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THE XYZ CONJECTURE AND THE FKG INEQUALITY

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Consider random variables $x_1, \cdots, x_n$, independently and uniformly distributed on the unit interval. Suppose we are given partial information, Γ, about the unknown ordering of the x's; e.g., $\Gamma = \{x_1 < x_{12}, x_7 < x_5, \cdots\}$. We prove the "XYZ conjecture" (originally due to Ivan Rival, Bill Sands, and extended by Peter Winkler, R. L. Graham, and other participants of the Symposium on Ordered Sets at Banff, 1981) that

$$P(x_1 < x_2 \mid \Gamma) \le P(x_1 < x_2 \mid \Gamma, x_1 < x_3).$$

The proof is based on the FKG inequality for correlations and shows by example that even when the hypothesis of the FKG inequality fails it may be possible to redefine the partial ordering so that the conclusion of the FKG inequality still holds.

1. Introduction. Let the incomes $x_1, \cdots, x_n$ of n individuals be initially ordered at random uniformly on all permutations. Suppose some partial information is available on the true ordering of the x's; e.g., $\Gamma = \{x_1 < x_{12}, x_7 < x_5, \cdots\}$. Then it is tempting to conjecture that for any Γ,

$$P(x_1 < x_2 \mid \Gamma) \le P(x_1 < x_2 \mid \Gamma, x_1 < x_3) \tag{1.1}$$

because if it is known that $x_1 < x_3$ then it seems more likely that x_1 is "small." Nevertheless, the conjecture (1.1) is surprisingly difficult to prove in spite of much effort by combinatorialists. Indeed the conjecture appears less tempting if one notes the fact that the following analogous conjecture is false:

$$P(x_1 < x_2 < x_4 \mid \Gamma) \le P(x_1 < x_2 < x_4 \mid \Gamma, x_1 < x_3 < x_4) \tag{1.1$'$}$$

even though one might analogously reason that if it is known that $x_1 < x_3 < x_4$ then it seems more likely that x_1 is "small" and x_4 "large." C. L. Mallows gave the following simple counterexample to (1.1)$'$: Let $n = 6$ and let $\Gamma = \{x_2 < x_5 < x_6 < x_3, x_1 < x_4\}$. Then we see that $P(x_1 < x_2 < x_4 \mid \Gamma) = 4/15$, $P(x_1 < x_2 < x_4 \mid \Gamma, x_1 < x_3 < x_4) = 1/4$ so (1.1) fails.

There has been much interest in the conjecture (1.1) called the XYZ conjecture in the combinatorics community [1] in connection with the theory of partial orders [5, 6]. Ivan Rival points out that "transitivity conjecture" is a more apt name. Peter Winkler [2] derives a number of interesting consequences of the conjecture (1.1).

The method of proof of (1.1) is interesting in itself and extends the technique used in [3], based on the FKG inequality [4] which arose in proving correlation inequalities in statistical physics. The FKG inequality asserts that if: $(\Omega, <)$ is a distributive lattice (i.e., Ω is a finite set with a partial order, $<$, which satisfies

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{1.2}$$

where for each pair $x, y \in \Omega$ there is a unique $x \wedge y$, $x \vee y$ which are respectively the largest (smallest) elements of Ω less (greater) than x and y); f and g are increasing real-valued functions on Ω (i.e., whenever $x < y$ then

$$f(x) \le f(y), \quad g(x) \le g(y) \tag{1.3}$$

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holds); μ is a positive log monotone density on Ω (i.e., for all x, y

$$\mu(x) \geq 0, \quad \mu(x)\mu(y) \leq \mu(x \wedge y)\mu(x \vee y) \tag{1.4}$$

holds), then

$$\textstyle\sum_{x\in\Omega} f(x)g(x)\mu(x) \sum_{y\in\Omega} \mu(y) \geq \sum_{x\in\Omega} f(x)\mu(x) \sum_{y\in\Omega} g(y)\mu(y). \tag{1.5}$$

It is observed in [4] that the Condition (1.4) is not necessary for (1.5). However, even when (1.4) fails for the "natural" ordering, $<$, it may be possible, as in the present proof below, to introduce a new partial order for which (1.4) holds and *the same* conclusion (1.5) follows. In the present case, the "natural" order, analogous to the technique in [3], is to take $x < y$ iff $x_1 \geq y_1$ and $x_i \leq y_i$, $i = 2, \cdots, n$ where $\Omega = \{x = (x_1, \cdots, x_n)\}$ as in Section 2, below. However, (1.4) fails for this order and it was necessary to find a new order (namely (2.1) below) for which the hypotheses (1.2) through (1.5) of the FKG inequality hold.

