varepsilon'\bar{v}/\varepsilon - \bar{x}h_{r'}(\varepsilon'-\varepsilon)/\varepsilon\|^2}\, d\bar{v}. \end{aligned} \tag{3.29}$$

To obtain an upper bound of (3.29), assume that $\|\bar{v}/h_{r'} - \bar{x}\| \leq \varepsilon$, $\|\bar{x}\| \leq 1$, and $r' \in \Lambda$; then, for sufficiently large r we have

$$\begin{aligned} \|\varepsilon'\bar{v}/\varepsilon - \bar{x}h_{r'}(\varepsilon' - \varepsilon)/\varepsilon\|^2 &= \|\bar{v} + c_{10}\varepsilon^{-1}\Delta^{-(r'-r)\beta}(\bar{v} - h_{r'}\bar{x})\|^2 \\ &\geq \|\bar{v}\|^2 - 2c_{10}\varepsilon^{-1}\Delta^{-(r'-r)\beta}\|\bar{v}\|(\|\bar{v}\| + h_{r'}\|\bar{x}\|) \\ &\geq \|\bar{v}\|^2 - 2c_{10}\varepsilon^{-1}\Delta^{-(r'-r)\beta}(1+\varepsilon)(2+\varepsilon)h_{r'}^2 \\ &\geq \|\bar{v}\|^2 - 6(1+\varepsilon)(2+\varepsilon)\varepsilon^{-1}c_{10}(\|x\| - 3\varepsilon)^{-2}\Delta^{-(r'-r)\beta}\log\log\Delta^{r'} \\ &\geq \|\bar{v}\|^2 - 6(1+\varepsilon)(2+\varepsilon)\varepsilon^{-1}c_{10}(\|x\| - 3\varepsilon)^{-2}\Delta^{-g_r\beta}\log\log\Delta^{r+g_r} \end{aligned}$$

(because the function $\Delta^{-x\beta}\log\log\Delta^x$ is decreasing for large x)

$$\equiv \|\bar{v}\|^2 - c_{12}(r),$$

and

$$\begin{aligned} \theta_{r,r'}\|\varepsilon'\bar{v}/\varepsilon - \bar{x}h_{r'}(\varepsilon' - \varepsilon)/\varepsilon\|^2 &\leq \theta_{r,r'}(\|2\bar{v}\| + \|\bar{x}\|h_{r'})^2 \\ &\leq \theta_{r,r'}(3+2\varepsilon)^2 h_{r'}^2 = (3+2\varepsilon)^2 9\lambda_j^{-2}a^{-2\beta}j\Delta^{-2(r'-r)\beta}h_{r'}^2 \\ &\leq 27(3+2\varepsilon)^2\lambda_j^{-2}a^{-2\beta}j(\|x\|_K - 3\varepsilon)^{-2}\Delta^{-2g_r\beta}\log\log\Delta^{r+g_r} \\ &\equiv c_{13}(r). \end{aligned} \tag{3.30}$$

Here, recall that $g_r = 2(\beta \log \Delta)^{-1} \log \log r$, so that $\lim_{r\to\infty} c_{12}(r) = \lim_{r\to\infty} c_{13}(r) = 0$. Therefore, from (3.29) and (3.30) we have

$$P(\bar{D}_{r'}) \le c_r(2\pi)^{-dj/2} \int_{\|\bar{v}/h_{r'} - \bar{x}\| \le \varepsilon} e^{-\|\bar{v}\|^2/2}\, d\bar{v} = c_r P(D_{r'}),$$

where

$$c_r = b_r^{-dj}(1 + c_{10}\varepsilon^{-1}\Delta^{-g_r\beta})^{dj} e^{c_{11}c_{13}(r) + c_{12}(r)/2} \downarrow 1$$

as $r \to +\infty$.

