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ON THE EXISTENCE OF NATURAL RATE OF ESCAPE FUNCTIONS FOR INFINITE DIMENSIONAL BROWNIAN MOTIONS

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It is proved that genuinely infinite dimensional Brownian motions on ℓ^p sequence spaces have natural rates of escape, provided the coordinates are independent. An analogous result holds for separable Hilbert space. Computations of Brownian rates of escape and further properties are considered.

1. Introduction. The main result presented here concerns the existence of natural rate of escape functions for a class of infinite dimensional Brownian motions. For any stochastic process $\langle X(t): t \geq 0\rangle$ taking values in a real Banach space $(V, \|\cdot\|)$, a nondecreasing function $\gamma: [0, \infty) \to (0, \infty)$ is called a *natural rate of escape* for X if $\liminf \|X(t)\|/\gamma(t) = 1$ as $t \to \infty$ with probability one (abbreviated w.p.1). The Brownian motions to which the theorem applies are those in ℓ^p sequence spaces $(1 \leq p \leq \infty)$ with independent coordinates (see (2.7), below). This result partially settles a conjecture made by Erickson [7], who surmised, after studying some examples, that *all* genuinely infinite dimensional Brownian motions possess natural rates of escape. According to a result of Dvoretzky and Erdös [6], for any genuinely $\mathbf{d}$-dimensional Brownian motion $X(t)$ with $3 \leq \mathbf{d} < \infty$, and for any function γ for which $t^{-1/2}\gamma(t) \downarrow 0$ as $t \to \infty$, $\liminf \|X(t)\|/\gamma(t)$ is 0 or ∞, w.p.1. The difference between the infinite and finite dimensional cases which we make use of is that $F(u) \equiv P[\|X(1)\| \leq u]$ is $o(u^n)$ as $u \downarrow 0$ for all n when X is genuinely infinite dimensional, whereas $F(u) \neq o(u^n)$ for $n \geq \mathbf{d}$ when X is $\mathbf{d}$-dimensional with $\mathbf{d} < \infty$.

To explain the importance of this difference, we need to look at the test given in [7].

THEOREM. (Erickson, 1980). *Let X be a genuinely $\mathbf{d}$-dimensional Brownian motion on a Banach space $(V, \|\cdot\|)$ with $3 \leq \mathbf{d} \leq \infty$, and let h be an admissible function. Fix $b > 1$ and put $\gamma(t) = t^{1/2}h(t)$. Then*

$$\liminf \|X(t)\|/\gamma(t) \gtreqless 1 \qquad \text{w.p.1} \tag{1.1}$$

depending on whether

$$\Sigma_k h(b^k)^{-2} P[\|X(1)\| \leq h(b^k)] \quad \begin{cases} \text{converges} \\ \text{diverges.} \end{cases} \tag{1.2}$$

The definitions of a Brownian motion and an admissible function are given in Section 2 below. The point to notice is that the existence of a rate of escape function can be shown by solving the following functional inequality problem:

(1.3) Given a nondecreasing function G with $G(0) = 0$, when does there exist a positive sequence $\langle h_k \rangle$ such that

$$\Sigma_k G(\theta h_k) \quad \begin{cases} \text{converges} & \text{if } \ 0 < \theta < 1 \\ \text{diverges} & \text{if } \ \theta \geq 1. \end{cases}$$

One easily checks that there is no solution if, for some n, $G(u) \sim u^n$ as $u \downarrow 0$. On the other hand, the constant $C = \inf\{u > 0: G(u) > 0\}$ works if it is positive. This suggests that if $G \to 0$ "rapidly," a solution may exist.

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This problem is of greater probabilistic import that just infinite dimensional Brownian motion, since most upper-lower class tests involve convergence or divergence of a series (or integral). Usually one determines by inspection a solution which typically involves a series of iterated logarithms (see Proposition 3.1 below). However, the most general sufficient conditions for existence of a solution which we have found are contained in the following.

PROPOSITION 3.2. *Let* $G: (0, \infty) \to (0, \infty)$ *be nondecreasing with*

(1.4a) $$G(u) = o(u^n) \quad \text{as } u \downarrow 0 \text{ for all } n > 0,$$

(1.4b) $$\log G(u) \quad \text{concave.}$$

Then there exists a solution $\langle h_n \rangle$ *to the problem* (1.3) *with the following properties*

(1.5a) $$h_n \downarrow 0 \quad \text{as } n \uparrow \infty$$

(1.5b) $$\limsup_{n\to\infty} \frac{h_{n+1}}{h_n} = 1.$$

The proof is in Section 3. One need not look far to find applications of this result, e.g. the following.

COROLLARY. *Let* $Y_1, Y_2, \cdots$ *be* i.i.d. *random variables with common distribution function F. If* $G(u) \equiv 1 - F(u^{-1})$ *satisfies* (1.4a) *and* (1.4b), *then there is a sequence of numbers* $a_1, a_2, \cdots$ *such that* $P[Y_n \geq \theta a_n \text{ i.o.}] = 0$ *or* 1 *depending on whether* $\theta > 1$ *or* $\theta \leq 1$.

Unfortunately, we have not been able to show that the solution of (1.3) given by Proposition 3.2 is sufficiently nice for use in Erickson's test, and have had to resort to a stronger log concavity property than (1.4b). See Proposition 3.3 below for this result.

Section 2 contains definitions and preliminary results, including an extension of Erickson's test which allows one to evaluate $\liminf \|X(t)/\gamma(t) - y\|$, where y is a fixed point in V. One surprising consequence of this result is that $\liminf\|X(t)/\gamma(t) - y\| = \liminf\|X(t)/\gamma(t) - z\|$ for all $y, z \in V$, w.p.1, provided V is separable (see Theorem 2.5). The main results are in Section 3, including the solutions to the functional inequality problem and its application to existence of rates of escape (Theorem 3.5). While the existence theorem is constructive in that it exhibits a solution, it is not very useful in explicitly calculating rate of escape functions. This latter topic is considered in Section 4, where we obtain useful estimates of small deviation probabilities for the ℓ^2 norm of Gaussian random vectors.

2. Preliminary definitions and results. By a Brownian motion on a Banach space $(V, \|\cdot\|)$, we mean a V-valued stochastic process $\langle X(t) : t \geq 0 \rangle$ which satisfies the following:

(2.1) $X(0) = 0$.

(2.2) The sample paths of X are $\|\cdot\|$-continuous.

(2.3) X has the Brownian scaling property, i.e. for $t > 0$

$$\text{Law}[t^{-1/2}X(t)] = \text{Law}\, X(1).$$

(2.4) X has stationary independent increments.

