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WHEN DO WEIGHTED SUMS OF INDEPENDENT RANDOM VARIABLES HAVE A DENSITY—SOME RESULTS AND EXAMPLES

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Let $\{X_n\}$ be a sequence of independent random variables and $\{a_n\}$ a positive decreasing sequence such that $\sum a_nX_n$ is a random variable. We show that under mild conditions on $\{X_n\}$

(i) if for every $\delta,\lambda > 0$

$$\sum_{n=1}^{\infty} \int_{\delta/a_n}^{\delta/a_{n+1}} \exp(-\lambda\xi^2 \textstyle\sum_{k=n+1}^{\infty} a_k^2)\, d\xi < \infty$$

then $P(\sum a_nX_n \in dx)$ has a density.

(ii) $\lim_{\xi\to\infty}|E(e^{i\xi\sum a_nX_n})| = 0$ for every $\{X_n\}$ iff $\lim_{N\to\infty} a_N^{-2} \sum_{n=N+1}^{\infty} a_n^2 = \infty$.

Several consequences, examples and counterexamples are given.

1. Introduction. Let $\{X_n\}$ be a sequence of independent random variables. We say that $\{X_n\}$ is range splitting if

(A) $$\sup_n E|X_n| < \infty, \quad \text{or}$$

(A′) $$E(X_n) = 0 \quad \text{and} \quad \sup_n E(X_n^2) < \infty$$

and there is a sequence of numbers $\{x_n\}$ and $\lambda_1, \lambda_2 > 0$ such that

(B) $$\inf_n P(X_n \ge x_n + \lambda_1) > 0 \quad \text{and} \quad \inf_n P(X_n \le x_n - \lambda_2) > 0.$$

Note that clearly every i.i.d sequence with $E|X_1| < \infty$ and which takes more than one value is range splitting.

Let $\{a_n\}$ be a sequence of real numbers and assume $\sum |a_n| < \infty$, $\sum a_n^2 < \infty$ for (A), (A′) respectively and let

$$X = \sum a_nX_n.$$

In this paper we study under what conditions on the sequence $\{a_n\}$ the distribution measure

(1) $$F_X(dx) = P(\sum a_nX_n \in dx)$$

will have a density, for every range splitting sequence. Of course, if any one of the X_n's has a distribution absolutely continuous with respect to Lebesgue measure (we will denote this by a.c. dx) i.e., has a density, then so does $F_X(dx)$.

However, if we rule out this trivial case, it is clear from the following theorem by P. Lévy [3] that some condition which splits up the range of the X_n's must be imposed if $F_X(dx)$ should have a density for every sequence $\{X_n\}$ which satisfies (A):

THEOREM (*P. Lévy*). *Let $\{X_n\}$ be a sequence of independent random variables. Then $X = \sum_{n=1}^{\infty} X_n$ has a distribution concentrated on a countable number of points iff there exists a sequence of numbers $\{\alpha_n\}$ such that $\sum \alpha_n < \infty$, $\sum_1^{\infty}(X_n \neq \alpha_n) < \infty$ and $\{X_n\}$ is distributed on a countable set.*

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If we restrict our attention for the moment to range splitting sequences $\{X_n\}$ where each X_n is distributed on a countable set, then the following theorem by Jessen and Wintner [2] is in force:

THEOREM (*Jessen-Wintner*). *Let* $X_1, X_2, \cdots$ *be independent random variables such that*

(i) $\sum_i^n X_j \to X$ a.s.
(ii) *each* X_j *is distributed on a countable set.*

Then the distribution of X *is of pure type, i.e., either* X *has a discrete distribution or the distribution of* X *is continuous but singular with respect to Lebesgue measure (singular* dx*) or the distribution of* X *is* a.c. dx.

Hence, combining the Jessen-Wintner Theorem with the theorem of P. Lévy we see that $F_X(dx)$ will either be continuous but singular dx or have a density for every range splitting sequence which is distributed on a countable set. The problem of course is to decide which one it is.

For instance, if we take $a_n = 1/2^n$ and $\{X_n\}$ a sequence of i.i.d's which take the values 0, 1 with probability 1/2, then of course

$$F_X(dx) = P\left(\sum_1^\infty X_n \frac{1}{2^n} \in dx\right) = dx \quad \text{on} \quad [0, 1].$$

On the other hand, we will show in Corollary 1 that for any sequence $\{a_n\}$ such that

$$\liminf_{N\to\infty} a_N^{-2} \sum_{n=N+1}^\infty a_n^2 < \infty$$

there is a sequence $\{p_n\}$, $1/16 \le p_n \le 1$ such that the independent sequence $\{X_n\}$, $X_n = \pm p_n$ with probability 1/2 has $P(\sum a_n X_n \in dx)$ singular dx. We also gave some examples of such singular distributions for sequences of exponential decay in [4].

This shows that if we want $P(\sum a_n X_n \in dx)$ to have a density for every range splitting $\{X_n\}$, then we must restrict the rate of decay of the a_n's. In this vein we proved the following theorem in [4].