Step 6. For each r,

$$\sum_{r<r'\le r+g_r, r'\in\Lambda} P(D_r \cap D_{r'}) \le d_r P(D_r), \tag{3.31}$$

where d_r is a constant such that $d_r \to 0$ as $r \to +\infty$. Just analogously with Step 5, we have

$$\begin{aligned} \det R_i \ge b_r &\ge (1 - \theta_{r,r+1})^j - (j! - 1)\theta_{r,r+1}^{j-1} \\ &\equiv c_{14} \text{ (independent of } r \text{ by definition)}, \end{aligned}$$

and under the assumptions of $\|\bar{u}/h_{r'} - \bar{x}\| < 2\varepsilon$, $\|\bar{x}\| < 1$, $r < r' \le r + g_r$, and $r' \in \Lambda$ we have

$$\begin{aligned} \theta_{r,r'}\|\bar{u}\|^2 &\le 9\lambda_j^{-2}a^{-2\beta}j\Delta^{-2\beta}(1 + 2\varepsilon)^2 h_{r'}^2 \\ &= 27(1 + 2\varepsilon)^2\lambda_j^{-2}a^{-2\beta}j(\|x\|_K - 3\varepsilon)^{-2}\Delta^{-2\beta} \log \log \Delta^{r+g_r} \\ &\le (\|x\|_K - 4\varepsilon)^2/4 \log \log \Delta^{r+g_r}, \qquad \text{by (3.15)}. \end{aligned}$$

Therefore, it follows from (3.18) and (3.28) that

$$\begin{aligned} P(\bar{D}_{r'}) &\le (2\pi)^{-dj/2}c_{14}^{-d/2} \int_{\|\bar{u}/h_{r'} - \bar{x}\| \le 2\varepsilon} e^{-(\bar{u},\bar{u})/2 + c_{11}\theta_{r,r'}(\bar{u},\bar{u})}\, d\bar{u} \\ &\le (2\pi)^{-dj/2}c_{14}^{-d/2}(2 + 4\varepsilon)^{dj}h_{r'}^{dj}\, e^{-(\|\bar{x}\| - 2\varepsilon)^2 h_{r'}^2/2 + (\|x\|_K - 4\varepsilon)^2 \log\log \Delta^{r+g_r}/4} \\ &\le (2\pi)^{-dj/2}(2 + 4\varepsilon)^{dj}c_{14}^{-d/2}3^{dj/2}(\|x\|_K - 3\varepsilon)^{-dj}(\log \log \Delta^{r+g_r})^{dj/2} \\ &\quad \times e^{-(\|x\|_K - 4\varepsilon)^2(\log\log \Delta^r - \log c_8)/2 + (\|x\|_K - 4\varepsilon)^2/4 \log\log \Delta^{r+g_r}}, \end{aligned}$$

hence,

$$\sum_{r<r'\le r+g_r, r'\in\Lambda} P(D_r \cap D_{r'}) \le d_r P(D_r),$$

where

$$\begin{aligned} d_r &= g_r(2\pi)^{-dj/2}(2 + 2\varepsilon)^{dj}c_{14}^{-d/2}3^{dj/2}(\|x\|_K - 3\varepsilon)^{-dj}(\log \log \Delta^{r+g_r})^{dj/2} \\ &\quad \cdot \exp\{-(\|x\|_K - 4\varepsilon)^2 (\log \log \Delta^r - \log c_8)/2 + (\|x\|_K - 4\varepsilon)^2/4 \cdot \log \log \Delta^{r+g_r}\} \\ &\to 0 \quad \text{as} \quad r \to +\infty. \end{aligned}$$

Step 7.

$$P(\limsup_{r\in\Lambda, r\to+\infty} D_r) = 1. \tag{3.32}$$

By Schwarz's inequality we have

$$\begin{aligned} \Big(\sum_{p\le r\le q, r\in\Lambda} P(D_r)\Big)^2 &= \Big(E\Big[\chi\Big(\bigcup_{p\le r\le q, r\in\Lambda} D_r\Big)\Big(\sum_{p\le r\le q, r\in\Lambda}\chi(D_r)\Big)\Big]\Big)^2 \\ &\le E\Big[\chi\Big(\bigcup_{p\le r\le q, r\in\Lambda} D_r\Big)\Big]E\Big[\Big(\sum_{p\le r\le q, r\in\Lambda}\chi(D_r)\Big)^2\Big] \\ &= P\Big(\bigcup_{p\le r\le q, r\in\Lambda} D_r\Big)\sum_{p\le r, r'\le q, r, r'\in\Lambda} P(D_r \cap D_{r'}), \end{aligned} \tag{3.33}$$

