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ON STRONGLY UNIMODAL INFINITELY DIVISIBLE DISTRIBUTIONS

BY MAKOTO YAMAZATO
Nagoya Institute of Technology

There are many results as to unimodality of infinitely divisible distributions. But few are known about strong unimodality of infinitely divisible distributions. In this paper, we consider two subclasses of infinitely divisible distributions and give necessary and sufficient conditions for strong unimodality of distributions belonging to such classes.

1. Introduction and results. Let μ be a probability distribution on R^1 and let $F_\mu(x)$ be its distribution function. The measure μ (or $F_\mu(x)$) is said to be *unimodal with mode m* if $F_\mu(x)$ is convex for $x < m$ and concave for $x > m$. The measure μ (or $F_\mu(x)$) is said to be *unimodal* if, for some m, it is unimodal with mode m.

Lapin asserted that the unimodality of distributions is closed under convolution (i.e., let $F(x)$ and $G(x)$ be unimodal distribution functions. Then $F*G(x) = \int F(x-y)\,dG(y)$ is unimodal.). But K. L. Chung in [2] pointed out that Lapin's assertion is incorrect. After this, Ibragimov [4] (c.f. [14]) studied under which condition the convolution of two unimodal distributions is unimodal. He called a distribution (function) *strongly unimodal* if its convolution with every unimodal distribution (function) is unimodal and gave a necessary and sufficient condition for strong unimodality.

Let g be a function on R^1. If g is positive on an interval I and $\log g$ is concave on I, then we say that g is log concave on I. We call g log concave if $I = \{x; g > 0\}$ is an interval and g is log concave on I.

THEOREM (Ibragimov). A unimodal distribution μ is strongly unimodal if and only if μ is degenerate (i.e., there is some a such that $\mu(\{a\}) = 1$) or μ is absolutely continuous (with respect to Lebesgue measure) and its density has a log concave version.

Note that the above version of a density of nondegenerate strongly unimodal distribution is PF_2 function (Pólya frequency function of order 2; see Karlin [6]). Many interesting properties of PF functions including a fact essentially equivalent to Ibragimov's Theorem are investigated by Schoenberg, Karlin, and others.

It is obvious from Ibragimov's Theorem that Γ- distributions with exponent $\lambda \ge 1$ (i.e., the density is of the form $\Gamma(\lambda)^{-1}\alpha^\lambda x^{\lambda-1}e^{-\alpha x}$ for $x > 0$, where $\lambda \ge 1$ and $\alpha > 0$) and normal distributions are strongly unimodal. These are examples of strongly unimodal infinitely divisible distributions (which we abbreviate i.d.d.). We do not know other strongly unimodal i.d.d. although many unimodal i.d.d. are obtained ([7], [9], [12] and [14]).

In this paper, we have an interest in finding other strongly unimodal i.d.d. and in characterizing the class of such distributions. We restrict our consideration to i.d.d. on $[0, \infty)$ with absolutely continuous Lévy measure. We denote the class of all such distributions by I_{ac}^+. Absolutely continuous means absolutely continuous with respect to Lebesgue measure in this paper. A characteristic function $\hat{\mu}(t)$ of such a distribution μ is represented as follows ([1]).

$$\hat{\mu}(t) = \exp \int_{0+}^{\infty} (e^{itu} - 1)u^{-1}k_\mu(u)\,du \tag{1}$$

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where $k_\mu(u) \geq 0$ and $\int_{0+}^{\infty} (1+u)^{-1} k_\mu(u)\, du < \infty$. Also, by the same $k_\mu(u)$, the Laplace transform $\check{\mu}(t) = \int_0^\infty e^{-tx}\mu(dx)$ of μ is represented as

$$\check{\mu}(t) = \exp \int_{0+}^{\infty} (e^{-tu} - 1) u^{-1} k_\mu(u)\, du. \tag{2}$$

We denote all quantities related to μ by the subscript μ as above.

Let us introduce the following two subclasses I_1 and I_2 of I_{ac}^+,

$$I_1 = \{\mu \in I_{ac}^+;\ k_\mu \text{ is log concave}\}$$

and

$$I_2 = \{\mu \in I_{ac}^+;\ \mathrm{k}_\mu \text{ is represented as (3) with some } a_\mu \text{ and } m_\mu\}.$$

For $u > 0$,

$$k(u) = e^{-au} + u \int_0^\infty e^{-ux} m(x)\, dx \tag{3}$$

where $a > 0$ and $0 \leq m(x) \leq 1$ a.e. $x > 0$ and

$$\int_{0+}^{1} x^{-1} m(x)\, dx < \infty.$$

Note that if a function g is log concave on an interval $(0, p)$, then g is non-decreasing on some interval $(0, q)$ and non-increasing on (q, p), and g is absolutely continuous on $(0, p)$.

Our main results are the following two theorems.

THEOREM 1. *Let $\mu \in I_1$. Then μ is strongly unimodal if and only if $k_\mu(0+) \geq 1$.*

THEOREM 2. *Let $\mu \in I_2$.*
(i) *If $a_\mu \leq b_\mu = \inf\{x; \int_0^x m_\mu(u)\, du > 0\}$, then μ is strongly unimodal.*
(ii) *If $b_\mu < a_\mu$ and $\int_{b_\mu}^{a_\mu} m_\mu(u)\, du < a_\mu - b_\mu$, then μ is not strongly unimodal.*

It should be noted that Theorem 2 completely determines whether a distribution of class I_2 is strongly unimodal. In Section 6, we construct an example of k of the form (3) which is not log concave. Thus the log concavity of k_μ does not follow from that of f_μ. This example also shows that Theorem 1 does not include Theorem 2. As a by-product of Theorems 1 and 2, we can get many new unimodal i.d.d. combining Theorem 1 or 2 with known results as to unimodal i.d.d.