(2.5) The following version of the strong Markov property holds for X:
For $t \geq 0$ put

$$\mathscr{F}_t^0 = \sigma[X(s)^{-1}(B) : 0 \leq s \leq t, B \text{ an open ball}]$$

and let $\mathscr{F}_t$ be the usual augmentation of $\mathscr{F}_t^0$ (see IV.48 and IV.52 of [5]). If λ is a stopping time with respect to the filtration $\langle \mathscr{F}_t : t \geq 0\rangle$, then conditional on $[\lambda < \infty]$, the process $Y(t) \equiv X(\lambda + t) - X(\lambda)$, $t \geq 0$ is a probabilistic replica of X which is independent of $\mathscr{F}_{\lambda+}$.

It is readily verified from (2.1) through (2.4) that X is a Gaussian process. A general existence theorem for processes satisfying (2.1) through (2.4) has been given by Kuelbs (Theorem 4 of [11]). See Proposition 2.1 of Cox [4] for a proof that (2.5) holds for the processes of [11] and the ℓ^p processes discussed next.

An independent coordinate ℓ^p valued Brownian motion ($1 \leq p \leq \infty$) is constructed in the following way. Take a sequence $\langle B_1(t), B_2(t), \cdots\rangle$ of independent standard one dimensional Brownian motions and a sequence $\langle \sigma_1, \sigma_2, \cdots\rangle$ of positive constants satisfying

$$\text{(2.6a)} \qquad \textstyle\sum_k \sigma_k^p < \infty \quad \text{if} \quad 1 \leq p < \infty,$$

$$\text{(2.6b)} \qquad \text{for some } u > 0, \qquad \textstyle\sum_k \exp[-u^2/2\sigma_k^2] < \infty \quad \text{if} \quad p = \infty.$$

Then the process

$$\text{(2.7)} \qquad X(t) = \langle \sigma_1 B_1(t), \sigma_2 B_2(t), \cdots\rangle$$

is a Brownian motion taking values in ℓ^p. The verifications of (2.1) through (2.5) have been given by Cox (see Propositions I.4.1 and I.4.2 of [4]). We note that the conditions (2.6) on the variance parameters are necessary and sufficient for X in (2.7) to be an ℓ^p valued process. An interesting refinement of (2.6b) is the following: after (3.4) of Hoffman-Jørgensen, et al. [9], put

$$\text{(2.8)} \qquad \mathscr{C} \equiv \inf\{u > 0 : \textstyle\sum_k \exp[-u^2/2\sigma_k^2] < \infty\},$$

then for all $t \geq 0$

$$\mathscr{C}\sqrt{t} = \inf\{r > 0 : P[\| X(t)\|_\infty \leq r] > 0\}.$$

Furthermore, X is a c_0 valued Brownian motion just in case $\mathscr{C} = 0$. Note that if $\mathscr{C} > 0$, the Gaussian measure Law $X(1)$ is neither Radonian nor tight, so many standard results (e.g. Kuelbs [11] or Borell [3]) are inapplicable.

A Brownian motion X is called *genuinely* **d**-*dimensional* ($1 \leq d \leq \infty$) if there is a closed linear subspace $V_0 \subseteq V$, referred to as the support space of X, such that

$$P[X(t) \in V_0 \quad \text{for all} \quad t \geq 0] = 1$$

but for any proper closed subspace $W \subseteq V_0$

$$P[X(t) \in W \quad \text{for} \quad t \geq 0] < 1,$$

and $\mathbf{d}$ = dimension of V_0. It is easily seen that a Brownian motion of the form (2.7) is genuinely infinite dimensional since $\sigma_k > 0$ for all k.

Following Erickson (page 325 of [7]), let us call a continuous function $h : (0, \infty) \to (0, \infty)$ *admissible* if h is eventually nonincreasing, $\gamma(t) \equiv t^{1/2}h(t)$ is eventually nondecreasing, and h is slowly varying at ∞, i.e. $h(ct)/h(t) \to 1$ as $t \to \infty$ for all $c > 0$.

We may now state the following extension of Theorem 2 from [7].

PROPOSITION 2.1. *Let X be a genuinely* **d**-*dimensional Brownian on a Banach space V with $3 \leq \mathbf{d} \leq \infty$. Let h be an admissible function and fix $y \in V$. Put $\gamma(t) = t^{1/2}h(t)$, and let $u > 0$.*

(a) *If for some $b > 1$*

$$\text{(2.9)} \qquad \textstyle\sum_k h(b^k)^{-2} P[\| X(1) - h(b^k)y \| \leq uh(b^k)] = \infty,$$

then $\liminf \| X(t)/\gamma(t) - y \| \leq u$ *as* $t \to \infty$, w.p.1.

(b) *Take $b > 1$ and put*

$$\text{(2.10)} \qquad \delta(y, b) = 4 \| y \| (1 - b^{-1}).$$

If the series in (2.9) *converges, then*

$$\liminf \| X(t)/\gamma(t) - y \| \geq u - \delta(y, b) \quad \text{as } t \to \infty, \qquad \text{w.p.1.}$$

The theorem follows easily from the following lemmas. For convenience, define

$$\lambda(y) = \liminf_{t\to\infty} \| X(t)/\gamma(t) - y \| \tag{2.11}$$

$$q(y, u, t) = h(t)^{-2} P[\| X(1) - h(t)y \| \leq uh(t)] \quad \text{where} \quad u, t \geq 0 \tag{2.12}$$

$$p(y, u, t_1, t_2) = P[\| X(t)/\gamma(t) - y \| \leq u \quad \text{for some} \quad t \in [t_1, t_2)] \quad \text{where} \quad u \geq 0 \quad \text{and} \quad 0 \leq t_1 < t_2. \tag{2.13}$$

Note that the random variable $\lambda(y)$ is constant w.p.1 by Kolmogorov's 0-1 law.

LEMMA 2.2. *If $b > 1$ and $u, \varepsilon > 0$ then* (2.14) *implies* (2.15) *implies* (2.16) *where*

$$P[\|X(t)/\gamma(t) - y\| < u \quad \text{i.o. (\textit{infinitely often})} \quad \text{as } t \to \infty] = 1 \tag{2.14}$$

$$\textstyle\sum_k p(y, u, b^k, b^{k+1}) = \infty \tag{2.15}$$

$$P[\|X(t)/\gamma(t) - y\| < u + \varepsilon \quad \text{i.o.} \quad \text{as } t \to \infty] = 1. \tag{2.16}$$

LEMMA 2.3. *Fix positive u, ε. For all $b > 1$ there exists k_0 such that $k \geq k_0$ implies*

$$p(y, u + \varepsilon, b^{k-1}, b^k) \geq A_1 q(y, u, b^k)$$

where

$$A_1 = (1 - b^{-1}) \Big/ \int_0^\infty P[\|X(t)\| \leq 2(u + \|y\|)]\, dt > 0.$$

LEMMA 2.4. *Fix positive u, ε. For all $b > 1$ satisfying $(1 - b^{-1})\|y\| < \varepsilon$ there is a k_0 such that $k \geq k_0$ implies*

$$p(y, u, b^k, b^{k+1}) \leq A_2 q(y, u + 4\varepsilon, b^k)$$

where

$$A_2 = (b^2 - 1)\varepsilon^{-2} \max\{8E\|X(1)\|^2, \tfrac{1}{2}\varepsilon^2 h(b)^2/(b^2 - b)\} < \infty.$$

The proofs of these lemmas are similar to the proofs of Theorem 1 and Lemmas 1 and 2 of Erickson [7]. The details are given in [4]. We note that the requirement $\mathbf{d} \geq 3$ is only needed to prove $A_1 > 0$.