THEOREM A. *Let* $\{a_n\}$ *be a sequence such that for some* $\alpha \ge \beta > 0$, $\alpha > 1/2$ *and* $\alpha - \beta < 1/2$

$$\mathrm{C}n^{-\alpha} \le |a_n| \le \bar{C}n^{-\beta}$$

where $0 < \mathrm{C} \le \bar{C}$. *Then* $P(\sum a_n X_n \in dx)$ *has a density for every range splitting* $\{X_n\}$. (*Of course we assume in addition either* $\sum |a_n| < \infty$ *and* (A) *or* $\sum a_n^2 < \infty$ *and* (A′).)

This looks like a very restrictive condition; however, we show in Corollaries 2 and 3 that for $\alpha - \beta \ge 1/2$, Theorem A fails even for the i.i.d sequence $X_n = \pm 1$ with probability 1/2.

We also give an example of a sequence for which the conclusion of Theorem A holds, but which decays faster than any polynomial. In Theorem 1 we give a necessary condition on $\{a_n\}$ for the conclusion of Theorem A to hold, and in Theorem 2 we give a sufficient condition on $\{a_n\}$ which guarantees the conclusion of Theorem A.

Throughout this paper we will always assume a sequence $\{a_n\}$ to satisfy

$$* \qquad a_n > 0 \quad \text{for all } n$$

$$** \qquad a_n \ge a_{n+1} \quad \text{for all } n.$$

There is no loss in generality since for * we always may transfer any negative signs to the corresponding X_n and for ** any rearrangement of the sequence $\{a_n X_n\}$ alters $\sum a_n X_n$ on a set of probability zero.

Furthermore, we will always assume

*** $$X = \sum a_n X_n \text{ is well defined,}$$

i.e., if $\sum a_n^2 < \infty$ but $\sum a_n = \infty$ we will mean all range splitting sequences which satisfy (A′).

For general references on the Jessen-Wintner and P. Lévy Theorem, see Breimann [1, pages 49–51].

2. Theorems and their corollaries.

We will need the following lemmas.

LEMMA 1. *Let p, q, λ be positive numbers such that $p + q \leq 1$. Then for $0 \leq |s| \leq \pi/4\lambda$*

$$[1 - (p + q)] + \sqrt{p^2 + q^2 + 2pq\cos(\lambda s)} \leq 1 - \frac{1}{4} pq\lambda^2 s^2$$

PROOF.

$$\begin{aligned}\frac{1 - [1 - (p + q) + \sqrt{p^2 + q^2 + 2pq\cos(\lambda s)}]}{s^2} &= \frac{(p + q) - \sqrt{p^2 + q^2 + 2pq\cos(\lambda s)}}{s^2} \\ &= \frac{(p + q)^2 - (p^2 + q^2 + 2pq\cos(\lambda s))}{s^2[(p + q) + \sqrt{p^2 + q^2 + 2pq\cos(\lambda s)}]} \\ &\geq \frac{2pq(1 - \cos(\lambda s))}{2(p + q)s^2} \\ &\geq pq\lambda^2 \frac{(1 - \cos(\lambda s))}{(\lambda s)^2}.\end{aligned} \tag{1}$$

It is not hard to show that

$$\frac{1 - \cos(x)}{x^2} > \frac{1}{4} \quad \text{for} \quad 0 \leq |x| \leq \frac{\pi}{4} \tag{2}$$

and therefore

$$1 - \frac{1}{4}x^2 \geq \cos(x) \quad \text{for} \quad 0 \leq |x| \leq \frac{\pi}{4}. \tag{3}$$

Now combine (1) and (3) to get the desired inequality. □

LEMMA 2. *Let $\{X_n\}$ be a range splitting sequence and let*

$$C = \frac{1}{4} p^* q^* (\lambda_1 + \lambda_2)^2, \quad \delta = \min\left(\frac{\pi \min(p^*, q^*)}{8r^*}, \frac{\pi}{4(\lambda_1 + \lambda_2)}\right).$$

Then for $0 \leq |s| \leq \delta$,

$$|E(e^{isX_n})| \leq \exp(-Cs^2)$$

where $r^* = \sup_n E|X_n|$,

$$p^* = \frac{1}{2} \inf_n P(X_n \geq x_n + \lambda_1), \quad q^* = \frac{1}{2} \inf_n P(X_n \leq x_n - \lambda_2)$$

and $\{x_n\}$, λ_1, λ_2 as in the definition of range splitting.