In order to prove Theorem 1, we show that the condition of Ibragimov's Theorem is satisfied using an integro-differential equation satisfied by one-sided i.d.d. We also show that the condition of Ibragimov's Theorem is satisfied in order to prove Theorem 2 by using an integral representation of a density of i.d.d. with k_μ of the form (3).

In Sections 2 and 3, we prove Theorem 1. In Section 4, we prepare two lemmas for the proof of Theorem 2. We will prove Theorem 2 in Section 5.

2. Proof of the "if" part of Theorem 1. If a function g is absolutely continuous, then we denote its Radon-Nikodym derivative by g^*. If g is differentiable, then we denote its derivative by g'. Let $\mu \in I_1$. Since k_μ is log concave, k_μ is absolutely continuous on $(0, p_\mu)$ where $p_\mu = \sup\{u;\ k_\mu(u) > 0\}$ and we can choose k_μ^* so that $k_\mu^* k_\mu^{-1}$ is non-decreasing on $(0, p_\mu)$. We always take such a version of k_μ^*. Let $\lambda = k_\mu(0+)$. If $\lambda > 0$, then $k_\mu^*(0+)$ exists including infinity.

The following two lemmas are essential tools for the proof of the "if" part of Theorem 1. Similar results are obtained for L distributions [8].

LEMMA 2.1. *Let $\mu \in I_1$. Assume that $\lambda > 1$ and k_μ is continuous on $(0, \infty)$. Then μ has a continuous density $f_\mu(x)$. The density f_μ is positive on $(0, \infty)$ and*

$$xf_\mu(x) = \int_0^\infty f_\mu(x-u)k_\mu(u)\,du = \lambda F_\mu(x) + \int_0^\infty F_\mu(x-u)\,k_\mu^*(u)\,du \tag{4}$$

for all x. Here, F_μ denotes the distribution function of μ.

PROOF. As in the case of L distributions (see [8], Lemma 2.4), we can easily show that for each $\alpha < \lambda$

$$|\hat{\mu}(t)| = o(|t|^{-\alpha}) \quad \text{as} \quad |t| \to \infty.$$

Thus $\hat{\mu}(t)$ is integrable and Lévy's inversion formula

$$F_\mu(x) - F_\mu(0) = (2\pi)^{-1}\int_{-\infty}^\infty (e^{ixt}-1)(it)^{-1}\hat{\mu}(t)\,dt$$

implies that $F_\mu(x)$ is continuously differentiable. Now, Steutel's results on one-sided i.d.d. (Corollary 4.2.2 and Theorem 4.2.5 of [9]) are applicable and hence f_μ satisfies the first equality of (4) and $f_\mu > 0$ on $(0, \infty)$. The second equality is obvious by the continuity of k_μ on $(0, \infty)$.

LEMMA 2.2. *Let $\mu \in I_1$. Assume that $\lambda > 1$, k_μ is continuous on $(0, \infty)$ and $k_\mu^*(0+) < \infty$. In the case that $p_\mu < \infty$, assume that $k_\mu^*(p_\mu -)$ exists. Then f_μ is C^2 on $(0, \infty)$ and the following equations hold*:

$$xf_\mu'(x) = (\lambda - 1)f_\mu(x) + \int_0^\infty f_\mu(x-u)k_\mu^*(u)\,du \tag{5}$$

and

$$xf_\mu''(x) = (\lambda - 2)f_\mu'(x) + \int_0^x f_\mu'(x-u)k_\mu^*(u)\,du \tag{6}$$

for $x > 0$.

For brevity, let us denote $f_\mu = f$, $k_\mu = k$ and $p_\mu = p$ in this section.

PROOF OF LEMMA 2.2. Since k^* is bounded on every finite interval and integrable and $F \in C^1(R)$ the right-hand side of (4) is continuously differentiable on $(0, \infty)$ and we have (5). Let $q_1 = \min\{u;\, k(u) \ge k(v) \text{ for all } v \ge 0\}$ and $q_2 = \max\{u;\, k(u) \ge 1\}$. Then $0 \le q_1 < q_2$. If $q_1 > 0$, then $f' > 0$ on $(0, q_1]$ by (5). If $q_1 = 0$, then μ belongs to class L and $f' > 0$ on $(0, q_2]$ by Theorem 5.1 of [8]. Combining these facts with the continuity of f, we have that f' is integrable. Let

$$\tilde{f}(x) = x^{-1}\left\{(\lambda-2)f'(x) + \int_0^x f'(x-u)k^*(u)\,du\right\}.$$

Since f' is continuous on $(0, \infty)$ and integrable and k^* is bounded on every finite interval, $\tilde{f}$ is well defined and continuous on $(0, \infty)$. For $0 < x_1 < x_2 < \infty$,

$$\int_{x_1}^{x_2} (x\tilde{f}(x) + f'(x))\,dx = (\lambda-1)(f(x_2) - f(x_1)) + \int_{x_1}^{x_2} dx \int_0^x f'(x-u)k^*(u)\,du$$

$$= (\lambda - 1)(f(x_2) - f(x_1)) + \int_0^{x_2} f(x_2 - u)k^*(u)\,du$$

$$+ \int_0^{x_1} f(x_1 - u)k^*(u)\,du$$

$$= x_2 f'(x_2) - x_1 f'(x_1)$$

by Fubini's theorem. Therefore f' is continuously differentiable on $(0, \infty)$ and we have (6).