where $\chi(A)$ is the indicator function of A. Applying Steps 5 and 6, we have

$$\sum_{p\le r,r'\le q,r,r'\in\Lambda} P(D_r \cap D_{r'})$$
$$\le \sum_{p\le r\le q,r\in\Lambda} P(D_r) + 2\sum_{p\le r<r'\le q,r'\le r+g_r,r,r'\in\Lambda} P(D_r\cap D_{r'})$$
$$+ 2\sum_{p\le r<r'\le q,r+g_r<r',r,r'\in\Lambda} P(D_r\cap D_{r'})$$
$$\le (1+2\sup_{r\ge p} d_r)\sum_{p\le r\le q,r\in\Lambda} P(D_r) + \sup_{r\ge p} c_r\Big(\sum_{p\le r\le q,r\in\Lambda} P(D_r)\Big)^2.$$

Combining this with (3.19) and (3.33) by letting q go to infinity, we have

$$P(U_{r\ge p,r\in\Lambda} D_r) \ge 1/\sup_{r\ge p} c_r \uparrow 1 \quad \text{as } p\to+\infty, \qquad \text{(by Step 5)}.$$

Step 8. Conclusion. With probability 1, x is a limit point of $\{f_n(t,\omega); n = 1, 2, \cdots\}$.

First, we recall from Step 4 of Lemma 1 that, with probability 1, there exists $n_1 = n_1(\varepsilon,\omega)$ such that $f_n \in B_{10\varepsilon}$ for all $n \ge n_1$. On the other hand, it follows from Step 7 that, with probability 1, there exists a subsequence $(n_{r_k};\ k = 1, 2, \cdots\}$ such that

$$\|\bar Z_{r_k}(\cdot,\omega)/h_{r_k} - \bar x(\cdot)\|_{K_a} \le \varepsilon. \tag{3.34}$$

Next, combining (3.8), (3.9), (3.13), (3.16) and (3.34), we have

$$\begin{aligned}\|f_{n_{r_k}}(\cdot,\omega) - x\|_C &\le \sup_{0\le t\le a}(\|f_{n_{r_k}}(t,\omega)\| + \|x(t)\|)\\ &\quad + \|f_{n_{r_k}}(\cdot,\omega) - x(\cdot)\|_{C_a}\\ &\le 12\,\varepsilon + \|f_{n_{r_k}}(\cdot,\omega) - \bar Z_{r_k}(\cdot,\omega)/h_{r_k}\|_{C_a}\\ &\quad + \|\bar Z_{r_k}(\cdot,\omega)/h_{r_k} - \bar x(\cdot)\|_{C_a} + \|\bar x - x\|_{C_a}\\ &\le 12\varepsilon + \varepsilon + \|\bar Z_{r_k}(\cdot,\omega)/h_{r_k} - \bar x(\cdot)\|_{K_a} + \|\bar x - x\|_{K_a}\\ &\le 15\varepsilon.\end{aligned}$$

4. Real analytic version: Proof of Theorem 2. In this section we will discuss nonrandom arguments. For $q > 0$ and $x \in C$, set

$$m(q;x) = m(\{0\le s\le 1;\ \|x(s)\| \ge \sigma(s)/\sqrt{q}\})$$

and
$$F(q) = \sup_{x\in B} m(q;x).$$

First, we will prove the following two lemmas concerning $F(q)$.