PROOF. Let $M = \dfrac{r^*}{\min(p^*, q^*)}$. Then

$$r^* \geq \int_{\{|X_n| \geq M\}} |X_n| P(d\omega) \geq \frac{r^*}{\min(p^*, q^*)} P(|X_n| \geq M)$$

from which we obtain

$$\min(p^*, q^*) \geq P(|X_n| \geq M)$$

and therefore

(1) $$\inf_n P(M \geq X_n \geq x_n + \lambda_1) \geq p^*$$

and

$$\inf_n P(-M \leq X_n \leq x_n - \lambda_2) \geq q^*.$$

Let

(2) $$A_n = \{M \geq X_n \geq x_n + \lambda_1\}, \quad B_n = \{-M \leq X_n \leq x_n - \lambda_2\}$$

and

$$p = P(A_n), \quad q = P(B_n).$$

We first establish the bound for the sequence $\{x_n\}$: if $x_n \geq 0$ then

$$E|X_n| \geq p(x_n + \lambda_1)$$

from which follows

$$\frac{r^*}{p^*} \geq x_n;$$

if $x_n < 0$ then

$$E|X_n| \geq q(|x_n| + \lambda_2)$$

from which follows

$$\frac{r^*}{q^*} \geq |x_n|$$

i.e.,

(3) $$|x_n| \leq \max\left(\frac{r^*}{p^*}, \frac{r^*}{q^*}\right) = \frac{r^*}{\min(p^*, q^*)}.$$

It is clear from (2) and (3) that on A_n, B_n

(4) $$|X_n - x_n| \leq M + \frac{r^*}{\min(p^*, q^*)} = \frac{2r^*}{\min(p^*, q^*)}$$

and furthermore

$$\frac{2r^*}{\min(p^*, q^*)} \geq X_n - x_n \geq \lambda_1 \quad \text{on} \quad A_n$$

$$-\frac{2r^*}{\min(p^*, q^*)} \leq X_n - x_n \leq \lambda_2 \quad \text{on} \quad B_n.$$

Let

(5) $$\delta = \min\left(\frac{\pi \min(p^*, q^*)}{8r^*}, \frac{\pi}{4(\lambda_1 + \lambda_2)}\right).$$

Now

(6) $$\begin{aligned} |E(e^{isX_n})| &\leq 1 - (p + q) + \left| \int_{A_n} e^{isX_n} P(d\omega) + \int_{B_n} e^{isX_n} P(d\omega) \right| \\ &= 1 - (p + q) + \left| \int_{A_n} e^{is(X_n - x_n)} P(d\omega) + \int_{B_n} e^{is(X_n - x_n)} P(d\omega) \right|. \end{aligned}$$

In what follows we will discuss the case $0 \leq s \leq \delta$; the case $-\delta \leq s < 0$ is done similarly. From (4) and (5) it follows that for $0 \leq s \leq \delta$

$$s\lambda_1 \leq \arg(e^{is(X_n - x_n)}) \leq \frac{\pi}{4} \text{ on } A_n; \quad -\frac{\pi}{4} \leq \arg(e^{is(X_n - x_n)}) \leq s\lambda_2 \text{ on } B_n. \tag{7}$$

Therefore

$$\int_{A_n} e^{is(X_n - x_n)} P(d\omega) = ae^{i\theta_1} \tag{8}$$

where $0 < a \leq p$, $s\lambda_1 \leq \theta_1 \leq \pi/4$ and

$$\int_{B_n} e^{is(X_n - x_n)} P(d\omega) = be^{i\theta_2}$$

where $0 < b \leq q$, $-\pi/4 \leq \theta_2 \leq -s\lambda_2$.

Combining (6) and (8) we obtain

$$\begin{aligned} |E(e^{isX_n})| &\leq 1 - (p + q) + |ae^{i\theta_1} + be^{i\theta_2}| \\ &= 1 - (p + q) + \sqrt{a^2 + b^2 + 2ab\cos(\theta_1 - \theta_2)} \\ &\leq 1 - (p + q) + \sqrt{p^2 + q^2 + 2pq\cos(s(\lambda_1 + \lambda_2))}. \end{aligned} \tag{9}$$

The last inequality follows since by (8) $\cos(\theta_1 - \theta_2) \geq 0$.

Now apply Lemma 1 to (9) and remember that $p \geq p^*$, $q \geq q^*$ to obtain

$$|E(e^{isX_n})| \leq 1 - \frac{1}{4}p^*q^*(\lambda_1 + \lambda_2)^2 s^2 \tag{10}$$

for $0 \leq |s| \leq \delta$.

Letting $C = \frac{1}{4}p^*q^*(\lambda_1 + \lambda_2)^2$ and the fact that $1 + x \leq e^x$ for $x > -1$ proves

$$|E(e^{isX_n})| \leq \exp(-Cs^2) \tag{11}$$

for $0 \leq |s| \leq \delta$. □

THEOREM 1. *Let* $X = \sum a_n X_n$, $F_X(dx) = P\{X \in dx\}$ *and* $\hat{F}_X(\xi)$ *the characteristic function of X then*

$$\lim_{\xi \to \infty} \hat{F}_X(\xi) = 0$$

for every range splitting $\{X_n\}$ iff

$$\lim_{N \to \infty} a_N^{-2} \sum_{n=N+1}^{\infty} a_n^2 = +\infty.$$

PROOF. ($\leftarrow$) Let C, δ be as in Lemma 2.

$$\hat{F}_X(\xi) = E(e^{i\xi \sum a_n X_n}) = \prod_{n=1}^{\infty} E(e^{i\xi a_n X_n}). \tag{1}$$

Let $N(\xi)$ be the last n such that

$$|\xi| a_{N(\xi)} > \delta. \tag{2}$$

Then by Lemma 2

$$\begin{aligned} |\hat{F}_X(\xi)| &\leq \prod_{n=N(\xi)+1}^{\infty} |E(e^{i\xi a_n X_n})| \leq \prod_{n=N(\xi)+1}^{\infty} \exp(-C\xi^2 a_n^2) \\ &= \exp(-C\xi^2 \textstyle\sum_{n=N(\xi)+1}^{\infty} a_n^2) \leq \exp(-C\delta^2 a_{N(\xi)}^{-2} \sum_{n=N(\xi)+1}^{\infty} a_n^2) \end{aligned} \tag{3}$$

where we use (2) for the last inequality.

