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FIRST HITTING TIME OF CURVILINEAR BOUNDARY BY WIENER PROCESS

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A function $f(t)$ such that $f(t)/\sqrt{t+1} \uparrow a$ is considered. We define $T = \inf\{t : |W(t)| = f(t)\}$, where $W(t)$ is the Wiener process starting from 0. A sufficient condition for $E\{T^{\mu}\}$ to be finite is given.

Let $w(t)$ be the Wiener process starting from zero and $f(t)$ be a positive increasing function of t. Put

$$T_f = \inf\{t : |w(t)| \geq f(t)\}.$$

Given $\mu > 0$, we are interested in sufficient conditions on f which ensure the finiteness of

$$m_{\mu}^{f} = E\{T_f^{\mu}\}.$$

If

(1) $$f(t) = c\sqrt{t+1},$$

then the problem is completely investigated. By the results of Shepp [2]

(2) $$m_{\mu}^{f} < \infty \quad \text{iff} \quad c < a(\mu),$$

where $a(\mu)$ is the first zero of the confluent hypergeometric function (here $(2m)!$ corrects an error in [2])

$$F_{\mu}(x) = M\left(-\mu, \frac{1}{2}, \frac{x^2}{2}\right) = \sum_{m=0}^{\infty} \frac{(-2x^2)^m \mu(\mu-1)\cdots(\mu-m+1)}{(2m)!}.$$

It is easy to see that $a(\mu)$ is a continuous decreasing function of μ and therefore there exists the inverse function $\mu(a)$. The result of Shepp (relation (2)) may be reformulated in the following way. For f given by (1),

$$m_{\nu}^{f} < \infty, \quad \text{iff} \quad \nu < \mu(c).$$

Suppose now that

(3) $$f(t)/\sqrt{t+1} \uparrow c.$$

It is obvious that $m_{\nu}^{f} < \infty$ if $\nu < \mu(c)$ and $m_{\nu}^{f} = \infty$ if $\nu > \mu(c)$. The question is whether it is possible that $m_{\mu(c)}^{f} < \infty$? The answer is positive and a sufficient condition for that is given by the following theorem:

THEOREM. *For f, satisfying* (3),

$$m_{\mu(c)}^{f} < \infty$$

if

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$$\int_0^\infty t^{\mu(c)-\mu(r(t))-1}\,dt < \infty, \tag{4}$$

where

$$r(t) = f(t)/\sqrt{t+1}.$$

PROOF. 1°. For $f(t) = c\sqrt{t+1}$ denote T_f by T_c. Consider the process $Y(u) = w(e^{2u})/e^u$. Let τ_c be the first exist time of $Y(u)$ from the boundaries $\pm c$. By the results of Breiman [1] the Laplace transform of τ_c is

$$\begin{aligned}\phi(\lambda) &= \int_0^\infty e^{-\lambda x}\,dP\{\tau_c > x\} \\ &= \exp(c^2/4)[D(-\lambda, 0) + D(-\lambda, 0)]/[D(-\lambda, c) + D(-\lambda, -c)],\end{aligned} \tag{5}$$

where $D(\lambda, z) = D_\lambda(z)$ is the parabolic cylinder function.

In Section 2 of [1] it was shown that $\Phi(\lambda)$ has only real simple poles on the negative axis.

Let $-2\beta(c)$ be the position of the largest pole and $-2\delta(c)$ be the positive of the second largest pole of $\phi(\lambda)$ (certainly $\delta(c) > \beta(c)$). Then

$$P\{\tau_c > x\} = \alpha \exp(-2\beta(c)x) + O(\exp(-2\delta(c)x)), \tag{6}$$

(see (2.4) of [1]). For the Wiener process, formula (6) becomes

$$P\{T_c > t\} = \alpha t^{-\beta(c)} + O(t^{-\delta(c)}),$$

(see (2.5) of [1]). From the above relation we see in particular that

$$\beta(c) = \mu(c). \tag{7}$$

Since $\beta(c)$ and $\delta(c)$ are continuous functions of c and

$$\alpha = \alpha(c) = 2D(\lambda, 0)\exp(c^2/4)\left(\frac{d}{d\lambda}(D(\lambda, c) + D(\lambda, -c))|_{\lambda=2\beta(c)}\right)^{-1},$$

then $\alpha(c)$ is a continuous function of c and $\alpha(c)$ is bounded on any segment which does not contain zero. Similarly, using (5) and standard techniques related to the Laplace transform, we can show

$$[P\{T_c > t\} - \alpha(c)t^{-\beta(c)}]/t^{-\beta(c)} = o(1)$$

uniformly in c, $c \in [a, b]$, $a, b > 0$. In particular there exists a constant $d = d(a, b)$ such that for any $c \in [a, b]$, $a, b > 0$

$$P\{T_c > t\} < dt^{-\beta(c)}. \tag{8}$$

2°. Now let $f(t)$ satisfy (3). We try to estimate $P\{T_f > x\}$. Consider the parabola $g(t) = c'\sqrt{t+1}$ where $c' = r(x)$. By virtue of (3), we have

$$\begin{aligned} g(t) &\ge f(t) \quad \text{for} \quad t \le x, \\ g(t) &\le f(t) \quad \text{for} \quad t \ge x. \end{aligned} \tag{9}$$

Formulae (9) show that $\{T_f > x\} \subset \{T_{c'} > x\}$ and

$$P\{T_f > x\} \le P(T_{c'} > x\}. \tag{10}$$

Put $d = d(f(0), c)$; then, by virtue of (8) and (10),

$$P\{T_f > x\} \le dx^{-\beta(r(x))}. \tag{11}$$

Compute

$$m_\nu^f = \int_0^\infty x^\nu \, dP\{T_f > x\} = \nu \int_0^\infty x^{\nu-1} P\{T_f > x\} \, dx. \tag{12}$$

Now let $\nu = \mu(c)$. By virtue of (11), the right hand side of (12) is finite if

$$\int_0^\infty x^{\mu(c)-\beta(r(x))-1} \, dx < \infty. \tag{13}$$

By (7), formula (13) is equivalent to (4).

COROLLARY. *If $d > 0$ and for some t_0,*

$$f(t) = c\sqrt{t+1}\,(1 - d/\log\log t), \quad \text{for} \quad t > t_0,$$

then $m^f_{\mu(c)} < \infty$.

PROOF. Let $-2e$ be the derivative of $\mu(\cdot)$ at the point c. Then for t big enough

$$\mu(r(t)) - \mu(c) > e(c - r(t)) = edc/\log\log t.$$

Put $\alpha = ecd$. For sufficiently large t

$$t^{\mu(r(t))-\mu(c)} \geq \exp(\alpha \log t/\log\log t) \geq \exp((\alpha \log t)/2) = t^{\alpha/2}.$$

Therefore the integrand in (4) is less than $t^{-1-\alpha/2}$ for t big enough, and that shows the finiteness of (4).

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