LEMMA 3. *$F(q)$ is a continuous function of q. Moreover, we have the following lemma.*

LEMMA 4. *$1 > F(q) > 0$ for $q > 1$, $F(q) = 0$ for $1 \ge q > 0$, $\lim_{q\uparrow+\infty} F(a) = 1$ and $F(q)$ is a strictly increasing function for $q \ge 1$.*

PROOF OF LEMMA 3. First, we notice that the function $m(q;x)$ on C is upper semi-continuous with respect to x. In fact, we have

$$\sup_{x\in U_\varepsilon(x_0)} m(q;x) \le m(\{0\le s\le 1;\ \|x_0(s)\| \gtrsim \sigma(s)/\sqrt{q} - \varepsilon\})$$
$$\downarrow m(q;x_0), \qquad \text{as } \varepsilon\downarrow 0,$$

where $U_\varepsilon(x_0)$ is an ε-neighborhood of x_0. In addition, we have $m(q; x/\|x\|_K) \ge m(q;x)$ if $0 < \|x\|_K < 1$. Therefore, it follows that there exists x_q with $\|x_q\|_K = 1$ such that $F(q) = m(q;x_q)$, for B is compact in C (Lemma 3 of [6]). Now, we will show that $F(q)$ is a right continuous function. Since $F(q)$ is a non-decreasing function, it is sufficient to prove that there exists a sequence $q_n \downarrow q$ such that $\lim_{n\to\infty} F(q_n) = F(q)$. Since B is compact, we can find a sequence $q_n \downarrow q$ such that x_{q_n} converges to some element $x_0 \in B$ in C. It means that for any $\varepsilon < 0$ there exists an n_0 such that $\| x_0(s) \| \geq \| x_{q_n}(s) \| - \varepsilon$ holds for all $n \geq n_0$ and $0 \leq s \leq 1$. Therefore, we have

$$\begin{aligned}\lim_{n\to\infty} F(q_n) &= \lim_{n\to\infty} m(q_n; x_{q_n}) \\ &\leq \lim_{n\to\infty} m(\{0 \leq s \leq 1; \| x_0(s) \| \geq \sigma(s)/\sqrt{q_n} - \varepsilon\}) \\ &= m(\{0 \leq s \leq 1; \| x_0(s) \| \geq \sigma(s)/\sqrt{q} - \varepsilon\}) \downarrow m(q; x_0), \quad \text{as } \varepsilon \downarrow 0 \\ &\leq F(q).\end{aligned}$$

This shows that $F(q)$ is right continuous.

It is rather difficult to show the left continuity of $F(q)$, for it depends on the property of the r.k. Hilbert space H. We denote by $H(I)$ the closed linear subspace of H spanned by $\{R(t,\cdot); t \in I\}$ for a subset I of $[0, 1]$. Since $H([0, 1]) = H$, it follows that for any $\varepsilon > 0$ there exists $x' = (x_1', \cdots, x_d') \in B$, where $x_i'(\cdot) = \sum_{j=1}^{n} a_i^{(j)} R(t_j, \cdot)$ such that

$$F(q) \leq m(q; x') + \varepsilon/2. \tag{4.1}$$

Since $R(t, \cdot)$ does not belong to $H([0, t-\varepsilon] \cup [t+\varepsilon, 1])$ for any $1 \geq t+\varepsilon > t-\varepsilon \geq 0$, there exists an interval I, whose length is less than $\varepsilon/2$, such that $\| x'/K(I^c) \|_K$ is positive but strictly less than $\| x' \|_K (\leq 1)$, where $x'/K(I^c)$ denotes the projection of x' onto the closed subspace $K(I^c) = H(I^c) \oplus \cdots \oplus H(I^c)$, $I^c = [0, 1] - I$. Now, set

$$x' = x_1' + x_2',$$

and $x'' = x_1'/\| x_1' \|_K$, where $x_1' = x'/K(I^c)$ and x_2' is the orthogonal complement of x'. Then, we have $\| x_1' \|_K < \| x' \|_K \leq 1$ and $\| x'' \|_K = 1$. Since x_2' is orthogonal to $K(I^c)$, this means that $x_2'(t) = (0, \cdots, 0)$ for $t \in I^c$, so $x''(t) = x'(t)/\| x_1' \|_K$ for $t \in I^c$. Combining with (4.1), we have

$$\begin{aligned}F(q) &\leq \varepsilon/2 + m(q; x') \\ &\leq \varepsilon + m(\{s \in I^c; \| x'(s) \| \geq \sigma(s)/\sqrt{q}\}) \\ &= \varepsilon + m(\{s \in I^c; \| x''(s) \| \geq \sigma(s)/\sqrt{q \| x_1' \|_K^2}\}) \\ &\leq \varepsilon + F(q \| x_1' \|_K^2) \\ &\leq \varepsilon + F(q') \text{ for } q \| x_1' \|_K^2 < q' < q.\end{aligned}$$

This shows that $F(q)$ is left continuous.