When the following theorem is applied to a natural rate of escape function γ, one obtains the rather surprising result that the normalized process $X(t)/\gamma(t)$ enters any ball of radius $(1 - \varepsilon)$ only finitely often as $t \to \infty$, but enters any ball of radius $(1 + \varepsilon)$ infinitely often as $t \to \infty$.

THEOREM 2.5. *Let X be a genuinely $\mathbf{d}$-dimensional Brownian motion on a separable Banach space V with $3 \leq \mathbf{d} \leq \infty$. If V_0 is the support space of X, and h is an admissible function, then*

$$P[\lambda(y) = \lambda(0) \quad \text{for all} \quad y \in V_0] = 1$$

where λ is given by (2.11) *with $\gamma(t) = t^{1/2}h(t)$.*

PROOF. Without loss of generality, assume $V_0 = V$. The result is a consequence of the preceding proposition and some probability estimates which are straightforward generalizations of (2.1.1) and (2.1.2) of Hoffman-Jørgensen, et al. [9]. First, we claim

$$q(y, u, t) \leq q(0, u, t) \tag{2.17}$$

all $y \in V$ and $u, t \geq 0$. Fix t and put $\mu = \text{Law } X(t)$. Since V is separable, μ is a Radon measure (see page 132 of Badrikian [1]) and so the Gaussian measure μ is 0-convex by Theorem 3.2 of Borell [3], i.e., for all pairs of Borel sets $A_1, A_2 \subseteq V$ and real numbers $\lambda \in [0, 1]$

$$\mu_*(\lambda A_1 + (1-\lambda)A_2) \geq \mu(A_1)^\lambda \mu(A_2)^{1-\lambda} \tag{2.18}$$

where μ_* is the inner measure induced by μ. From this it follows that if $K \subseteq V$ is closed and convex and $z_0 \in V$ is fixed, then the set

$$M = \{z \in V : \mu(z + V) \geq \mu(z_0 + V)\}$$

is convex. If K is symmetric in the sense that $z \in K$ implies $-z \in K$, then $\mu(-z_0 + K) = \mu(z_0 + K)$, so $0 \in M$. Applying this to centered balls in V, we see that μ-measure of a translated ball never exceeds μ-measure of a centered ball. This proves (2.17).

Now let H be the generating Hilbert space for μ, i.e. $H \subseteq V$ and (i, H, V) is an abstract Wiener space, where $i : H \to V$ is the inclusion map, and the measure induced on V by extending the canonical Gaussian cylinder measure on H to the completion of H under the measurable norm $\|\cdot\|$ is exactly μ (the completion being V). Consult Section 2 of Kuelbs [10] and Section 1.4 of Kuo [13] for an exposition of this theory. We denote the inner product of H by $\langle\cdot,\cdot\rangle$ and the norm by $|\cdot|$. If $z \in H$, then $\langle z, \cdot\rangle$ is defined a.e. $[\mu]$, and the translated measure $\mu_z = \mu(z + \cdot)$ is absolutely continuous with respect to μ with density

$$\frac{d\mu_z}{d\mu}(x) = \exp[-\tfrac{1}{2}|z|^2 - \langle z, x\rangle].$$

Hence, using the Cauchy-Schwarz inequality and symmetry of μ, one obtains

$$\begin{aligned}\mu\{x : \|x\| \leq r\} &\leq \left(\int_{\|x\|\leq r} e^{-\langle z,x\rangle}\mu(dx)\right)^{1/2} \cdot \left(\int_{\|x\|\leq r} e^{\langle z,x\rangle}\mu(dx)\right)^{1/2} \\ &= \int_{\|x\|\leq r} e^{-\langle z,x\rangle}\mu(dx) \\ &= \exp[|z|^2/2]\mu_z\{x : \|x\| \leq r\}.\end{aligned}$$

Hence, for $y \in H$

$$\begin{aligned} q(y, u, t) &\geq \exp[-h(t)^2|y|^2/2]\cdot q(0, u, t) \\ &\geq \exp[-M|y|^2/2]\cdot q(0, u, t)\end{aligned} \tag{2.19}$$

where $M = \sup\{h(t)^2 : t \geq 0\}$.

Now fix $y \in H$. Assume $\lambda(0) < R$ w.p.1, then by Proposition 2.1, part (b), $\sum_k q(0, R, b^k) = \infty$ for all $b > 1$, since $\delta(0, b) \equiv 0$. Hence, $\sum_k q(y, R, b^k) = \infty$ for all $b > 1$ by (2.19). Part (a) of Proposition 2.1 implies $\lambda(y) \leq R$, w.p.1. On the other hand, assume $\lambda(0 > r$, w.p.1. Proposition 2.1, Part (a), implies $\sum_k q(0, r, b^k) < \infty$ for all $b > 1$, so $\sum_k q(y, r, b^k) < \infty$ for all $b > 1$ by (2.17). Letting $b \downarrow 1$ in Part (b) of Proposition 2.1 proves that $\lambda(y) \geq r$, w.p.1.

Thus, for all $y \in H$, we have shown that $P[\lambda(y) = \lambda(0)] = 1$. A separability argument can now be applied to complete the proof of the theorem.

The next result suggests that when looking for natural rate of escape functions, it is sufficient to look in the class of functions $\gamma(t)$ of the form $t^{1/2}h(t)$ where h is admissible.

PROPOSITION 2.6. *If X and V are as in Proposition 2.1, then there exist admissible functions h_1 and h_2 such that*

$$P[\liminf_{t\to\infty} \|X(t)\|/\gamma_1(t) \geq 1] = 1 \tag{2.20a}$$

$$P[\liminf_{t\to\infty} \|X(t)\|/\gamma_2(t) \leq 1] = 1 \tag{2.20b}$$

where $\gamma_i(t) = t^{1/2}h_i(t)$ for $i = 1, 2$.

PROOF. Putting $y = 0$, $u = 1$, and $b = e$ in the test series (2.9), we see that the assertion will follow if admissible h_i are constructed for which

$$\sum_k h(e^k)^{-2} P[\|X(1)\| \le h(e)^k] = \begin{cases} S < \infty & \text{if } h = h_1 \\ \infty & \text{if } h = h_2. \end{cases} \tag{2.21}$$

We describe the construction of the h_i and leave to the reader the verification of (2.21) and admissibility. Define for $v > 0$

$$G(v) = v^{-2} P[\|X(1)\| \le v]. \tag{2.22}$$

Note that if $0 < C = \inf\{v > 0 : G(v) > 0\}$, then $h_1 \equiv (2C)^{-1}$ and $h_2 \equiv 2C$ will solve the problem. Hence, we assume $G > 0$ on $(0, \infty)$.