PROOF OF "IF" PART OF THEOREM 1. FIRST STEP. Let $\lambda > 1$ and k be continuous on $(0, \infty)$. Let $k^*(0+)$ be finite. Assume that $k^*(p-)$ exists when $p < \infty$. Then the assumption of Lemma 2.2 is fulfilled. Since f' is integrable, integration by parts applied to (5) yields

$$xf'(x) = -f(x) + \int_0^x f'(x-y)k(y)\,dy. \tag{7}$$

By (5) and (7), we have

$$\begin{aligned} x^2[f'(x)]^2 &= (\lambda - 1)xf(x)f'(x) - f(x)\int_0^x f(x-y)k^*(y)\,dy \\ &\quad + \int_0^x \int_0^x f(x-y)k^*(y)f'(x-z)k(z)\,dy\,dz. \end{aligned} \tag{8}$$

By (4) and (6), we have

$$x^2 f(x)f''(x) = (\lambda - 2)xf(x)f'(x) + \int_0^x \int_0^x f(x-y)k(y)f'(x-z)k^*(z)\,dy\,dz. \tag{9}$$

Let $A(x) = [f'(x)]^2 - f(x)f''(x)$. The function $A(x)$ is continuous on $(0, \infty)$ by Lemma 2.2. By (8) and (9),

$$x^2 A(x) = (\lambda - 1)f(x)^2 + \int_0^x dz \int_0^z dy\, G(x, y, z)H(y, z) \tag{10}$$

where $G(x, y, z) = f(x-y)f'(x-z) - f(x-z)f'(x-y)$ and $H(y, z) = k^*(y)k(z) - k^*(z)k(y)$. Since $k(x)$ is log concave, $H(y, z)$ is nonnegative if $0 < y < z < p$. Now we prove that (*) there is some $\delta > 0$ such that $A(x) > 0$ on $(0, \delta)$. This statement is proved for L distributions ([8] Theorem 1.3 (iii)). Thus we assume that $q_1 > 0$ where q_1 is as in the proof of Lemma 2.2. Then $f' > 0$ on $(0, q_1]$. Moreover, for $0 < x < q_1$,

$$xf'(x) \geq (\lambda - 1)f(x) \tag{11}$$

by (5). We have by (6),

$$xf''(x) \leq (\lambda - 2)f'(x) + Mf(x), \quad 0 < x < q_1 \tag{12}$$

where $M = \sup|k^*(u)|$. By (11) and (12), we have

$$xA(x) \geq f(x)(f'(x) - Mf(x)). \tag{13}$$

(11) shows that the right-hand side of (13) is positive for sufficiently small $x > 0$. Thus we have (*). Now, let us assume that μ is not strongly unimodal. That is, there is some x_0 such that $A(x_0) < 0$. Then by (*) and the continuity of $A(x)$, there exists some $0 < x_1 < x_0$ such that $A(x_1) = 0$ and $A(x) > 0$ on $(0, x_1)$. By (10), we have

$$(\lambda - 1)f(x_1)^2 + \int_0^{x_1} dz \int_0^z dy\, G(x_1, y, z)H(y, z) = 0. \tag{14}$$

Since $\lambda > 1$ and $G(x_1, y, z)H(y, z) \geq 0$ for $0 < y < z < x_1$, (14) shows that $f(x_1) = 0$. This contradicts Lemma 2.1. Thus μ is strongly unimodal.

SECOND STEP. We consider the general case. If $p < \infty$, then let

$$\ell_n(x) = \min\{nx + \lambda(1 + 1/n), n(p - 1/n - x)\} \quad \text{for } 0 < x \leq p\ 1/n$$
$$= 0 \quad \text{for } x > p - 1/n$$

and if $p =$ in, then we let

$$\ell_n(\mathrm{x}) = \mathrm{nx} + \lambda(1 + 1/\mathrm{n}) \quad \text{for } x > 0.$$

Let

$$k_n(x) = \min\{(1 + 1/n)k(x), \ell_n(x)\}$$

for $x > 0$. Then $k_n(x)$ is continuous on $(0, \infty)$, $k_n(0+) = (1 + 1/n)\lambda > 1$, $k_n^*(0+) \leq n < \infty$. If $p < \infty$, then $k_n^*(p - 1/n) - = -n > -\infty$. The distribution function $F_n(x)$ corresponding to $k_n(x)$ converges weakly to $F(x)$ since for all $n \geq 1$, $k_n(x)$ is bounded by $2k(x)$ and $k_n(x)$ approaches $k(x)$ as $n \to \infty$ except at p. Note that $k_n(x)$ is log concave. For, let $k_n(x) = (1 + 1/n)k(x)$. Then for sufficiently small $h > 0$,

$$k_n(x)^2 - k_n(x + h)k_n(x - h) \geq (1 + 1/n)\{k(x)^2 - k(x + h)k(x - h)\} \geq 0.$$

If $k_n(x) = \ell_n(x)$, then we have for sufficiently small $h > 0$,

$$k_n(x)^2 - k_n(x + h)k_n(x - h) \geq \ell_n(x)^2 - \ell_n(x + h)\ell_n(x - h) \geq 0.$$

Therefore $F_n(x)$ is strongly unimodal by the first step. It is easy to see that the strong unimodality is closed under weak convergence. Thus $F(x)$ is strongly unimodal. This completes the proof.

3. Proof of the "only if" part of Theorem 1. We prove the "only if" part of the theorem by a method analogous to that of Wolfe which is used to analyze the continuity properties of L distributions ([13]). Suppose that μ is strongly unimodal and $\lambda < 1$. Then μ is absolutely continuous and the density f_μ has a bounded version. We can assume $\int_0^1 u^{-1}k_\mu(u)\,du = \infty$ so that μ is absolutely continuous ([10]). If k_μ is non-increasing, then f_μ is unbounded since μ belongs to the class L (c.f. [8], Theorem 1.6). If μ does not belong to the class L, then there is q_1 such that k_μ is non-decreasing on $[0, q_1]$ and non-increasing on $[q_1, \infty)$. Choose α so that $\lambda < \alpha < \min\{1, k_\mu(q_1)\}$ and fix it. Let

$$k_1(u) = k_\mu(u) \quad \text{on} \quad [0, \delta_1] \cup [\delta_2, \infty),$$
$$= \alpha \quad \text{on} \quad [\delta_1, \delta_2]$$

where $\delta_1 = \min\{u; k_\mu(u) = \alpha\}$ and $\delta_2 = \max\{u; k_\mu(u) = \alpha\}$. Let

$$k_2(u) = \alpha - k_\mu(u) \quad \text{on} \quad [0, \delta_1],$$
$$= 0 \quad \text{on} \quad [\delta_1, \infty)$$

and let

$$k_3(u) = 0 \quad \text{on} \quad [0, \delta_1] \cup [\delta_2, \infty)$$
$$= k_\mu(u) - \alpha \quad \text{on} \quad [\delta_1, \delta_2].$$