2. Proof of (1.1). Let $\Omega = \{1, 2, \cdots, N\}^n$ be the set of $\mathbf{x} = (x_1, \cdots, x_n)$ where each $x_i \in \{1, 2, \cdots, N\}$. We will later let $N \to \infty$ as in [3]. We say $\mathbf{x} < \mathbf{y}$ if and only if

$$x_1 \geq y_1, \quad x_i - x_1 \leq y_i - y_1, \quad i = 2, \cdots, n. \tag{2.1}$$

It is easy to verify that $<$ is a partial order on Ω and that

$$(\mathbf{x} \wedge \mathbf{y})_i = \min(x_i - x_1, y_i - y_1) + \max(x_1, y_1), \quad i = 1, \cdots, n \tag{2.2}$$

$$(\mathbf{x} \vee \mathbf{y})_i = \max(x_i - x_1, y_i - y_1) + \min(x_1, y_1), \quad i = 1, \cdots, n. \tag{2.3}$$

To prove that $(\Omega, <)$ is a distributive lattice; i.e., (1.2) holds, note that the ith component, $(\mathbf{x} \wedge \mathbf{y})_i$, of $\mathbf{x} \wedge \mathbf{y}$ belongs to $1, \cdots, N$ and similarly for the ith component of $\mathbf{x} \vee \mathbf{y}$ in (2.3). Thus $\mathbf{x} \wedge \mathbf{y}$ and $\mathbf{x} \vee \mathbf{y}$ belong to Ω. To prove the distributive property (1.2) we note that from (2.2) and (2.3), with $\mathbf{w} = \mathbf{y} \vee \mathbf{z}$

$$\begin{aligned}(\mathbf{x} \wedge (\mathbf{y} \vee \mathbf{z}))_i &= \min(x_i - x_1, w_i - w_1) + \max(x_1, w_1) \\ &= \min(x_i - x_1, \max(y_i - y_1, z_i - z_1)) + \max(x_1, \min(y_1, z_1))\end{aligned} \tag{2.4}$$

since $w_i - w_1 = \max(y_i - y_1, z_i - z_1)$. Also the rhs of (1.2) has

$$\begin{aligned}((\mathbf{x} \wedge \mathbf{y}) \vee (\mathbf{x} \wedge \mathbf{z}))_i &= \max((\mathbf{x} \wedge \mathbf{y})_i - (\mathbf{x} \wedge \mathbf{y})_1, (\mathbf{x} \wedge \mathbf{z})_i - (\mathbf{x} \wedge \mathbf{z})_1) \\ &\quad + \min((\mathbf{x} \wedge \mathbf{y})_1, (\mathbf{x} \wedge \mathbf{z})_1) \\ &= \max(\min(x_i - x_1, y_i - y_1), \min(x_i - x_1, z_i - z_1)) \\ &\quad + \min(\max(x_1, y_1), \max(x_1, z_1)).\end{aligned} \tag{2.5}$$

For any three real numbers a, b, c we have

$$\min(a, \max(b, c)) = \max(\min(a, b), \min(a, c)) \tag{2.6}$$

and its dual

$$\max(a, \min(b, c)) = \min(\max(a, b), \max(a, c)). \tag{2.7}$$

It follows from (2.7), replacing a, b, c by x_1, y_1, z_1 respectively that the last summands in (2.4) and (2.5) are equal, and from (2.6), replacing a, b, c by $x_i - x_1$, $y_i - y_1$, $z_i - z_1$ respectively that the first summands in (2.4) and (2.5) are equal. Thus the left sides of (2.4) and (2.5) are equal and (1.2) holds; i.e., $(\Omega, <)$ is a distributive lattice. Peter Winkler has found a simpler proof of this based on the map: $(x_1, \cdots, x_n) \to (-x_1, x_2 - x_1, \cdots, x_n - x_1)$.