From the assumption that the support of $X(1)$ is of dimension no less than 3, one concludes that

$$\limsup_{v \downarrow 0} (G(v)/v) < \infty.$$

Hence, choosing an arbitrary $p > 1$, h_1 may be defined for $x \ge 1$ by

$$h_1(x) = (\log x)^{-p}.$$

The situation for h_2 is not quite so simple. Take a positive, strictly increasing function G_0 defined on $(0, 1]$ for which $G_0 \le G$ on $(0, 1]$. Define a sequence of numbers $a_1, a_2, \cdots$ by $a_m = G_0^{-1}(2^{-m})$ and note that $a_m > a_{m+1} \to 0$ as $m \to \infty$. Let $k_1, k_2, \cdots$ be a sequence of positive integers satisfying

$$k_{m+1} - k_m > 2^m \text{ for } m \ge 1$$

$$a_{m+1} k_{m+1} > a_m k_m \quad \text{for} \quad m \ge 1$$

$$a_m \exp[k_m/2] \uparrow \infty \text{ as } m \to \infty.$$

Clearly such a sequence exists. Now, define h_2 by

$$h_2(\exp(k_m)) = a_m$$

for $m \ge 1$, and if $\exp(k_m) < t < \exp(k_{m+1})$,

$$h_2(t) = c_n/(b_n + \log t)$$

where

$$b_m = (a_{m+1} k_{m+1} - a_m k_m)/(a_m - a_{m+1})$$

$$c_m = a_m b_m + a_m k_m.$$

The definitions of b_m and c_m are such that h_2 is continuous.

3. The existence theorem. The first three results of this section give sufficient conditions for the existence of a solution to the functional inequality problem stated in (1.3). In all three cases, the solution is exactly exhibited via some "formula" (see (3.5), (3.9), and (3.16)). However, only the formula of Proposition 3.1 has actually proved useful in applications, such as in Section 4. Although the solution given in Proposition 3.2 is not suitable for our purposes, we give it primarily because of its simplicity and generality. It also serves as a motivation for the more complicated Proposition 3.3, which is the result used in the existence theorem. We make the following definitions:

$$L_0(t) = \max\{t, 1\}$$

$$L_1(t) = L(t) = \max\{\log t, 1\}$$

$$L_n(t) = L(L_{n-1}(t)) \quad \text{for} \quad n \ge 2.$$

PROPOSITION 3.1. *Let $G:(0, \infty) \to (0, \infty)$ be nondecreasing. Define for $r > 0$*

$$g(r) = -\log G(\exp(-r)) \tag{3.1}$$

and assume the following

$$\exists r_0 \text{ such that } g \text{ is convex on } (r_0, \infty) \tag{3.2}$$

$$\lim_{r\to\infty} g'(r) = \infty \tag{3.3}$$

$$\text{for some integer } N \geq 0, \limsup_{t\to\infty} \frac{L_{N+1}(t)}{g'(g^{-1}(t))} \equiv \nu_0 < \infty. \tag{3.4}$$

Then the function

$$h(t) \equiv \exp[-g^{-1}(\textstyle\sum_{k=2}^{N+2} L_k(t))], \tag{3.5}$$

defined for all t sufficiently large, is an admissible function for which

$$\sum_k G(\theta h(e^k)) \quad \begin{cases} \text{converges } if\ \theta < \exp[-\nu_0] \\ \text{diverges } if\ \theta \geq 1. \end{cases} \tag{3.6}$$

REMARKS. Note that (3.2) implies that g' exists on (r_0, ∞) except on a set of Lebesgue measure 0 (on which it can be defined by right continuity). Under (3.2), Condition (3.3) is equivalent to (1.4a). Also, (3.3) guarantees that g is eventually strictly increasing so that g^{-1} is well defined on some interval (u_0, ∞).

PROOF. Let r_1 be the maximum of the three numbers r_0, $\exp_{(N+1)}[1]$, and $\sup\{r: g'(r) < 1\}$. Here $\exp_{(N+1)}$ is the iterated exponential function, defined in the obvious way. Now h is well defined for $t \geq \exp_2[g(r_1)] \equiv t_1$. We first show that h is admissible. It follows from the definition that h is decreasing on (t_1, ∞). Further, the defining formula may be written in the form

$$h(t) = C \exp\left[-\int_{t_1}^{t} \tau^{-1}\varepsilon(\tau)d\tau\right]$$

where C is some constant and

$$\varepsilon(\tau) = (g^{-1})'(\textstyle\sum_{k=2}^{N+2} L_k(\tau))\cdot[\sum_{k=1}^{N+1}(\prod_{j=1}^{k} L_j(\tau))^{-1}].$$

Since $(g^{-1})'(u) \to 0$ as $u \to \infty$ by (3.3), it follows that $\varepsilon(\tau) \to 0$ as $\tau \to \infty$. (In fact, $\varepsilon(\tau)(\log \tau) \to 0$). Hence, h is slowly varying at ∞ (see page 282 of Feller [8]). Also,

$$t^{1/2}h(t) = C' \exp\left[-\int_{t_1}^{t} \tau^{-1}\{1/2 - \varepsilon(\tau)\}d\tau\right].$$

Since the quantity in braces is eventually positive, it follows that $t^{1/2}h(t)$ is eventually increasing. Thus, h is admissible.

We now establish the convergence claim in (3.6). Fix $\theta < \exp[-\nu_0]$ and put $\mu = -\log\theta < \nu_0$. Let $\nu \in (0, \mu)$. By (3.4), there exists $t_2 \geq t_1$ such that for all $t \geq t_2$

$$g'(g^{-1}(\textstyle\sum_{j=0}^{N} L_j(t))) \geq \nu^{-1}L_{N+1}(t). \tag{3.7}$$

In view of the monotonicity of G and h, the integral test applies. Hence, convergence of the series in (3.6) is implied by convergence of

$$I(\theta) \equiv \int_{\log t_1}^{\infty} G(\theta h(e^x))dx. \tag{3.8}$$

Now, make the substitution $x = \log t$ and use (3.3) and (3.5) to obtain

$$I(\theta) \le \int_{t_1}^{\infty} \exp[-\mu g'(g^{-1}(\textstyle\sum_{k=2}^{N+2} L_k(t)))][\prod_{k=0}^{N+1} L_k(t)]^{-1} dt.$$

If (3.7) is applied to this latter expression, we see that $I(\theta) < \infty$ would follow from convergence of

$$\int_{t_2}^{\infty} (\textstyle\prod_{k=0}^{N+1} L_k(t))^{-1}(L_{N+2}(t))^{-\mu\nu^{-1}}\, dt.$$

But this later integral converges since $\mu\nu^{-1} > 1$.