Let $k_4(u) = k_1(u) + k_2(u)$. Let $F_i(x)$ $(i = 1, 2, 3, 4)$ be distribution functions with characteristic functions

$$\hat{F}_i(t) = \exp \int_0^\infty (e^{itu} - 1)u^{-1}k_i(u)\,du, \qquad (i = 1, 2, 3, 4).$$

The distribution functions $F_2(x)$ and $F_4(x)$ are L distribution functions. Since $\alpha < 1$, $F_2(x)$

and $F_4(x)$ have unbounded densities $f_2(x)$ and $f_4(x)$ respectively. Assume that $F_1(x)$ has a bounded density $f_1(x)$, then since

$$f_4(x) = \int_0^x f_1(x-y) f_2(y)\, dy,$$

$f_4(x)$ must be bounded. This is absurd, and thus $f_1(x)$ must be unbounded. Since $F_3(0+) > 0$, we have

$$f_\mu(x) \geq f_1(x) F_3(0+)$$

and thus $f_\mu(x)$ is unbounded. This completes the proof.

4. Two lemmas. Let G be a proper or defective distribution function (i.e., $G(\infty) \leq 1$. See [3] page 127.) on $(0, \infty)$. We say that G is a distribution function on $(0, \infty]$. We define an integral of a function $f(x)$ with respect to G by

$$\int_{(0,\infty]} f(x)\, dG(x) = \int_0^\infty f(x)\, dG(x) + \lim_{x\to\infty} f(x)(1 - G(\infty))$$

if the right hand side exists. We say that a real valued function $\psi(t)$ on $[0, \infty)$ is a Stieltjes transform of G if it satisfies

$$\psi(t) = \int_{(0,\infty]} x(x+t)^{-1}\, dG(x) \tag{15}$$

for $0 \leq t < \infty$. This definition of Stieltjes transform is slightly different from the original one (see [11]).

LEMMA 4.1. (Steutel) (i) *A function $\psi(t)$ on $[0, \infty)$ is a Stieltjes transform of some distribution function $G(x)$ on $(0, \infty]$ if and only if ψ is represented as*

$$\psi(t) = \exp\left\{-\int_0^\infty \frac{t}{\lambda(\lambda+t)} m(\lambda)\, d\lambda\right\} \tag{16}$$

for $0 \leq t < \infty$, where $0 \leq m(\lambda) \leq 1$ a.e. and

$$\int_0^1 m(\lambda)\lambda^{-1}\, d\lambda < \infty.$$

(ii) *Let $M(\lambda) = \int_0^\lambda m(u)\, du$. Then G and M uniquely determine each other.*

PROOF. See Steutel [9] page 44.

REMARK ON LEMMA 4.1. (i) $\psi(t)$ is a Stieltjes transform of a distribution function $G(x)$ on $(0, \infty]$ if and only if $\psi(t)$ is a Laplace transform of a distribution which is absolutely continuous except at the origin with density

$$\int_{[b,\infty)} u e^{-ux}\, dG(u) \quad \text{for} \quad x > 0$$

and has a mass $1 - G(\infty)$ at the origin.

(ii) Since

$$t\{\lambda(\lambda+t)\}^{-1} = \lambda^{-1} - (\lambda+t)^{-1} = \int_0^\infty (1 - e^{-tx}) e^{-\lambda x}\, dx$$

for λ, $t > 0$, we can rewrite (16) as

$$\exp \int_{0+}^{\infty} (e^{-tu} - 1)u^{-1}k(u)\, du$$

with $u^{-1}k(u) = \int_0^\infty e^{-ux}m(x)\, dx$.

Let $\psi(t)$ be the Stieltjes transform of a distribution function $G(x)$ on $(0, \infty]$. If $G(b-) = 0$ and $G(b) = G(a-) > 0$ for $0 < b < a$, then we can extend the domain of ψ to $(-a, -b) \cup (-b, \infty)$ and $\psi(t)$ is strictly decreasing on $(-a, -b)$. Thus, $\psi((-a)+)$ exists if we allow infinity.

LEMMA 4.2. *Let ψ, G, and M be quantities defined in Lemma* 4.1. *Let $b = \inf\{x; G(x) > 0\}$ and $c = \inf\{\lambda; M(\lambda) > 0\}$. Then the following hold*:
(i) *$b = c$.*
(ii) *Let $G(b) > 0$. Then, for $a > b$, $G(a) = G(b)$ and $\psi((-a)+) \le 0$ if and only if $M(a) = a - b$.*

PROOF. *First step.* Let $x_{nk} = b + kn2^{-n}$ for $n = 1, 2, \cdots$ and $k = 0, 1, \cdots, 2^n$ and let $x_{n,2^n+1} = \infty$. For each n, we define a distribution function G_n on $(0, \infty]$ by

$$\begin{aligned} G_n(x) &= 0 && \text{if } \ 0 < x < b, \\ &= G(x_{nk}) && \text{if } \ x_{nk} \le x < x_{n,k+1} \quad (0 \le k \le 2^n). \end{aligned} \tag{17}$$