PROOF OF LEMMA 4. We will prove Lemma 4 in several steps.

Step 1. $F(q) = 0$ for $q \leq 1$. In fact, by Schwarz's inequality, we have

$$\begin{aligned}\| x(s) \|^2 &= \textstyle\sum_{i=1}^{d} (x_i(s))^2 = \sum_{i=1}^{d} (x_i(\cdot), R(s, \cdot))_H^2 \\ &\leq \sigma^2(s) \textstyle\sum_{i=1}^{d} \| x_i \|_H^2 = \sigma^2(s) \| x \|_K^2.\end{aligned} \tag{4.2}$$

Therefore, if $x \in B$, $\{0 \leq s \leq 1; \| x(s) \| \geq \sigma(s)/\sqrt{q}\}$ is empty for any $q < 1$. This means $F(q) = 0$ for $q < 1$, and by continuity of $F(q)$ we have $F(1) = 0$.

Step 2. $F(q) > 0$ for $q > 1$, and $\lim_{q\to+\infty} F(q) = 1$. To see this, set $y(\cdot) = (R(t, \cdot)/\sigma(t), 0, \cdots, 0)$ for a fixed $t > 0$. Then, clearly we have $\| y(\cdot) \|_K = 1$, $F(q) \geq m(q; y) > 0$, and $\lim_{q\to+\infty} m(q, y) = 1$.

Step 3. $F(q) < 1$ for all $q > 0$. Since there exists an $x_q \in B$ such that $F(q) = m(q; x_q)$, it is sufficient to show that for $q > 1$ and $x \in B$, $\{0 \leq s \leq 1; \| x(s) \| \geq \sigma(s)/\sqrt{q}\}$ *is isolated from the origin. To show this, assume that there exists a sequence* $s_n \downarrow 0$ such that $\| x(s_n) \| \geq \sigma(s_n)/\sqrt{q}$ and $s_{n+1}/s_n \leq 1/2$. Then, we will find a contradiction. In fact, define $y^{(n)} =$

$(y_1^{(n)}, \cdots, y_d^{(n)}) \in K$ as follows:

$$y_i^{(n)} = \sum_{j=1}^{n} a_j x_i(s_j) R(s_j, \cdot),$$

and

$$a_j = (\sigma^2(s_j) j)^{-1}, \tag{4.3}$$

where $x_i(s)$ is the ith component of $x(s)$. Then, we have

$$\begin{aligned}(x, y^{(n)})_K &= \textstyle\sum_{i=1}^{d} (x_i, y_i^{(n)})_H = \sum_{i=1}^{d} \sum_{j=1}^{n} a_j x_i^2(s_j) \\ &= \textstyle\sum_{j=1}^{n} a_j \| x(s_j) \|^2 \\ &\geq \textstyle\sum_{j=1}^{n} a_j \sigma^2(s_j)/q \\ &= \textstyle\sum_{j=1}^{n} (jq)^{-1} \uparrow +\infty, \qquad \text{as } n \to +\infty.\end{aligned} \tag{4.4}$$

On the other hand, by (4.2) we have

$$\begin{aligned}\| y^{(n)} \|_K^2 &= \textstyle\sum_{i=1}^{d} \| y_i \|_H^2 = \sum_{i=1}^{d} \sum_{j,k}^{n} a_j a_k x_i(s_j) x_i(s_k) R(s_j, s_k) \\ &\leq \textstyle\sum_{j,k}^{n} a_j a_k R(s_j, s_k) \| x(s_j) \| \, \| x(s_k) \| \\ &\leq \textstyle\sum_{j,k}^{n} a_j a_k R(s_j, s_k) \sigma(s_j) \sigma(s_k) \\ &= \textstyle\sum_{j,k}^{n} (jk)^{-1} R(s_j, s_k)/(\sigma(s_j)\sigma(s_k)) \\ &\leq \textstyle\sum_{j=1}^{\infty} j^{-2} + 2 \sum_{j<k} (jk)^{-1} R(s_j, s_k)/(\sigma(s_j)\sigma(s_k)).\end{aligned}$$