The divergence claim in (3.6) follows from monotonicity of G and the easily verfied fact that $I(1) = \infty$.

PROOF OF PROPOSITION 3.2. From the log concavity of G, it follows that g is absolutely continuous on every bounded subinterval of some interval (r_1, ∞), where g is given by (3.1). Also, g' has only jump discontinuities, and we may define it at these jumps by right continuity. Assumption (1.4a) implies $\lim g'(u) = \infty$ as $u \to \infty$, and we may assume then that $g'(r) > 0$ for $r > r_1$. Define

$$(g')^{-1}(x) = \inf\{y \ge r_1 : g'(y) \ge x\}$$

and note that this function is nondecreasing. Take $h_1 > 0$ and $\delta > 0$ sufficiently large that $\delta G(h_1) > r_1$. Define $\{h_n\}$ by the recursion

$$h_{n+1} = \exp[-(g')^{-1}(\delta \textstyle\sum_{j=1}^{n} G(h_j))]. \tag{3.9}$$

Clearly $h_2 \ge h_3 \ge \cdots$. Put $S = \sum_{j=1}^{\infty} G(h_j)$, and suppose $S < \infty$. Then for $n \ge 1$

$$h_{n+1} \ge \exp[-(g')^{-1}(S)] \equiv M > 0$$

and so

$$S \ge \textstyle\sum_{j=1}^{\infty} G(M) = \infty.$$

Hence, $S = \infty$, and it futher follows that $\lim h_n = 0$. This establishes divergence of $\Sigma G(\theta h_n)$ for $\theta \ge 1$. Put $\lambda(u) = g(-\log u)$ and fix $\theta \in (0, 1)$. Now λ is convex by (1.4b), so for $n \ge 2$

$$\lambda(\theta h_n) \ge \lambda(h_n) - (1-\theta)h_n\lambda'(h_n) \ge \lambda(h_n) + (1-\theta)\delta \textstyle\sum_{k=1}^{n-1} G(h_k)$$

where we have used $g'((g')^{-1}(u)) \le u$. Hence

$$\textstyle\sum_{n=2}^{\infty} G(\theta h_n) \le \sum_{n=2}^{\infty} G(h_n)\exp[-\mu \sum_{j=1}^{n-1} G(h_j)]$$

where $\mu = \delta(1-\theta) > 0$. But the series on the right is convergent (see Section 104.2 on page 119 of Pierpont [15]).

To establish (1.5b), assume it false, i.e. for some θ_0 in (0, 1)

$$h_{n+1} < \theta_0 h_n$$

for all n sufficiently large, say $n \ge n_0$. But then we have the contradiction

$$\infty = \textstyle\sum_{n \ge n_0} G(h_{n+1}) \le \sum_{n\ge n_0} G(\theta h_n) < \infty$$

which completes the proof.

Our final result on this subject states that if in addition g is convex (which implies λ convex), then we can obtain an admissible solution. The idea of the proof is to produce a solution to the following integral equation analog of (3.9)

$$h(t) = \exp\left[-(g')^{-1}\left(\int_0^t G(h(\tau))\, d\tau\right)\right].$$

This solution is given in (3.15) and (3.16) under rather strict assumptions on g. It is then verified that the solution has the right properties, and then that any function g satisfying the conditions of the proposition can be suitably approximated by a "nice" function.

PROPOSITION 3.3. *Let $G:(0,\infty)\to(0,\infty)$ be nondecreasing near 0 with $G(0+)=0$. Put $g(u)=-\log G(\exp[-u])$ for $u\in\mathbb{R}$. Assume g is eventually convex and that*

$$\lim_{n\to\infty} g'(u)=\infty. \tag{3.10}$$

Then there exists an admissible function h such that $\Sigma_k G(\theta h(e^k))$ converges if $0<\theta<1$ and diverges if $\theta>1$.

PROOF. We begin by assuming g satisfies the following additional assumptions:

$$g\in C^2[0,\infty) \tag{3.11}$$

$$g'(0)=0 \tag{3.12}$$

$$g''>0 \tag{3.13}$$

$$\text{for some}\quad a>0,\quad \text{eventually } g''(u)>\exp[-au]. \tag{3.14}$$

Define a function $H:(0,1]\to[1,\infty)$ by

$$H(y)=\exp\left[\int_0^{-\log y} e^{g(x)}g''(x)\,dx\right]. \tag{3.15}$$

Then H is continuous and strictly decreasing by (3.13), and also $H(y)\to\infty$ as $y\downarrow 0$ (use (3.10) and (3.14)). Thus

$$h(u)=H^{-1}(u) \tag{3.16}$$

is continuous and decreases to 0 as $u\to\infty$. Hence, it suffices to consider convergence or divergence of $I(\theta)$ given in (3.8) with $t_1=1$. In the integral

$$J(v)=\int_1^v \exp[-g(-\log h(w))]\,\frac{dw}{w}, \tag{3.17}$$

make the change of variables $w=H(e^{-u})$ to obtain

$$J(v)=g'(-\log h(v))\to\infty \quad\text{as}\quad v\to\infty.$$

The limit relation follows from (3.10). This shows $I(\theta)$ diverges for $\theta\geq 1$ since $J(v)\to I(1)$ as $v\to\infty$. If $0<\theta<1$, let $\mu=-\log\theta$, and use convexity of g to obtain

$$\begin{aligned}\int_1^v \exp[-g(\mu-\log h(w))]\,\frac{dw}{w}&\leq\int_1^v \exp[-g(-\log h(w))-\mu g'(-\log h(w))]\,\frac{dw}{w}\\&=\int_1^v \exp[-g(-\log h(w))]\exp[-\mu J(w)]\,\frac{dw}{w}.\end{aligned}$$

Note that (3.17) was used at the last step. If one makes the change of variables $y=J(w)$, then the last integral becomes

$$\int_0^{J(v)} e^{-\mu y}\,dy$$

which is no greater than μ^{-1}. Hence, $I(\theta)<(-\log\theta)^{-1}$ if $0<\theta<1$, and is convergent, in particular.

By making the appropriate substitutions, one can verify that

$$h(t)=\exp\left[-\int_1^t \varepsilon(w)\,\frac{dw}{w}\right]$$

where

$$\varepsilon(t) = \exp[-g(-\log h(t))]/g''(-\log h(t)).$$

It follows from (3.10) and (3.14) that $\varepsilon(t) \to 0$ as $t \to \infty$. Thus, h is slowly varying. That $t^{1/2}h(t)$ is eventually nonincreasing is proved as in Proposition 3.1. Hence, h is an admissible function.