Then, G_n converges weakly to G and $b_n = \inf\{x; G_n(x) > 0\}$ approaches b as $n \to \infty$ respectively. The Stieltjes transform ψ_n of G_n is of the form

$$\psi_n(t) = p_{n0} + \textstyle\sum_{k=1}^{\ell(n)} p_{nk} \dfrac{\lambda_{nk}}{\lambda_{nk} + t} \tag{18}$$

where p_{nk} $(k = 1, 2, \cdots, \ell(n))$ is the magnitude $G_n(\lambda_{nk}) - G_n(\lambda_{nk}-)$ of the kth jump of G_n and $p_{n0} = 1 - G_n(\infty)$ is the jump of G_n at infinity. Assume that $\psi_n(-\infty) = 0$ if $\lim_{t\to-\infty}\psi_n(t) = 0$ and note that $\psi_n((-\lambda_{nk})+) = \infty$ and $\psi_n((-\lambda_{nk})-) = -\infty$ for $k = 1, 2, \cdots, \ell(n)$. Then, differentiating both sides of (18) with respect to t, we get that $\psi_n(t)$ has $\ell(n)$ zeros at $-\infty \le -\mu_{n,\ell(n)} < \cdots < -\mu_{n1} < 0$ such that $\lambda_{n1} < \mu_{n1} < \lambda_{n2} < \mu_{n2} < \cdots < \lambda_{n,\ell(n)} < \mu_{n,\ell(n)} \le \infty$. This implies the representation

$$\psi_n(t) = \textstyle\prod_{k=1}^{\ell(n)} \dfrac{\lambda_{nk}}{\lambda_{nk} + t} \dfrac{\mu_{nk} + t}{\mu_{nk}}.$$

Since $\lambda(\lambda + t)^{-1}$ is the Laplace transform of an exponential distribution, we have that

$$\frac{\lambda}{\lambda + t} \frac{\mu + t}{\mu} = \exp \int_0^\infty (e^{-tu} - 1)(e^{-\lambda u} - e^{-\mu u})u^{-1}\, du$$

for $0 < \lambda < \mu$. This equality with (ii) of the remark on Lemma 4.1 implies that

$$\psi_n(t) = \exp\left\{-\int_0^\infty \frac{t}{\lambda(\lambda + t)} m_n(\lambda)\, d\lambda\right\} \tag{19}$$

where

$$\begin{aligned} m_n(\lambda) &= 1 && \text{if } \ \lambda_{nk} \le \lambda \le \mu_{nk} \quad (k = 1, 2, \cdots, \ell(n)), \\ &= 0 && \text{otherwise.} \end{aligned}$$

Note that in the above representation, m_n and G_n uniquely determine each other (see [9] page 44 Lemma 2.12.1). Let $M_n(\lambda) = \int_0^\lambda m_n(u)\, du$ and let $c_n = \inf\{\lambda; M_n(\lambda) > 0\}$. It is obvious that $b_n = \lambda_{n1} = c_n$. The representation (19) shows that $M_n(a) = a - b_n$ if and only if $\lambda_{n1} < a \le \mu_{n1}$. Since ψ_n is strictly decreasing on $(-\lambda_{n2}, -\lambda_{n1})$, $\psi_n((-\lambda_{n1})-) = -\infty$ and $\psi_n(-\mu_{n1}) = 0$, it is easy to see that $G_n(a) = G_n(b_n)$ and $\psi_n((-a)+) \le 0$ if and only if $\lambda_{n1} < a \le \mu_{n1}$. Thus we have shown that the lemma is true if G is a step function.

Second step. Let us prove (i). It is shown in [9] page 46–47 that $M_n(\lambda)$ converges weakly to $M(\lambda)$. Note that, in general, if $\{F_n\}_{n=1,2,\dots}$ is a sequence of non-negative and non-decreasing functions converging to a function F at the continuity points of F as $n \to \infty$, then $\liminf_{n\to\infty}\inf_x\{x;\ F_n(x) > 0\} \le \inf\{x;\ F(x) > 0\}$. It follows that $b \le c$ from the first step. Now, let us show that $b \ge c$. For $\lambda \ge c$, let

$$\begin{aligned} \tilde{m}_n(\lambda) &= m(\lambda) && \text{if } 0 < m(\lambda) < 1,\\ &= 1 - 1/n && \text{if } m(\lambda) = 1,\\ &= 1/n && \text{if } m(\lambda) = 0. \end{aligned}$$

Let

$$\lambda_{nk} = c + kn2^{-n} \quad (k = 1, 2, \cdots, 2^n)$$

and

$$\mu_{n0} = c, \qquad \mu_{nk} = \lambda_{nk} + \int_{\lambda_{nk}}^{\lambda_{n,k+1}} \tilde{m}_n(u)\, du$$

for $k = 1, 2, \cdots, 2^n - 1$. Then $\lambda_{nk} < \mu_{nk} < \lambda_{n,k+1}$ for $k = 1, 2, \cdots, 2^n - 1$. Define $m_n(\lambda)$ by

$$\begin{aligned} m_n(\lambda) &= 1 && \text{if } \lambda_{nk} < \lambda \le \mu_{nk} && (k = 1, 2, \cdots, 2^n - 1)\\ &= 0 && \text{if } \mu_{nk} < \lambda \le \lambda_{n,k+1} && (k = 0, 1, \cdots, 2^n - 1),\\ &= 1 && \text{if } \lambda > c + n. \end{aligned}$$

It is shown in [9] pages 48–49 that $M_n(\lambda) = \int_0^\lambda m_n(u)\, du$ and $G_n(x)$, corresponding to $M_n(\lambda)$, converge weakly to $M(\lambda)$ and $G(x)$ as $n \to \infty$ respectively. Since $\inf\{\lambda;\ M_n(\lambda) > 0\} = c + n2^{-n}$ converges to c as $n \to \infty$, we have $b \ge c$ and thus $b = c$.