Now let $f(\mathbf{x})$ be the indicator $\chi(x_1 \leq x_2)$ of the event $x_1 \leq x_2$,

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } x_1 \leq x_2 \\ 0 & \text{else} \end{cases} \tag{2.8a}$$

and $g(\mathbf{x}) = \chi(x_1 \le x_3)$; i.e.,

(2.8b) $$g(\mathbf{x}) = \begin{cases} 1 & \text{if } x_1 \le x_3 \\ 0 & \text{else.} \end{cases}$$

That f is increasing follows easily from (2.1) since if $\mathbf{x} < \mathbf{y}$ and $f(\mathbf{x}) = 1$ then $0 \le x_2 - x_1 \le y_2 - y_1$ and so $f(\mathbf{y}) = 1$. Similarly g is increasing, so (1.3) holds. We now take $\mu(\mathbf{x})$ to be the indicator, $\chi(\Gamma)$, of Γ; i.e.,

(2.9) $$\mu(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \text{ satisfies the inequalities } \Gamma \\ 0 & \text{else} \end{cases}$$

where Γ is the set of inequalities representing the partial information or partial ordering of the x_i. To prove (1.4) it is necessary only to show that if $\mu(\mathbf{x}) = \mu(\mathbf{y}) = 1$ then also $\mu(\mathbf{x} \wedge \mathbf{y}) = \mu(\mathbf{x} \vee \mathbf{y}) = 1$. This is what fails for the simpler natural partial order referred to at the end of Section 1. So suppose $\mu(\mathbf{x}) = \mu(\mathbf{y}) = 1$; i.e., that $\mathbf{x}$ and $\mathbf{y}$ satisfy the inequalities in Γ. If $x_i < x_j$ is one of the inequalities in Γ then we have $x_i < x_j$, $y_i < y_j$. From (2.2) we have

(2.10) $$(\mathbf{x} \wedge \mathbf{y})_i \le \min(x_j - x_1, y_j - y_1) + \max(x_1, y_1) = (\mathbf{x} \wedge \mathbf{y})_j$$

and similarly, $(\mathbf{x} \vee \mathbf{y})_i \le (\mathbf{x} \vee \mathbf{y})_j$. Since $\mathbf{x} \wedge \mathbf{y}$ and $\mathbf{x} \vee \mathbf{y}$ both satisfy each inequality in Γ, $\mu(\mathbf{x} \wedge \mathbf{y}) = \mu(\mathbf{x} \vee \mathbf{y}) = 1$ and so (1.4) holds. The hypothesis of the FKG inequality is verified and so (1.5) holds.

From (1.5) we obtain

(2.11) $$P(x_1 \le x_2, x_1 \le x_3, \Gamma)P(\Gamma) \ge P(x_1 \le x_2, \Gamma)P(x_1 \le x_3, \Gamma).$$

Letting $N \to \infty$, the probability that $x_i = x_j$ for some $i \ne j$ tends to zero and so (2.11) also holds for permutations induced by $x_1, \cdots, x_n$. Dividing by $P(\Gamma)P(x_1 < x_3, \Gamma)$ we obtain (1.1) and the XYZ conjecture is proved. We remark that taking $\Omega = \{1, \cdots, N\}^n$ with $N \to \infty$ rather than $\Omega = S_n$, the set of permutations on n objects, was needed because $(S_n, <)$ itself is not a distributive lattice.

3. Conjectures. It is tempting in general to assume that if a correlation inequality is true then it can be proved by using the FKG inequality with a proper choice of Ω, $<$, f, g, and μ. This assumption motivated a search for the ordering in (2.1). On the other hand it is generally false that (1.4) is necessary for (1.5) as was shown in [4]. In our context, is it true or false that

(3.1) $$P(x_1 < x_3, x_2 < x_3 \mid \Gamma) \le P(x_1 < x_3, x_2 < x_3 \mid x_1 < x_4, x_2 < x_4, \Gamma)$$

or that

(3.2) $$P(x_1 < x_2 \mid \Gamma) \le P(x_1 < x_2 \mid \Gamma, x_1 < x_3, x_4 < x_3)$$

holds in general? It would be nice to have a systematic way to use the FKG inequality for true correlation inequalities as well as a systematic way to generate counterexamples to false ones.

A few days after the preceding paragraph was written, C. L. Mallows, who had formulated the question of whether (3.1) holds, gave the simple counterexample $\Gamma = \{x_1 < x_4, x_2 < x_3\}$, $n = 4$. He also gave the counterexample $\Gamma = \{x_1 < x_3, x_4 < x_2\}$, $n = 4$ to (3.2). He then posed the following general question. If A, B, C are each events of the same form as Γ, based on $x_1, \cdots, x_n$, for which pairs A, B is it true that

(3.3) $$P(A \mid C)P(B \mid C) \le P(A \cap B \mid C)$$

holds for all n and C? Peter Winkler [7] has now obtained a complete answer to this question, showing that if (3.3) holds for any pair A, B and all C and n then (3.3) can be proved from (1.1) which is thus, in a sense, the basic inequality, and, speaking loosely, $A = \{x_1 < x_2\}$, $B = \{x_1 < x_3\}$ is the basic case for which (3.3) holds for all C and n.

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