We now need only remove the extra restrictions (3.11) through (3.14) to complete the proof of the proposition. Suppose g_0 satisfies the conditions in the statement of the Proposition (i.e. g_0 eventually concave, and $g_0' \to \infty$). Then we claim that there exists g satisfying these conditions and also (3.11) through (3.14) for which there exists constants u_0, u_1, and u_2 such that for all $u \geq u_0$

$$g(u - u_1) \leq g_0(u) \leq g(u + u_2). \tag{3.18}$$

Let us assume such a g exists, and let h be given by (3.15) and (3.16). It is then clear that

$$\int^{\infty} \exp[-g_0(-\log \theta h(y))]\frac{dy}{y}$$

converges for $\theta < \exp[-u_1]$ and diverges for $\theta > \exp[u_2]$. Hence, some constant multiple of h works to solve the problem for g_0.

Now to establish that such a function g exists, fix $\lambda > 0$, and let ϕ be a bounded, nonnegative function on $\mathbb{R}$ with two continuous derivatives, support in $[0, \lambda]$, and integral equal 1. For $x \geq 0$, put

$$g_1(x) = \int_0^{\lambda} g_0(x + z)\phi(z)\,dz.$$

Then $g_1 \in C^2[0, \infty)$, and there exists an x_0 such that for all $x > x_0$

$$g_1(x - \lambda) \leq g_0(x) \leq g_1(x)$$

since g_0 eventually increasing. Furthermore, it is easily checked that g_1 is eventually convex, and $g_1' \to \infty$. Now put

$$g_2(x) = g_1(x) + e^{-x}.$$

Then g_2 is eventually convex, satisfies (3.10), (3.11), and (3.14), and eventually $g_2'' > 0$. Also

$$g_0(x) \leq g_1(x) \leq g_2(x).$$

Fix any $\mu > 0$, then

$$g_1(x) - g_2(x - \mu) \geq \mu g_1'(x - \mu) - e^{\mu}e^{-x}$$

and the second member tends to ∞ as $x \to \infty$. Hence, for all x sufficiently large

$$g_2(x - \mu - \lambda) \leq g_1(x - \lambda) \leq g_0(x).$$

Now g_2 satisfies all the requirements of the function g we sought for (3.18), except that (3.13) only holds eventually for g_2 and (3.12) may not hold. However, it is a simple matter to modify g_2 so as to obtain g satisfying these properties. This completes the proof of the Proposition.

It is well known that if X is a zero mean Gaussian random vector on a separable Banach space $(V, \|\cdot\|)$, then

$$F(u) = P[\|X\| \leq u] \tag{3.19}$$

is log concave (see (1.13) on page 321 of [9]). However, in order to apply Proposition 3.3, we need the stronger concavity condition stated there. The following result, due to Pearlman [14], shows that this condition holds if the unit ball of V is sufficiently symmetrical.

LEMMA 3.4. *Let Z by an $\mathbb{R}^d$ valued Gaussian random vector with zero mean and covariance matrix equal to the $d \times d$ identity matrix. Let $K \subseteq \mathbb{R}^d$ be a closed convex subset which is symmetric about the coordinate hyperplanes, i.e. for any j with $1 \le j \le d$, if $(x_1, \cdots, x_j, \cdots, x_d) \in K$, then $(x_1, \cdots, -x_j, \cdots, x_d) \in K$. Then*

$$\psi(u) \equiv P[Z \in e^{-u}K]$$

is log concave.

PROOF. First note that it suffices to prove the Lemma for K of the form

$$K = \{x \in \mathbb{R}^d\colon \max_{1\le i\le n}(\textstyle\sum_{j=1}^d a_{ij}|x_j|) \le 1\}$$

where the a_{ij} are nonnegative constants, for $1 \le i \le n$, $1 \le j \le d$. Now put $Y_j = \log|Z_j|$ for $1 \le j \le d$, then

$$P[Z \in e^{-u}K] = \psi(u) = P[\max_{1\le i\le n}(\textstyle\sum_{j=1}^d a_{ij}\exp[u + Y_j]) \le 1]$$

$$= \int_{\mathbb{R}^d} h(u, y)f(y)\,dy$$

where

$$f(y) = (2/\pi)^{d/2}\textstyle\prod_{j=1}^d \exp[y_j - \frac{1}{2}e^{-2y}y_j]$$

is just the density for Y, and $h(u, y)$ is the indicator of the subset of $\mathbb{R}^{d+1}$ given by

$$\{(u, y)\colon \max(\textstyle\sum_{j=1}^d a_{ij}\exp[u + y_j]) \le 1\}.$$

This set is clearly closed and convex, so h is log concave (note that $-\infty$ is an allowable value for a concave function, as discussed on page 123 of [2]). Also, f is clearly log concave. Hence, it follows from Corollary 4.1 of Borell [2] that ψ is log concave, completing the proof.

THEOREM 3.5. *Lex X be an independent coordinate ℓ^p-valued Brownian motion with $1 \le p \le \infty$. Then X has a natural rate of escape in ℓ^p.*

PROOF. Put $G(r) = r^{-2}P[\|X(1)\|_p \le r]$ for $r > 0$ and $g(u) = -\log G(\exp(-u))$. In view of Erickson's test, we need only show that g satisfies the two conditions of Proposition 3.3. To show that g is eventually convex, it suffices to show that $-\log P[\|X(1)\|_p \le e^{-u}]$ is convex in u, but this latter function is the limit as $n \to \infty$ of the increasing sequence

$$g_n(u) = \begin{cases} -\log P[\sum_{j=1}^n \sigma_j^p |B_j(1)|^p \le e^{-pu}] & \text{if } p < \infty \\ -\log P[\max_{1\le j\le n}|\sigma_j B_j(1)| \le e^{-pu}] & \text{if } p = \infty \end{cases}$$

where $\sigma_j B_j$ is the jth coordinate of X (see (2.7)). Now the set

$$K_p = \{x \in \mathbb{R}^n\colon (\textstyle\sum_{j=1}^n |\sigma_j x_j|^p)^{1/p} \le 1\}, \quad 1 \le p < \infty$$

is closed, convex, and symmetric about the coordinate hyperplanes. The analogously defined set K_∞ also satisfies these properties. Hence, g_n is convex by Lemma 3.4, and so also is g. Now clearly $G(u) = o(u^n)$ as $u \downarrow 0$ for all n, so g satisfies (3.10). This completes the proof.

COROLLARY 3.6. *Let X be a Brownian motion on a separable Hilbert space. Then X has a natural rate of escape.*

PROOF. There exists an orthonormal sequence $e_1, e_2, \cdots$ such that the one-dimensional motions $\langle e_1, X\rangle$, $\langle e_2, X\rangle$, $\cdots$ are independent (see e.g. Theorem 4 of Kuelbs [11]), and

$$X = \textstyle\sum_{j=1}^\infty \langle e_j, X\rangle e_j.$$

The proof is completed as in the ℓ^2 case of Theorem 3.5.