Third step. We prove (ii). Suppose that $G(a) = G(b) > 0$ and $\psi((-a)+) \le 0$. Then, noting that $\psi(t)$ is strictly decreasing on $(-a, -b)$, we have that $\psi(t) < 0$ on $(-a, -b)$. Noting that $x(x + t)^{-1}$ is a decreasing function of $x > -t$ if $t < 0$, we have, for $-a < t < -b$,

$$\begin{aligned} \psi(t) &\ge \frac{b}{b+t} G(b) + \textstyle\sum_{x_{nk}>a} \dfrac{x_{nk}}{x_{nk}+t} \{G(x_{nk}) - G(x_{n,k-1})\}\\ &= \int_{[b,\infty]} \frac{x}{x+t}\, dG_n(x) = \psi_n(t) \end{aligned}$$

where $\{x_{nk}\}$, G_n and ψ_n are quantities defined in the first step. Note that $b_n = b$ and $G_n(a) = G_n(b)$. The above inequality shows that $\psi_n(t) < 0$ on $(-a - b)$. Thus $\psi_n((-a)+) \le 0$ and then, by the first step, $M_n(a) = a - b$. Since $M_n(a)$ converges to $M(a)$ as $n \to \infty$, we have that $M(a) = a - b$. Conversely, assume that $M(a) = a - b$. Since

$$\exp\int_b^a \frac{t}{\lambda(\lambda + t)}\, d\lambda = \frac{b(a+t)}{a(b+t)},$$

we can rewrite $\psi(t)$ by (i) as

$$\psi(t) = \frac{b(a+t)}{a(b+t)} \int_{[a,\infty]} \frac{x}{x+t}\, d\tilde{G}(x) \tag{20}$$

where $\tilde{G}(x)$ is some distribution function on $[a, \infty]$. Moreover, we can rewrite (20) as

$$\psi(t) = \frac{b(a-b)}{a(b+t)} \int_{[a,\infty]} \frac{x}{x-b} \, d\tilde{G}(x) + \frac{b}{a} \int_{[a,\infty]} \frac{x}{x+t} \frac{x-a}{x-b} \, d\tilde{G}(x) \tag{21}$$

for $-a < t < \infty$. Thus, $\psi(t)$ is the Stieltjes transform of a distribution function $\tilde{\tilde{G}}$ which satisfies $\tilde{\tilde{G}}(a) = \tilde{\tilde{G}}(b) \geq 0$. It follows from the uniqueness theorem for Stieltjes transforms ([11], page 336) that $\tilde{\tilde{G}} = G$. Note that

$$\frac{a-b}{b+t} + \frac{x-a}{x+t} < 0$$

for $(t, x) \in (-a, -b) \times (a, \infty)$. We have, by (21), that $\psi(t) < 0$ on $(-a, -b)$ and thus $\psi((-a)+) \leq 0$. This completes the proof.

5. Proof of Theorem 2. Let $\mu \in I_2$. Lemma 4.1, 4.2 and the remark following Lemma 4.1 yield the decomposition $\mu = \nu_1 * \nu_2$ where ν_1 is exponentially distributed with density ae^{-ax} and ν_2 is absolutely continuous except at the origin with density

$$w(x) = \int_{[b,\infty)} ue^{-ux} \, dG(u) \quad \text{for} \quad x > 0$$

and has a mass $1 - G(\infty)$ at the origin. Here, $a = a_\mu$ and $G(x)$ is a distribution function on $[b, \infty]$ such that $b = \inf\{x;\, G(x) > 0\} = b_\mu$ and the corresponding function $m(\lambda)$, which appears in Lemma 4.1, coincides with m_μ. Thus μ is absolutely continuous with density

$$f(x) = \int_{[b,\infty]} \frac{au(e^{-ax} - e^{-ux})}{u-a} \, dG(u). \tag{22}$$

The representation (22) of f shows that f belongs to $C^\infty((0, \infty))$. The first and the second derivatives of f satisfy the equations

$$f'(x) = -af(x) + \phi(x) \tag{23}$$

and

$$f''(x) = a^2 f(x) - a\phi(x) + \phi'(x) \tag{24}$$

respectively, where $\phi(x) = \int_{[b,\infty]} aue^{-ux} \, dG(u)$. Let $A(x) = f'(x)^2 - f(x)f''(x)$. By (22)-(24), we have

$$A(x) = \phi(x)^2 + \xi(x)f(x), \tag{25}$$

where $\xi(x) = -\phi'(x) - a\phi(x)$.

Now we can prove (i). Let $a \leq b$. Then $\xi(x) \geq 0$ for $x > 0$. Thus $A(x) \geq 0$ for $x > 0$ and (i) is true by Ibragimov's theorem.

Let us prove (ii). Suppose that $b < a$ and $\int_b^a m_\mu(u) \, du < a - b$. Let

$$\phi_1(x) = \int_{[b,a)} aue^{-ux} \, dG(u),$$

$$\xi_1(x) = \int_{[b,a)} au(u-a)e^{-ux} \, dG(u)$$

and

$$f_1(x) = \int_{[b,a)} \frac{au(e^{-ax} - e^{-ux})}{u-a} \, dG(u)$$

and let $\phi_2(x) = \phi(x) - \phi_1(x)$, $\xi_2(x) = \xi(x) - \xi_1(x)$ and $f_2(x) = f(x) - f_1(x)$. Then $A(x)$ is the sum of

$$A_1(x) = \phi_1(x)^2 + \xi_1(x)f_1(x),$$
$$A_2(x) = 2\phi_1(x)\phi_2(x) + \xi_1(x)f_2(x) + \xi_2(x)f_1(x)$$

and

$$A_3(x) = \phi_2(x)^2 + \xi_2(x)f_2(x).$$

By the definitions of these functions, we easily obtain the following.

$$\phi_1(x) = O(e^{-bx}), \quad \xi_1(x) = O(e^{-bx}),$$
$$\phi_2(x) \sim a^2e^{-ax}(G(a) - G(a-)) \quad \text{and} \tag{26}$$
$$\xi_2(x) = o(e^{-ax})$$

as $x \to \infty$. If $a \leq u \leq \infty$, then

$$\frac{u(1 - e^{(a-u)x})}{u - a} \leq \min\left\{\frac{u}{u - a}, ux\right\} \leq 1 + ax.$$