Since $N(\xi) \to \infty$ as $|\xi| \to \infty$ the result follows from the hypothesis.

To prove ($\to$) we will prove the converse. Suppose

$$\liminf a_N^{-2} \sum_{n=N+1}^{\infty} a_n^2 < \infty.$$

Select a subsequence $\{n_j\}$ such that

(4) $$\sup_j a_{n_j}^{-2} \sum_{k=n_j+1} a_k^2 \le R < \infty$$

and let $\{\ell_j\}$ be the sequence such that

(5) $$2^{-(\ell_j+1)} \le a_{n_j} < 2^{-\ell_j}.$$

The sequence $\{\ell_j\}$ contains infinitely many odd or even integers; i.e., by choosing the n_j's appropriately we may assume all the ℓ_j's to be either even or odd. We will assume all the ℓ_j's are odd; the even case is done similarly. Furthermore, we may assume that the sequence $\{n_j\}$ was chosen such that

(6) $$\ell_{j+1} \ge \ell_j + 4.$$

Now let

(7) $$I_j = \{n \ge n_j + 1 \mid 2^{-(\ell_j+1)} \le a_n < 2^{-\ell_j}\}.$$

By (6) the I_j's are disjoint and $n_{j+1} \notin I_j$.

For $n \notin \cup_{j=1}^{\infty} I_j$ and $2^{-(\ell+1)} \le a_n < 2^{-\ell}$ define

(8) $$p_n = \begin{cases} (a_n 2^{\ell+1})^{-1} & \text{for } \ell \text{ odd} \\ (a_n 2^{\ell+2})^{-1} & \text{for } \ell \text{ even} \end{cases}$$

For $n \in I_j$ define

(9) $$p_n = (a_n 2^{\ell_j+3})^{-1}.$$

From the fact that the I_j's are disjoint and $a_{n_{j+1}} \notin I_j$ it follows that p_{n_j} is defined according to definition (8) and therefore

(10) $$2^{-4} \le p_n < 1 \quad \text{for all} \quad n$$

(11) $$p_{n_j} a_{n_j} = 2^{-(\ell_j+3)}$$

(12) $$p_n a_n \le 2^{-(\ell_j+3)} \quad \text{for} \quad n \ge n_j + 1.$$

Now let $\{X_n\}$ be a sequence of independent random variables such that

(13) $$X_n = \pm p_n \text{ with probability } 1/2.$$

Clearly $\{X_n\}$ is range splitting.

Let $X = \sum a_n X_n$; then

(14) $$\hat{F}_X(\xi) = \prod_{n=1}^{\infty} \cos(\xi p_n a_n).$$

Let $\xi_j = \pi 2^{\ell_j+1}$. Then

(15) $$\hat{F}_X(\xi_j) = \pm \prod_{n \ge n_j+1} \cos(\pi 2^{\ell_j+1} p_n a_n).$$

This follows from the fact that $p_n a_n = 2^{-\text{even integer}}$ and for $n < n_j$, $p_n a_n \ge 2^{-(\ell_j+1)}$. From (11) and (12) it is also clear that

(16) $$\pi 2^{\ell_j+1} p_n a_n \le \frac{\pi}{4} \quad \text{for} \quad n \ge n_j + 1.$$

For $0 \le x \le \frac{\pi}{4}$ we have the inequality

$$\frac{1-\cos(x)}{x^2} = \left(\frac{\sin(x)}{x}\right)^2 \frac{1}{1+\cos(x)} < 1$$

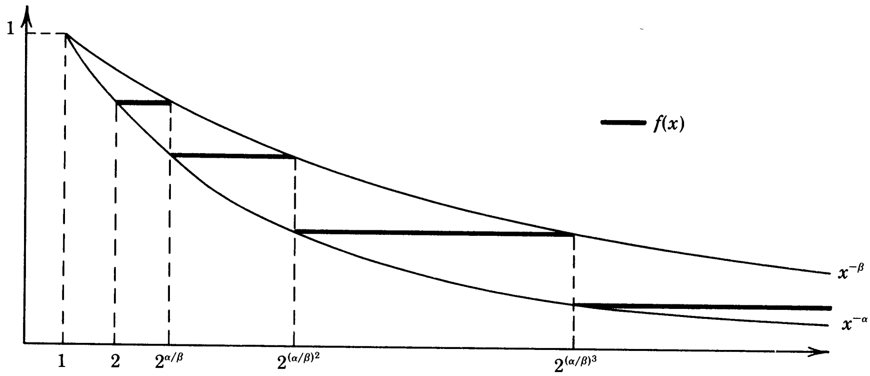

FIG. 1

from which follows

$$1 - x^2 \leq \cos(x) \quad \text{for} \quad 0 \leq x \leq \frac{\pi}{4}.$$

Also for $-.64 \leq y \leq 0$

$$\ln(1+y) \geq \frac{|\ln(.36)|}{.64} y$$

and therefore letting $C_0 = \dfrac{|\ln(.36)|}{.64}$ we have

$$\cos(x) \geq 1 - x^2 \geq \exp(-C_0 x^2) \quad \text{for} \quad 0 \leq x \leq \frac{\pi}{4}. \tag{17}$$