By definition we have $2s_k \leq s_j$ for $j < k$; therefore, applying the inequality (3.20), it follows that

$$\| y^{(n)} \|_K^2 \leq \textstyle\sum_{j=1}^{\infty} j^{-2} + 6 \sum_{j=1}^{\infty} j^{-2} \sum_{k=1}^{\infty} 2^{-\beta k}. \tag{4.5}$$

This means that $\| y^{(n)} \|_K^2$ is bounded; however, (4.4) and (4.5) contradict the Schwarz inequality $(x, y^{(n)})_K \leq \| x \|_K \| y^{(n)} \|_K \leq \| y^{(n)} \|_K$.

Step 4. $F(q)$ is strictly increasing for $q > 1$. Since there exists an $x_q \in B$ for each q such that $F(q) = m(q; x_q)$ and we have $1 > F(q) > 0$ for $q > 1$ by Steps 2 and 3, it follows that $m(q; x_q) < m(q'; x_q) \leq F(q')$ for $q' > q$.

Next, we will prove the following two lemmas, still concerned with nonrandom arguments, on which the proof of Theorem 2 is essentially based. For a function f in C and a continuous and non-increasing function h having a finite positive q of (2.2), set

$$m(t; h, f) = m(\{0 \leq s \leq t; \| f(s) \| > \sigma(s) h(s)\}),$$

$$f_n(t) = f(t/n)/(\sqrt{q}\, \sigma(1/n) h(1/n)), \qquad n = 1, 2, \cdots$$

and

$$f_n^{(\varepsilon)}(t) = f(t/n)/(\sqrt{q}\, \sigma(1/n) h(\varepsilon/n)), \qquad \varepsilon > 0.$$

Then, we have

LEMMA 5. *Assume that for a fixed $q > 0$, the set $\{f_n(t); n = 1, 2, \cdots\}$ is pre-compact in C and that all the limit points are contained in B, the unit ball of the Hilbert space K. Then,*

$$\limsup_{t \downarrow 0} m(t; h, f)/t \leq F(q). \tag{4.6}$$

LEMMA 6. *Assume that for a fixed $q > 0$, and for each rational $\varepsilon > 0$, the set $\{f_n^{(\varepsilon)}(t); n = 1, 2, \cdots\}$ is pre-compact in C and that the set of all the limit points coincides with B.*

Then,

$$\limsup_{t\downarrow 0} m(t; h, f)/t \geq F(q). \tag{4.7}$$

PROOF OF LEMMA 5. It is sufficient for the proof to show a contradiction if we assume that

$$\limsup_{t\downarrow 0} m(t; h, f)/t \geq F(q+\varepsilon) + 3\varepsilon \tag{4.8}$$

holds for some $\varepsilon > 0$. From (4.8) there exists a sequence $t_n \downarrow 0$ such that

$$m(t_n; h, f)/t_n \geq F(q+\varepsilon) + 2\varepsilon. \tag{4.9}$$

Now, take an integer k_n such that $(k_n+1)^{-1} < t_n \leq k_n^{-1}$. Then, if necessary, choosing a subsequence of $\{t_n\}$, we can assume (from the assumption of Lemma 5) that $\{f_{k_n}(t); n = 1, 2, \cdots\}$ converges to some continuous $x \in B$. This means that there exists n_0 such that for all $n \geq n_0$ we have

$$\| x - f_{k_n} \|_C \leq \sigma(\varepsilon)(1/\sqrt{q} - 1/\sqrt{q+\varepsilon}). \tag{4.10}$$