Here we used the fact that $u/(u - a)$ is decreasing on $[a, \infty)$ and ux is an increasing function of u. The last inequality yields the estimate

$$f_2(x) \leq a(1 + ax)e^{-ax} \quad \text{for all} \quad x \geq 0. \tag{27}$$

Since $e^{(u-a)x} - 1 \geq (u - a)x$, we have that

$$f_1(x) \leq x\phi_1(x). \tag{28}$$

By (27) and (28), we have that

$$|A_2(x)| \leq \phi_1(x)\{2\phi_2(x) + x\xi_2(x)\} + a(1 + ax)e^{-ax}|\xi_1(x)|.$$

Then we have

$$A_2(x) = O(xe^{-(a+b)x}) \quad \text{as} \quad x \to \infty. \tag{29}$$

By (26) and (27), we have that

$$A_3(x) = o(xe^{-2ax}) \quad \text{as} \quad x \to \infty. \tag{30}$$

Let $d = \inf\{x;\ G(x) > G(b)\}$ and let ψ and M be quantities that are defined for G in Lemma 4.1. Note that $M(\lambda) = \int_0^\lambda m_\mu(u)\,du$. We consider the following three cases:

(I) $d \geq a$.

(Ia) $G(a-) < G(a)$,

(Ib) $G(a-) = G(a)$ and $0 < \psi((-a)+) \leq \infty$.

(II) $d = b$.

(III) $b < d < a$.

These three cases are not empty and exhaust all possibilities. For, if $b \leq d < a$, then $G(a) > G(b)$. Thus $M(a) < a - b$ by Lemma 4.2 (ii). Since $M(a) = \int_b^a m_\mu(x)\,dx$, the assumption of (ii) of Theorem 2 is fulfilled. Let $d \geq a$. Then $G(a-) = G(b) > 0$. If $G(a) > G(a-)$, then the assumption of (ii) of Theorem 2 is also fulfilled by Lemma 4.2 (ii). Suppose that $G(a) = G(a-) = G(b) > 0$. Then the domain of ψ can be extended to $(-a, -b) \cup (-b, \infty)$ and $\psi((-a)+)$ exists if we allow infinity. Lemma 4.2 (ii) shows that the assumption of (ii) of Theorem 2 holds if and only if $0 < \psi((-a)+) \leq \infty$ in this case.

Case (I). Let $G(b) = q$. Note that G is flat on (b, a). Thus we have

$$\phi_1(x) = abqe^{-bx}, \qquad \xi_1(x) = abq(b - a)e^{-bx} \tag{31}$$

and

$$f_1(x) = abq(e^{-ax} - e^{-bx})/(b - a).$$

We also obtain that

$$A_1(x) = (abq)^2 e^{-(a+b)x}. \tag{32}$$

Then we easily obtain by (25), (31) and (32) that

$$\begin{aligned} A(x)e^{(a+b)x} &= \{A_1(x) + \xi_1(x) f_2(x)\} e^{(a+b)x} + \eta(x) \\ &= abq(b-a)\left\{\frac{ab}{b-a} q + f_2(x) e^{ax}\right\} + \eta(x) \end{aligned} \tag{33}$$

where $\eta(x) = \{f_1(x)\xi_2(x) + 2\phi_1(x)\phi_2(x) + A_3(x)\} e^{(a+b)x}$. It is easy to see by (26), (30) and (31) that

$$\lim_{x\to\infty} \eta(x) = 2a^3 bq(G(a) - G(a-)). \tag{34}$$

Let $G(a) = G(a-)$ and $0 < \psi((-a)+) \le \infty$. Since $u/(u-t)$ is positive and increasing as $t \uparrow a$ for each $a < u < \infty$,

$$\begin{aligned} \lim_{x\to\infty} e^{ax} f_2(x) &= \lim_{x\to\infty} \int_{(a,\infty]} \frac{au}{u-a}(1 - e^{(a-u)x})\, dG(u) \\ &= \int_{(a,\infty]} \frac{au}{u-a}\, dG(u) = \lim_{t\uparrow a} \int_{(a,\infty]} \frac{au}{u-t}\, dG(u). \end{aligned}$$

These equalities show that

$$\lim_{x\to\infty}\left\{\frac{ab}{b-a} q + f_2(x) e^{ax}\right\} = a\psi((-a)+).$$

Thus we have by (33) with (34) that

$$\lim_{x\to\infty} A(x)e^{(a+b)x} = a^2 bq(b-a)\psi((-a)+) < 0. \tag{35}$$

If $G(a) > G(a-)$, then

$$\lim_{x\to\infty} e^{ax} f_2(x) \ge \lim_{x\to\infty} a^2 x(G(a) - G(a-)) = \infty.$$

Thus we obtain by (33) with (34) that

$$\lim_{x\to\infty} A(x)e^{(a+b)x} = -\infty. \tag{36}$$

Now (35) and (36) establish Theorem 2 (ii) in Case (I).

Let us consider Case (II) and (III). We have already estimated $A_2(x)$ and $A_3(x)$. In order to estimate $A_1(x)$, we write $A_1(x)$ as

$$A_1(x) = \int_{[b,a)^2} \tfrac{1}{2} a^2 uv P(u, v, x)\, dG(u)\, dG(v) \tag{37}$$

where

$$P(u, v, x) = \{(a-u)^2 e^{-(a+u)x} + (a-v)^2 e^{-(a+v)x} - (u-v)^2 e^{-(u+v)x}\}/(a-u)(a-v). \tag{38}$$

Choose δ so that $0 < \delta < (a-d)/2$ and fix it. Let $D_1 = [b, a-\delta]^2$ and $D_2 = [b, a]^2 \backslash D_1$. Since $P(u, v, x) \le 2e^{-(u+v)x}$ on $[b, a]^2$, we have that

$$\begin{aligned} \int_{D_2} P(u, v, x)\, dG(u)\, dG(v) &\le 2 \int_{[b,a)\times[a-\delta,a)} e^{-(u+v)x}\, dG(u)\, dG(v) \\ &= o(e^{-(a+b-2\delta)x}) \quad \text{as} \quad x \to \infty. \end{aligned} \tag{39}$$