Now we use (16) and (17) in (15) to conclude

$$\begin{aligned} |\hat{F}_X(\xi_j)| &\geq \textstyle\prod_{n \geq n_j+1} \exp(-C_0(\pi 2^{l_j+1})^2 p_n^2 a_n^2) = \exp(-C_0 \pi^2 (2^{l_j+1})^2 \sum_{k=n_j+1}^{\infty} p_n^2 a_n^2) \\ &\geq \exp(-C_0 \pi^2 16 a_{n_j}^{-2} \textstyle\sum_{k=n_j+1}^{\infty} a_k^2) \geq \exp(-C_0 \pi^2 16 R) \end{aligned} \tag{18}$$

where the next to last inequality follows from (10) and (11) and R from (4).

Since (18) holds for all j it follows that

$$\limsup_{\xi \to \infty} |\hat{F}_X(\xi)| > 0. \quad \square$$

COROLLARY 1. *If* $\liminf_{n \to \infty} a_N^{-2} \sum_{n=N+1}^{\infty} a_n^2 < \infty$, *then there exists a sequence of positive numbers* $\{p_n\}$ *with* $2^{-4} \leq p_n \leq 1$ *such that if* $\{X_n\}$ *is the range splitting sequence which takes the values*

$$X_n = \pm p_n \quad \textit{with probability} \quad ½$$

then $F_X(dx) = P(\sum a_n X_n \in dx)$ *is continuous but singular* dx.

PROOF. The sequence $\{p_n\}$ was constructed in the proof of Theorem 1. Since the Jessen-Wintner Theorem applies to $\sum a_n X_n$ and $\{X_n\}$ is range splitting, it follows that $F_X(dx)$ is either continuous and singular dx or a.c. dx. Since we showed in the proof of Theorem 1, that $\limsup |\hat{F}_X(\xi)| > 0$ it follows that $F_X(dx)$ cannot have a density. $\square$

In the next two Corollaries we give counterexamples to Theorem A for $\alpha - \beta \geq ½$. In Corollary 2 we show that if $f(x)$ is the step function of Figure 1 and $a_n = f(n)$ then for $\alpha - \beta \geq ½ + \varepsilon$ and $\beta \geq ½ + ¼\, \varepsilon$ we have $\liminf a_N^{-2} \sum_{n=N+1}^{\infty} a_n^2 < \infty$.

COROLLARY 2. *Let* $\varepsilon > 0$. *If* $\beta \geq ½ + ¼\,\varepsilon$ *and* $\alpha - \beta \geq ½ + \varepsilon$, *then the sequence* $a_1 = 1$, $a_n = 2^{-\alpha\left(\frac{\alpha}{\beta}\right)^k}$ *for* $2^{\left(\frac{\alpha}{\beta}\right)^k} \leqslant n < 2^{\left(\frac{\alpha}{\beta}\right)^{k+1}}$ *and* $k = 0, 1, \cdots$ *satisfies*

$$n^{-\alpha} \leq a_n \leq n^{-\beta}$$

and

$$\liminf a_N^{-2} \textstyle\sum_{n=N+1}^{\infty} a_n^2 < \infty.$$

PROOF. Clearly $n^{-\alpha} \leq a_n \leq n^{-\beta}$. Let N_k = greatest integer less than $2^{\left(\frac{\alpha}{\beta}\right)^{k+1}}$; then

$$a_{N_k}^{-2} \textstyle\sum_{n=N_k+1}^{\infty} a_n^2 \leq 2^{2\alpha\left(\frac{\alpha}{\beta}\right)^k} \textstyle\sum_{n=k+1}^{\infty} \left(2^{\left(\frac{\alpha}{\beta}\right)^{n+1}} - 2^{\left(\frac{\alpha}{\beta}\right)^{n}}\right) 2^{-2\alpha\left(\frac{\alpha}{\beta}\right)^n}$$

$$= \textstyle\sum_{n=k+1}^{\infty} 2^{2\alpha\left(\frac{\alpha}{\beta}\right)^k + \left(\frac{\alpha}{\beta}\right)^{n+1} - 2\alpha\left(\frac{\alpha}{\beta}\right)^n} \left(1 - 2^{\left(\frac{\alpha}{\beta}\right)^n - \left(\frac{\alpha}{\beta}\right)^{n+1}}\right)$$

$$\leq \textstyle\sum_{n=k+1}^{\infty} \left[2^{2\alpha + \left(\frac{\alpha}{\beta}\right)^{n+1-k} - 2\alpha\left(\frac{\alpha}{\beta}\right)^{n-k}}\right]^{\left(\frac{\alpha}{\beta}\right)^k} \tag{1}$$

$$= \textstyle\sum_{n=1}^{\infty} \left[2^{2\alpha + \left(\frac{\alpha}{\beta}\right)^n \alpha\left(\frac{1}{\beta} - 2\right)}\right]^{\left(\frac{\alpha}{\beta}\right)^k}.$$