On the other hand, letting $I[x] = 1$ if $x > 1$, $= 0$ otherwise, we have

$$\begin{aligned} m(1/n; h, f) &\leq \varepsilon/n + m(\{\varepsilon/n \leq s \leq 1/n; \|f(s)\| > \sigma(h)h(s)\}) \\ &= \varepsilon/n + \int_{\varepsilon/n}^{1/n} I[\|f(s)\|/(\sigma(s)h(s))]\, ds \\ &= \varepsilon/n + n^{-1}\int_{\varepsilon}^{1} I[\|f(t/n)\|/(\sigma(t/n)h(t/n))]\, dt \\ &\leq \varepsilon/n + n^{-1}\int_{\varepsilon}^{1} I[\|f_n(t)\|/(\sigma(t)q^{-1/2})]\, dt. \end{aligned} \tag{4.11}$$

Consider k_n instead of n in (4.11) and take account of (4.10); then, we have

$$\begin{aligned} m(t_n; h, f)/t_n &\leq m(k_n^{-1}; h, f)(k_n+1) \\ &= m(k_n^{-1}; h, f) + \varepsilon + \int_{\varepsilon}^{1} I[\|f_{k_n}(t)\|/(\sigma(t)q^{-1/2})]\, dt \\ &\leq m(k_n^{-1}; h, f) + \varepsilon + \int_{\varepsilon}^{1} I[\|x(t)\|/(\sigma(t)(q+\varepsilon)^{-1/2})]\, dt \\ &\leq m(k_n^{-1}; h, f) + \varepsilon + F(q+\varepsilon). \end{aligned}$$

This inequality, however, contradicts with (4.9) if we take sufficiently large n such that $m(k_n^{-1}; h, f) < \varepsilon$.

PROOF OF LEMMA 6. It is sufficient for the proof to show that for any $x \in B$ and any rational $\varepsilon > 0$ $(\varepsilon < q)$

$$\limsup_{n\to\infty} n m(1/n; h, f) \geq m(q-\varepsilon; x) - \varepsilon \tag{4.12}$$

holds; recall that $m(q; x) = m(\{0 \leq s \leq 1; \|x(s)\| \geq \sigma(s)/\sqrt{q}\})$. From our assumption there exists a subsequence $\{j_n\}_{n=1}^{\infty}$ such that $\{f_{j_n}^{(\varepsilon)}(t): n = 1, 2, \cdots\}$ converges to $x \in B$ in C. This means that there exists an n_0 such that for any $n \geq n_0$

$$\| x - f_{j_n}^{(\varepsilon)} \|_C < \sigma(\varepsilon)(1/\sqrt{q-\varepsilon} - 1/\sqrt{q}). \tag{4.13}$$

By an argument to that around (4.11), we have

$$
\begin{aligned}
j_n m(1/j_n;\, h, f) &\geq j_n m(\{\varepsilon/j_n \leq s \leq 1/j_n;\, \| f(s)\| > \sigma(s)h(s)\}) \\
&= j_n \int_{\varepsilon/j_n}^{1/j_n} I[\| f(s)\|/(\sigma(s)h(s))]\, ds \\
&= \int_{\varepsilon}^{1} I[\| f(t/j_n)\|/(\sigma(t/j_n)h(t/j_n))]\, dt \\
&\geq \int_{\varepsilon}^{1} I[\| f_{j_n}^{(\varepsilon)}(t)\|/(\sigma(t)q^{-1/2})]\, dt.
\end{aligned} \tag{4.14}
$$

Combining (4.13) and (4.14), we have

$$
\begin{aligned}
j_n m(1/j_n;\, h, f) &\geq m(\{\varepsilon \leq t \leq 1;\, \|x(t)\| \geq \sigma(t)/\sqrt{q-\varepsilon}\}) \\
&\geq m(q-\varepsilon;\, x) - \varepsilon \quad \text{for} \quad n \geq n_0.
\end{aligned}
$$

We have completed the proof of Lemma 6.

PROOF OF THEOREM 2. Since the function $h(\varepsilon t)$ has the same q in (2.2) as $h(t)$, Theorem 1 is also valid for

$$\{f_n^{(\varepsilon)}(t, \omega) = Y(t/n, \omega)/(\sqrt{q}\,\sigma(1/n)h(\varepsilon/n)); \qquad n = 1, 2, \cdots\}.$$

Therefore, we obtain the proof of Theorem 2 from Theorem 1, Lemma 5 and Lemma 6.

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