4. Computation of rates of escape. In this section we show how rate of escape functions may be computed for a large class of independent coordinate ℓ^2-valued Brownian motions. We limit attention to the case $p = 2$ for the sake of convenience, but similar results may be obtained if $2 < p < \infty$. A general result for $p = \infty$ is given in Theorem 5 of Kuelbs [12]. See also Example 3, page 334, and Theorem 3 of Erickson [7]. In general, we can only obtain the rate of escape function up to a constant multiple, where the constant is bounded by some computable numbers.

First, we must obtain some usable probability estimates. Let $\sigma:[1, \infty) \to (0, \infty)$ be a nonincreasing function satisfying $\int \sigma(x)^2\, dx < \infty$. Put $\phi(x) = x^{1/2}\sigma(x)$ for $1 \le x < \infty$, and assume for some $x_0 \ge 1$ that $\phi \in C^1(x_0, \infty)$. We will need the following quantities:

$$K = \liminf_{x\to\infty}[-\phi(x)/\phi'(x)]$$

$$L = \limsup_{x\to\infty}[-\phi(x)/\phi'(x)]$$

$$M = \limsup_{x\to\infty}[-\phi(x)/(x\phi'(x))].$$

THEOREM 4.1. *Suppose that ϕ satisfies the following*

$$\phi(x) \le 1 \quad \textit{for} \quad x > x_0 \tag{4.1}$$

$$\phi'(x) < 0 \quad \textit{for} \quad x > x_0 \tag{4.2}$$

$$K \in (0, \infty] \tag{4.3}$$

$$M \in [0, \infty). \tag{4.4}$$

Now let $\eta_1, \eta_2, \cdots$ be independent standard normal random variables and put

$$F(r) = P[\textstyle\sum_{j=1}^{\infty} |\sigma(j)\eta_j|^2 \le r^2].$$

Then there are constants α_1 and α_2 such that for all $\delta > 0$ there is a u_1 such that for all $u \ge u_1$

$$f(u + \alpha_1 - \delta) \le -\log F(e^{-u}) \le f(u + \alpha_2 + \delta) \tag{4.5}$$

where

$$f(u) = \int_{u_0}^{u} \phi^{-1}(e^{-w})\, dw, \quad u_0 = -\log \phi(x_0). \tag{4.6}$$

REMARK. The most stringent conditions are (4.3) and (4.4). These two conditions roughly say that for some $a > 0$, $b > ½$

$$\mathcal{O}(e^{-ax}) \le \sigma(x) \le \mathcal{O}(x^{-b}). \tag{4.7}$$

PROOF. Note first that f is eventually convex and $f' \to \infty$. Also

$$\liminf_{u\to\infty} f''(u) = K, \quad \limsup_{u\to\infty} f''(u) = L, \quad \limsup_{u\to\infty} f''(u)/f'(u) = M.$$

We shall make frequent use of these relations.

According to Theorems 4.1 and 4.3 of Hoffman-Jørgensen, Shepp, and Dudley [9], the following bounds are valid for $r \in (0, 1]$, $s \in (0, r)$, $x > x_0 + 1$, and $y > x_0 + 1$

$$\begin{aligned} &A_1 + \log\left(1 - (r^2 - s^2)^{-1} \int_x^{\infty} \sigma(\xi)^2\, d\xi\right) \\ &\quad + \int_1^x \log \phi(\xi)\, d\xi - \frac{1}{2}\log x + (x + 1)\log s - \frac{1}{2} s^2\sigma(x + 1)^{-2} \\ &\le \log F(r) \le A_2 + \int_1^y \log \phi(\xi)\, d\xi + \log \phi(y) + (y - 1)\log r \end{aligned} \tag{4.8}$$

where A_1 and A_2 are constants. We will make the substitutions $u = -\log r$, $v = -\log s$, $y = \phi^{-1}(e^{-u})$, and $x = \phi^{-1}(e^{-v})$. If we integrate by parts and use (4.6), there results

$$\int_1^y \log \phi(\xi)\, d\xi + y \log r = A_3 - f(u) = -f(u) + o(f'(u)). \tag{4.9}$$

The last equality follows since $f'(u) = \phi^{-1}(\exp(-u)) \to \infty$ as $u \to \infty$. A similar expression holds for x and v.

We begin treating the major terms in (4.8) individually. Fix $\eta > 0$. Now, for all x sufficiently large

$$1 - (r^2 - s^2)^{-1} \int_x^\infty \sigma(\xi)^2\, d\xi \geq 1 - \frac{1}{2}(M + \eta)e^{-2v}(e^{-2u} - e^{-2v})^{-1} = e^{-u} \tag{4.10}$$

where the last equality results from the substitution

$$\begin{aligned} v &= u + ½ \log[(1 + ½(M + \eta) - e^{-u})(1 - e^{-u})^{-1}] \\ &= u + ½ \log(1 + ½M) + o(1). \end{aligned} \tag{4.11}$$

The other problematic term in the lower bound is estimated as follows

$$-½s^2\sigma(x + 1)^{-2} = -½e^{-2v}(f'(v) + 1)\phi(f'(v) + 1)^{-2}.$$

Using (4.3), we have for all x sufficiently large that

$$\phi(x + 1) \geq \phi(x) \exp[-(K^{-1} + \eta)]$$

and hence that

$$-½s^2\sigma(x + 1)^{-2} \geq -½ \exp[2(K^{-1} + \eta)]f'(v) + \mathcal{O}(1). \tag{4.12}$$

If we collect (4.9) through (4.12) and put them into (4.8), we obtain

$$\begin{aligned} &-f(v) - 2v - ½ \exp[2(K^{-1} + y)]f'(V) + o(f'(v)) \\ &\qquad\qquad \leq \log F(e^{-u}) \leq -f(u) + o(f'(u)). \end{aligned} \tag{4.13}$$

We will now make use of the following inequalities, valid for all u sufficiently large and all $\alpha > 0$:

$$u \leq (K^{-1} + \eta) f'(u) \tag{4.14a}$$

$$f(u) + \alpha f'(u) \leq f(u + \alpha) \tag{4.14b}$$

$$v \leq u + ½ \log(1 + ½M) + \eta. \tag{4.14c}$$

When these are used in (4.13), the result is

$$-f(u + 2K^{-1} + ½ \log(1 + ½M) + ½ \exp(2K^{-1}) + o(1)) \leq \log F(e^{-u}) \leq -f(u + o(1)).$$

The claim in (4.5) follows from this.