Since $\beta > ½$ and therefore $1/\beta - 2 < 0$ it follows that

$$2\alpha + \left(\frac{\alpha}{\beta}\right)^n \alpha\left(\frac{1}{\beta} - 2\right)$$

is strictly decreasing with n and therefore we obtain that

$$\lim_{k\to\infty} \textstyle\sum_{n=1}^{\infty} \left(2^{2\alpha + \left(\frac{\alpha}{\beta}\right)^n \alpha\left(\frac{1}{\beta} - 2\right)}\right)^{\left(\frac{\alpha}{\beta}\right)^k} = 1, 0 \tag{2}$$

for $2\alpha + (\alpha/\beta)\,\alpha\,(1/\beta - 2) =, < 0$, respectively.

Combining (1) and (2) we see that for $2\alpha + (\alpha/\beta)\,\alpha\,(1/\beta - 2) \leq 0$

$$\liminf a_N^{-2} \textstyle\sum_{n=N+1}^{\infty} a_n^2 < \infty. \tag{3}$$

Now $2\alpha + (\alpha/\beta)\,\alpha\,(1/\beta - 2) \leq 0$ implies $\alpha \geq 2\beta^2/(2\beta - 1)$ which together with $\alpha \geq \beta + ½ + \varepsilon$ shows that if

$$\beta \geq \frac{1 + 2\varepsilon}{4\varepsilon}, \tag{4}$$

then (3) holds. □

In the following Corollary we show that if the function from Figure 1 is modified as shown in Figure 2 and $\{n_k\}$ chosen appropriately, then for $\beta > ½$ and $\alpha - \beta \geq ½$ the sequence $a_n = f(n)$ satisfies

$$\liminf_N a_N^{-2} \textstyle\sum_{n=N+1}^{\infty} a_n^2 < \infty.$$

COROLLARY 3. *Let* $\beta > ½$. *For any* $\alpha - \beta \geq ½$ *there exists a strictly increasing sequence of integers* $\{n_k\}$ *such that if* $a_1 = 1$,

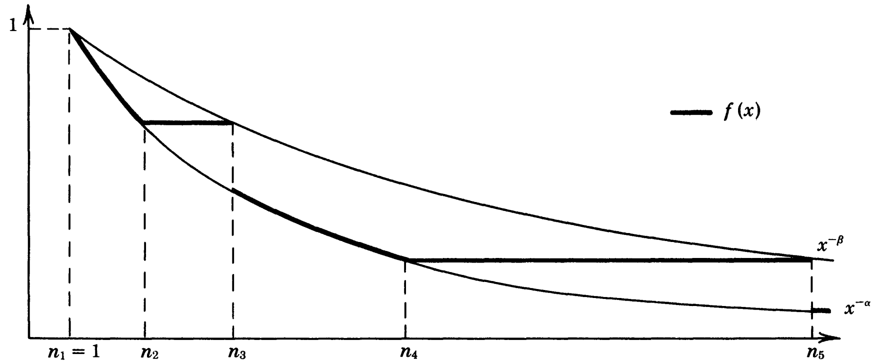

FIG. 2

$$a_n = \begin{cases} n^{-\alpha} & \text{for } n_{2k-1} < n \le n_{2k} \\ (n_{2k})^{-\alpha} & \text{for } n_{2k} < n \le n_{2k+1} \end{cases}$$

$k = 1, 2, \cdots$. Then $n^{-\alpha} \le a_n \le n^{-\beta}$ and

$$\liminf_N a_N^{-2} \textstyle\sum_{n=N+1}^{\infty} a_n^2 < \infty.$$

Furthermore, this sequence $\{a_n\}$ can be modified to a sequence $\{\bar{a}_n\}$ such that

$$2^{-4} n^{-\alpha} \le \bar{a}_n \le n^{-\beta}$$

and $P(\sum \bar{a}_n \bar{X}_n \varepsilon\, dx)$ is continuous but singularly dx, where $\bar{X}_n$ is the i.i.d. *sequence* ± 1 *with probability* ½.

PROOF. Set $n_1 = 1$. Suppose $n_1, \cdots, n_{2k}$ have been chosen. We want $a_n = (n_{2k})^{-\alpha}$ for $n_{2k} < n \le n_{2k+1}$. So let n_{2k+1} be the last n such that

$$n^{-\alpha} \le a_n = (n_{2k})^{-\alpha} \le n^{-\beta}.$$

Since the lower inequality is automatically satisfied, this implies

$$n \le (n_{2k})^{\frac{\alpha}{\beta}}$$

from which follows

$$n_{2k+1} = [(n_{2k})^{\frac{\alpha}{\beta}}] \tag{1}$$

where [ ] denotes integer part.