Case (II). Let us estimate the integral of $P(u, v, x)$ on D_1 with respect to $dG\, dG$. Choose $c \in (b, (a+b)/2 - \delta)$ and fix it. We estimate the integral of the nonnegative terms (the first and the second terms in the braces of the right hand side of (38)) and the integral of the nonpositive term (the third term) of $P(u, v, x)$ separately. The integral of the nonnegative terms is shown to be bounded by a constant times $e^{-(a+b)x}$ by estimating the integrand. In order to estimate the integral of the nonpositive term, restrict the domain of integration to $[b, c]^2$ and then estimate the integrand. The result is that

$$(40) \qquad \int_{D_1} P(u, v, x)\, dG(u)\, dG(v) \leq N_1 e^{-(a+b)x} - N_2 e^{-2cx}$$

where N_1 is a constant and $N_2 = \int_{[b,c]^2} ((u - v)/(a - b))^2\, dG(u)\, dG(v)$. Since $d = b$, N_2 is positive. Therefore, noting that $2c < a + b - 2\delta < a + b < 2a$, we have by (29), (30), (39) and (40) that $A(x) < 0$ for large $x > 0$.

Case (III). Choose $c \in (d, a - 2\delta)$ and fix it. Since $b < d < a$, G must have a jump $q > 0$ at b. Bearing this fact in mind, we can estimate the integral of $P(u, v, x)$ on D_1 with respect to $dG\, dG$ as in the case (II). We have

$$(41) \qquad \int_{D_1} P(u, v, x)\, dG(u)\, dG(v) \leq N_3 e^{-(a+b)x} - N_4 e^{-(b+c)x}$$

where N_3 is a constant and $N_4 = \int_{[b,c]} q((u - b)/(a - b))^2\, dG(u)$. Here we restricted the domain of integration of the non-positive term to $\{b\} \times [b, c]$. By the choice of c, the constant N_4 is positive. Therefore, by (29), (30), (39) and (41), we have $A(x) < 0$ for large $x > 0$ since $b + c < a + b - 2\delta < a + b < 2a$. This completes the proof appealing to Ibragimov's theorem.

6. Remarks.

REMARK *1. An example.* There exists $k(u)$ which satisfies the condition of Theorem 2 but does not satisfy the condition of Theorem 1. Let

$$k(u) = e^{-au} + e^{-bu} - e^{-cu}$$

for $0 < a < b < c$. The function $k(u)$ is positive for $u > 0$. We have

$$[k'(u)]^2 - k(u)k''(u) = -(a - b)^2 e^{-(a+b)u} - (b - c)^2 e^{-(b+c)u} + (c - a)^2 e^{-(c+a)u}.$$

Since $0 < a < b < c$, $[k'(u)]^2 - k(u)k''(u) < 0$ for large $u > 0$. Thus $k(u)$ is not log concave. However, since

$$u^{-1}k(u) = u^{-1}e^{-au} + \int_b^c e^{-xu}\, dx,$$

$k(u)$ satisfies the condition of Theorem 2.

REMARK *2. An application of Lemma* 4.1. It is shown in [14] that all L distributions are unimodal. Thus all stable distributions are unimodal. Let us prove this fact as an application of Lemma 4.1. Let μ be a stable distribution with exponent $\beta (0 < \beta < 2)$. The characteristic function $\hat{\mu}(t)$ of μ is represented as

$$\hat{\mu}(t) = \exp\left\{ c_1 \int_{0+}^{\infty} g(t, u) u^{-1-\beta}\, du + c_2 \int_{-\infty}^{0-} g(t, u) \, |u|^{-1-\beta}\, du \right\}$$

where $c_1, c_2 \geq 0$ and $g(t, u) = e^{itu} - 1 - itu(1 + u^2)^{-1}$. Note that $c_1 u^{-1-\beta} = d_1 \int_{0+}^{\infty} \lambda^{\beta} e^{-u\lambda}\, d\lambda$ for $u > 0$ where $d_1 \geq 0$ is a constant. Let $a_n = (n d_1^{-1})^{1/\beta}$. Define $m_n(\lambda)$ $(n = 1, 2, \cdots)$ by

$$\begin{aligned} m_n(\lambda) &= 0 \quad \text{if} \quad \lambda < a_n \\ &= 1 \quad \text{if} \quad \lambda \geq a_n \end{aligned}$$

for $n = 1, 2, \cdots$ and $m_0(\lambda)$ by

$$m_0(\lambda) = d_1\lambda^\beta - \textstyle\sum_{n=1}^{\infty} m_n(\lambda).$$

Then $m_0(\lambda) \le 1$ and $d_1\lambda^\beta = m_0(\lambda) + \sum_{n=1}^{\infty} m_n(\lambda)$. Note that for $n \ge 1$, the distribution with characteristic function

$$\exp \int_{0+}^{\infty} g(t, u)\, du \int_0^{\infty} m_n(\lambda) e^{-u\lambda}\, d\lambda$$

is an exponential distribution, which is strongly unimodal. Since $m_0(\lambda) \le 1$, the distribution with characteristic function

$$\exp \int_{0+}^{\infty} g(t, u)\, du \int_0^{\infty} m_0(\lambda) e^{-u\lambda}\, d\lambda$$

has a completely monotone density by Lemma 4.1 and the remark following it. Apply the same argument to $c_2 |u|^{-1-\beta}$. Then, we can decompose μ as $\mu_1 * \mu_2$ where μ_1 is strongly unimodal and μ_2 is unimodal as a convolution of two unimodal distributions with mode at the origin and support on the positive and negative axes respectively.

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NAGOYA INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MATHEMATICS
GOKISO-CHO, SHOWA-KU
NAGOYA, 466 JAPAN