THEOREM 4.2. *Assume $\sigma(\cdot)$ satisfies the conditions of Theorem 4.1, and that $L = \infty$ implies $K = \infty$. Let $B_1, B_2, \cdots$ be independent standard one dimensional Brownian motions and suppose*

$$X(t) = \langle \sigma(1)B_1(t), \sigma(2)B_2(t), \cdots \rangle$$

is an ℓ^2-valued Brownian motion. Let f be defined by (4.6). Then

$$C_1 \leq \liminf_{t\to\infty} \|X(t)\|_2/\gamma(t) \leq C_2$$

where

$$\gamma(t) = t^{1/2}\exp[-f^{-1}(\log\log t)]$$
$$C_1 = \exp[-2K^{-1}]$$
$$C_2 = \{\exp[\exp(2K^{-1})](1 + \tfrac{1}{2}M)\}^{1/2}.$$

PROOF. It follows immediately from (4.13) and (4.14c) that for any $\eta > 0$

$$-f(v) - \tfrac{1}{2}\exp[2(K^{-1} + \eta)]f'(v) + o(f'(v)) \le \log(e^{2u}F(e^{-u})) \le -f(u) + 2u + o(f'(u))$$

where v is given by (4.11), and the inequalities hold for all u sufficiently large. By (4.14a), $u \le (K^{-1} + \eta)f'(u)$ eventually. Also, if $L < \infty$, then for $\beta \ge 0$

$$f(u) \ge f(u-\beta) + \beta f'(u-\beta) \ge f(u-\beta) + \beta f'(u) - \beta^2(L+\eta)$$

for all u sufficiently large. Hence,

$$-f(u) + 2u \le -f(u) + 2K^{-1}f'(u) + o(f'(u)) \le -f(u - 2K^{-1}) + o(f'(u)).$$

If $L = \infty$, then $K = \infty$, and the latter inequality still holds. Hence, if we put $G(x) = x^{-2}F(x)$, it follows just as in the proof of Theorem 4.1 that

$$f(u - 2K^{-1} + o(1)) \le -\log G(e^{-u}) \le f(u + \tfrac{1}{2}\log(1 + \tfrac{1}{2}M) + \tfrac{1}{2}\exp(2K^{-1}) + o(1)).$$

In order to apply Proposition 3.1, we need to compute lim sup $L(f(u))/f'(u)$. Since $K > 0$ and $f' \to \infty$, we have for any $\varepsilon > 0$ that there is a u_1 such that for all $u \ge u_1$

$$f'(u) \le f''(u)\exp[\varepsilon f'(u)].$$

If this inequality is integrated from u_1 to u, we can obtain

$$Lf(u)/f'(u) \le \varepsilon - L(\varepsilon)/f'(u) - L(1 - f(u_1)/f(u))/f'(u).$$

Hence

$$\limsup_{u\to\infty} Lu/f'(f^{-1}(u)) = 0$$

and it follows from Proposition 3.1 that if we put

$$h(t) = \exp[-f^{-1}(L_2 t)]$$

then h is admissible and

$$\sum_{k=1}^{\infty} G(\theta h(e^k)) \begin{cases} = \infty & \text{if } \theta > \{\exp[\exp(2K^{-1})](1 + \tfrac{1}{2}M)\}^{1/2} \\ < \infty & \text{if } \theta < \exp[-2K^{-1}]. \end{cases}$$

The theorem follows from this and Erickson's test.

REMARK. The computation of h in the theorem involves an inversion of ϕ, an indefinite integration, and then another inversion. There is an alternative procedure which involves one less inversion and is better suited to approximation methods. If $\xi = \phi^{-1}(e^{-u})$ then

$$f(u) = C - \xi\log\phi(\xi) + \int_{x_0}^{\xi}\log\phi(y)\,dy.$$

Hence, if $\xi = \xi(r)$ is the solution of

$$-\xi\log\phi(\xi) + \int_{x_0}^{\xi}\log\phi(y)\cdot dy = r \tag{4.15}$$

(where we have altered f by the constant $C = o(f')$), then

$$-\log\phi(\xi(r)) = f^{-1}(r)$$

so that

$$h(t) = \phi(\xi(\log \log t)). \tag{4.16}$$

We put these techniques to use in an example.

PROPOSITION 4.3. *Let*

$$\sigma(x) = x^p \exp[-\alpha x^q]$$

where $\alpha > 0$ *and* $0 < q \le 1$, *and let* X *be an* ℓ^2*-valued Brownian motion as in Theorem* 4.2. *Then*

$$C_1 \le \liminf_{t\to\infty} \| X(t) \|_2/\gamma(t) \le C_2$$

where

$$\begin{aligned}
\gamma(t) &= t^{1/2}(\log\log t)^{(p+1/2)/(q+1)} \exp[-\alpha c^{-q}(\log\log t)^q] \\
c &= \alpha q/(q+1) \\
C_1 &= \begin{cases} k & \text{if } 0<q<1 \\ k\exp[p - 3/2] & \text{if } q=1 \end{cases} \\
C_2 &= \begin{cases} ke & \text{if } 0<q<1 \\ k\exp[(p+1/2)\alpha + e^{-2\alpha}] & \text{if } q=1 \end{cases} \\
k &= \exp[(p+1/2)(q+1)^{-1}\log c].
\end{aligned}$$

PROOF. We have

$$\begin{aligned}
\phi(x) &= \exp[-\alpha x^q + (p+1/2)\log x] \\
K = L &= \begin{cases} \infty & \text{if } 0<q<1 \\ \alpha^{-1} & \text{if } q=1 \end{cases} \\
M &= 0.
\end{aligned}$$

The equation (4.15) becomes

$$C\xi^{q+1} - d\xi = r \tag{4.17}$$

where $C = \alpha q/(q+1)$ and $d = p + 1/2$. Consider the approximate solutions

$$\begin{aligned}
\xi_0 = \xi_0(r) &= \exp[(q+1)^{-1}\log(r/c)] \\
\xi_1(r) &= \xi_0(1 + d/(\alpha q\xi_0 - d)) \\
\xi_2(r) &= \exp[(q+1)^{-1}\log((r + d\xi_0)/c)].
\end{aligned}$$

Note that ξ_0 is obtained by neglecting the $d\xi$ term in (4.17), ξ_1 is a one step Newton-type approximation starting with ξ_0, and ξ_2 is obtained by replacing $d\xi$ with $d\xi_0$ in (4.17). Now, it is straightforward to verify that $\xi_2 \le \xi \le \xi_1$ eventually, and that

$$\lim_{r\to\infty}\phi(\xi_1(r))/\phi(\xi_2(r)) = 1.$$

Hence, either of $\phi(\xi_i(\log\log t))$, $i = 1, 2$, may be used in place of the exact h given in (4.16). Now, if $q = 1$, then one readily verifies that $\phi(\xi_1(\log\log t))$ is asymptotically equivalent to $\exp[-\alpha d]\phi(\xi_0(\log\log t))$. If $0 < q < 1$, then $\phi(\xi_1(\log\log t))$ is asymptotically equivalent to $\phi(\xi_0(\log\log t))$. The proposition follows from this and the formulae in Theorem 4.2.

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