Now suppose we selected $n_1, \cdots, n_{2k-1}$. We want n_{2k} to satisfy the following property

$$\textstyle\sum_{n=1+n_{2k-1}}^{n_{2k}} a_n^2 \ge \frac{3}{4} \sum_{n=1+n_{2k-1}}^{\infty} a_n^2. \tag{2}$$

To find n_{2k} which satisfies (2), observe that this means

$$\tfrac{1}{4} \textstyle\sum_{n=1+n_{2k-1}}^{n_{2k}} a_n^2 \ge \sum_{n=1+n_{2k}}^{\infty} a_n^2. \tag{3}$$

Now from $n^{-\alpha} \le a_n \le n^{-\beta}$

$$\textstyle\sum_{n=1+n_{2k-1}}^{n_{2k}} a_n^2 \ge \displaystyle\int_{1+n_{2k-1}}^{n_{2k}} x^{-2\alpha}\, dx \tag{4}$$

and

$$\sum_{n=1+n_{2k}}^{\infty} a_n^2 \leq \int_{n_{2k}}^{\infty} x^{-2\beta}\, dx.$$

Therefore from (3) and (4) it follows that to satisfy (2), we simply select n_{2k} so large that

$$(5) \qquad \tfrac{1}{4} \int_{1+n_{2k-1}}^{n_{2k}} x^{-2\alpha}\, dx \geq (2\beta - 1)^{-1}(n_{2k})^{-2\beta+1}$$

which is clearly possible.

Now from (2) we obtain

$$(6) \qquad \begin{aligned} \sum_{n=1+n_{2k-1}}^{\infty} a_n^2 \leq \tfrac{4}{3} \sum_{n=1+n2k-1}^{n_{2k}} a_n^2 \leq \tfrac{4}{3} \int_{n_{2k-1}}^{\infty} x^{-2\alpha}\, dx \\ = \tfrac{4}{3}(2\alpha - 1)^{-1}(n_{2k-1})^{-2\alpha+1}. \end{aligned}$$

From (1), it follows that for every k there is $0 \leq \varepsilon < 1$ such that $\varepsilon + n_{2k+1} = (n_{2k})^{\frac{\alpha}{\beta}}$. Therefore

$$n_{2k} = (\varepsilon + n_{2k+1})^{\frac{\beta}{\alpha}}$$

and therefore

$$(7) \qquad a_{n_{2k+1}} = (n_{2k})^{-\alpha} = (\varepsilon + n_{2k+1})^{-\beta}.$$

From (7) it follows that for k sufficiently large

$$(8) \qquad \tfrac{1}{2}(n_{2k+1})^{-\beta} \leq a_{n_{2k+1}} \leq (n_{2k+1})^{-\beta}.$$

Hence from (6) and (8) we get for k sufficiently large

$$(9) \qquad (a_{n_{2k+1}})^{-2} \sum_{n=1+n_{2k+1}}^{\infty} a_n^2 \leq \tfrac{16}{3}\, (2\alpha - 1)^{-1}(n_{2k+1})^{1-2(\alpha-\beta)}.$$

Now it follows from (9) that if $(\alpha - \beta) \geq \tfrac{1}{2}$

$$(10) \qquad \liminf_N a_N^{-2} \sum_{n=N+1}^{\infty} a_n^2 < \infty.$$

For the second part, it follows from Corollary 1 that there is a sequence $\{p_n\}$ with $2^{-4} \leq p_n \leq 1$ such that $P(\sum a_n p_n \bar{X}_n \varepsilon\, dx)$ is continuous but singular dx. It can be seen from the proof of Theorem 1, where we constructed the sequence $\{p_n\}$ that $\{a_n p_n\}$ is decreasing, i.e., $\bar{a}_n = a_n p_n$ is the desired sequence. □

THEOREM 2. *If for all* $\lambda, \delta > 0$

$$\sum_{n=1}^{\infty} \int_{\delta/a_n}^{\delta/a_{n+1}} \exp(-\lambda \xi^2 \textstyle\sum_{k=n+1}^{\infty} a_k^2)\, d\xi < \infty,$$

then $F_X(dx) = P(\sum a_n X_n\, dx)$ *has a density for every range splitting* $\{X_n\}$.

PROOF. Let δ, C be as in Lemma 2 and let $N(\xi)$ be the last n such that

$$(1) \qquad |\xi|\, a_n \geq \delta.$$

Then by Lemma 2

$$(2) \qquad |\hat{F}_X(\xi)| \leq \prod_{n=N(\xi)+1}^{\infty} \exp(-C\xi^2 a_n^2) = \exp(-C\xi^2 \textstyle\sum_{n=N(\xi)+1}^{\infty} a_n^2).$$

For $\qquad \delta/a_n < |\xi| \leq \delta/a_{n+1}, \quad N(\xi) = n.$

Therefore for any $p > 0$

(3) $$\int_{-\infty}^{\infty} |\hat{F}_X(\xi)|^p \, d\xi \leq 2\delta/a_1 + 2\sum_{n=1}^{\infty} \int_{\delta/a_n}^{\delta/a_{n+1}} \exp(-pC\xi^2 \sum_{k=n+1}^{\infty} a_k^2) \, d\xi < \infty.$$

From (3) it is clear that in particular

(4) $$\hat{F}_X(\xi) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Let $\phi(x)$ be the inverse Fourier transform of $\hat{F}_X(\xi)$, i.e.,

(5) $$\phi(x) = \int_{\mathbb{R}} e^{ix\xi} \hat{F}_X(\xi) \, d\xi.$$

Then $\phi(x)$ is the density of $F_X(dx)$ by Theorem 8.39 in [1, page 178]. □

Corollary 4. *If for all* $\lambda > 0$

$$\sum_{n=1}^{\infty} \left(\frac{1}{a_{n+1}} - \frac{1}{a_n} \right) \exp(-\lambda a_n^{-2} \sum_{k=n+1}^{\infty} a_k^2) < \infty$$

then $P(\sum a_n X_n \varepsilon \, dx)$ *has a density for every range splitting* $\{X_n\}$.

Proof. We observe that for

$$\delta/a_n \leq |\xi| \leq \delta/a_{n+1}, \quad \exp(-\lambda\xi^2 \sum_{k=n+1}^{\infty} a_k^2) \leq \exp(-\lambda\delta^2 a_n^{-2} \sum_{k=n+1}^{\infty} a_k^2)$$

and apply Theorem 2. □

Remark. It should be noted that Theorem A is an easy consequence of Corollary 4.

Corollary 5. *Let* $\{X_n\}$ *be a range splitting sequence and* C, *as in Lemma* 2. *Suppose* $\{a_n\}$ *satisfies the hypothesis of Theorem* 2. *Then for any measurable set* B

$$P(\sum a_n X_n \varepsilon B) \leq C_0 |B|$$

where $|\ |$ *denotes Lebesgue measure and*

$$C_0 = 2\delta/a_1 + 2\sum_{n=1}^{\infty} \int_{\delta/a_n}^{\delta/a_{n+1}} \exp(-C\xi^2 \sum_{k=n+1}^{\infty} a_k^2) \, d\xi.$$

Proof. From (3) in the proof of Theorem 2

$$\int_{\mathbb{R}} |\hat{F}_X(\xi)| \, d\xi \leq C_0$$

and therefore the density $\phi(x)$ which is the inverse Fourier transform of $\hat{F}_X(\xi)$ satisfies $\|\phi\|_\infty \leq C_0$. □

3. A sequence which decays faster than any polynomial. We give an example of a sequence which decays faster than any polynomial, but still satisfies the hypothesis of Corollary 4. Let $a_1 = a_2 = 1$. For $n = 3, 4, \cdots$

$$a_n = \frac{\sqrt{\ln(n)}}{\sqrt{n}\, n^{\ln(n)}}.$$

Then

(1) $$\sum_{n=N}^{\infty} a_n^2 \geq \int_N^{\infty} \frac{\ln(x)}{x} e^{-2(\ln(x))^2} \, dx = -\frac{1}{4} e^{-2(\ln(x))^2} \Big|_N^{\infty} = \frac{1}{4} N^{-2\ln(N)}.$$

Therefore there exists a constant $C > 0$ such that

$$a_N^{-2} \sum_{n=N+1}^{\infty} a_n^2 \geq \frac{N\, N^{2\ln(N)}}{\ln(N) 4(N+1)^{2\ln(N+1)}} \geq C\, N^{1-\varepsilon} \tag{2}$$

for N sufficiently large. The last inequality follows from the fact that

$$\lim_{N\to\infty} \frac{N^{2\ln(N)}}{(N+1)^{2\ln(N+1)}} = 1.$$

So for any $\lambda > 0$ we obtain

$$\sum_{N=1}^{\infty} \left(\frac{1}{a_{N+1}} - \frac{1}{a_N}\right) \exp(-\lambda a_N^{-2} \sum_{n=N+1}^{\infty} a_n^2) \leq \sum_{N=1}^{\infty} \frac{(N+1)^{\ln(N+1)}\sqrt{N+1}}{\sqrt{\ln(N+1)}} \exp(-\lambda C N^{1-\varepsilon})$$

$$\leq \sum_{N=L}^{\infty} (N+1)^{2\ln(N+1)} \exp(-\lambda C N^{1-\varepsilon})$$

$$= \sum_{N=1}^{\infty} \exp(-\lambda C N^{1-\varepsilon} + 2\ln^2(N+1)) < \infty.$$

REFERENCES

BREIMAN, L. (1968). *Probability*. Addison-Wesley, Reading, Mass.

JESSEN, B. and WINTNER, A. (1935). Distribution functions and the Riemann zeta function. *Trans. Amer. Math. Soc.* **38** 48–88.

LÉVY, P. (1931). Sur les séries dont les termes sont des variables éventuellement indépendantes. *Studia Math.* **3** 119–155.

REICH, J. I. (1982). Some results on distributions arising from coin tossing. *Ann. Probability* **10,** 780–786.

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