The Annals of Probability
1982, Vol. 10, No. 4, 1019–1028

THE EXPECTED RATIO OF THE SUM OF SQUARES TO THE SQUARE OF THE SUM[1]

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Let $\{X_i, i = 1, 2, \cdots\}$ be a sequence of positive i.i.d. random variables. Define $S_n = \sum_{i=1}^n X_i$ and $T_n = \sum_{i=1}^n X_i^2$. We study the rate, if any, at which $E[S_n^{-2}T_n] \to 0$.

1. Introduction. Let $X_1, X_2, \cdots$ be non-negative independent random variables with common distribution function F. Assume $F(0) < 1$ and let $H(x) = 1 - F(x) = P(X_1 > x)$. Define $S_n = \sum_{i=1}^n X_i$ and $T_n = \sum_{i=1}^n X_i^2$. Let $R_n = T_n S_n^{-2}$ if $S_n > 0$, $R_n = 1$ otherwise. It is shown in Theorem 4 that $ER_n \to 0$ iff $\int_0^x H(y)\,dy$ is slowly varying. The purpose of the rest of this paper is to study the rate at which $ER_n \to 0$.

The question which led us to this study was posed by Professor A. Joffe (private communication) in connection with his work (1978) on branching processes. He specifically sought moment conditions on X_1 under which $ER_n = O((\ln n)^{-1})$ as $n \to \infty$. The answer to this question is provided by Corollary 2; namely it is sufficient that $E\{X_1 \ln X_1\} < \infty$ (where $0 \ln 0 = 0$). Theorem 6 gives a further sufficient condition for $ER_n = O\{(\ln n)^{-1}\}$.

Note that the quantity R_n is related to one of the "self-normalized sums" studied by Logan, Mallows, Rice and Shepp (1973). Specifically, $R_n = (S_n\,(2))^{-2}$. Observe, however, that they assume $EX_1 = 0$ when $E\,|\,X_1\,| < \infty$, which is incompatible with our non-negativity assumption. Cohn and Hall (1982) have also recently obtained some results on ER_n in the course of their study on weighted sums of random variables. For some readers, the main interest in our theorems will be in the methodology. The various techniques used below may also be useful in studying expectations of other random variables.

We use the following conventions throughout this paper. Let a_n and b_n be sequences of positive numbers. We write $a_n = o(b_n)$ if $a_n b_n^{-1} \to 0$ as $n \to \infty$; $a_n = O(b_n)$ if $a_n b_n^{-1} \le c$ for all n where c is a positive constant; $a_n \sim b_n$ if $a_n b_n^{-1} \to 1$ as $n \to \infty$; and $a_n \approx b_n$ if $a_n = O(b_n)$ and $b_n = O(a_n)$. Also, the symbols C, C_1, C_2, etc. always denote positive constants. If we write $a_n \le C_i b_n$ for all n, we mean that there exists $C_i > 0$ such that $a_n \le C_i b_n$ for all n, unless C_i has already been determined by earlier considerations. Finally, when we consider limits, we always mean as the variable goes to plus infinity.

By Jensen's inequality, $n^{-1} \le R_n \le 1$ a.s. for all n so that $n^{-1} \le ER_n \le 1$. The exact behaviour of ER_n depends very much on the nature of H as the following heuristic argument suggests. Suppose $H(x) = x^{-b}$ for $x > 1$, where $b > 0$. The random variables $H(X_i)$ are uniformly distributed in (0, 1). Suppose $H(X_1)$, $H(X_2)$, $\cdots$, $H(X_n)$ have the "typical" values n^{-1}, $2n^{-1}$, $\cdots$, 1 in some order. Then a direct calculation gives

$$R_n = \sum_{k=1}^n k^{-2b^{-1}} \{\sum_{k=1}^n k^{-b^{-1}}\}^{-2}.$$

The asymptotic behaviour (in the sense of $\approx$) of these typical values of R_n is given in Table 1 for various values of b. Table 1 also gives the asymptotic behaviour of ER_n as determined later in the paper. Note that the latter entries are somewhat larger than the former when $1 \le b < 2$ but they are the same when $b < 1$ or $b \ge 2$.

Received July 1980; revised February 1982

[1] Research supported in part by the Natural Sciences and Engineering Research Council of Canada. The first author is now at the University of Waterloo.

AMS 1970 *subject classifications*. Primary 60F99.

Key words and phrases. Limit theorems, expectation.

TABLE 1
Asymptotic Behaviour of Typical R_n and of ER_n when $H(x) = x^{-b}$.

b	Typical R_n	ER_n
$b > 2$	n^{-1}	n^{-1}
$b = 2$	$n^{-1} \ln n$	$n^{-1} \ln n$
$1 < b < 2$	$n^{2b^{-1}-2}$	n^{1-b}
$b = 1$	$(\ln n)^{-2}$	$(\ln n)^{-1}$
$0 < b < 1$	1	1

2. The case of finite mean. We begin with estimates which make explicit the relationship between ER_n and the rate of growth of S_n. Define $M_n = \max(X_1, X_2, \cdots, X_n)$.

LEMMA 1. (a) *Suppose a_n is a sequence of positive reals such that*

$$P[S_n - M_n \le a_n] = o\left(na_n^{-2} \int_0^{a_n} vH(v)\,dv\right) \tag{1}$$

Then

$$\begin{aligned} ER_n &\le 2na_n \int_0^\infty v(v+a_n)^{-3}H(v)\,dv(1+o(1)) \\ &\le 6na_n^{-2} \int_0^{a_n} vH(v)\,dv(1+o(1)). \end{aligned} \tag{2}$$

(b) *Suppose a_n is a sequence such that $P(S_{n-1} > a_n) \to 0$. Then*

$$\begin{aligned} ER_n &\ge 2na_n \int_0^\infty v(v+a_n)^{-3}H(v)\,dv(1+o(1)) \\ &\ge \frac{1}{4} na_n^{-2} \int_0^{a_n} vH(v)\,dv(1+o(1)). \end{aligned} \tag{3}$$

PROOF. First note that for $a > 0$

$$\begin{aligned} a\int_a^\infty v^{-2}H(v)\,dv &\le a\int_a^\infty v^{-2}H(a)\,dv = H(a) \\ &= 2a^{-2}\int_0^a vH(a)\,dv \le 2a^{-2}\int_0^a vH(v)\,dv. \end{aligned}$$

Therefore

$$\begin{aligned} a\int_0^\infty v(v+a)^{-3}H(v)\,dv &\le a^{-2}\int_0^a vH(v)\,dv + a\int_a^\infty v^{-2}H(v)\,dv \\ &\le 3a^{-2}\int_0^a vH(v)\,dv. \end{aligned} \tag{4}$$

On the other hand,

$$a^{-2}\int_0^a vH(v)\,dv \le 8a\int_0^a v(v+a)^{-3}H(v)\,dv \tag{5}$$
$$\le 8a\int_0^\infty v(v+a)^{-3}H(v)\,dv.$$

The second inequalities in (2) and (3) follow from (4) and (5) respectively. Observe that

$$ER_n \le E(R_n, S_n - M_n \ge a_n) + P(S_n - M_n < a_n). \tag{6}$$

Now,

$$\begin{aligned} E(R_n, S_n - M_n \ge a_n) &\le \textstyle\sum_{j=1}^n E(X_j^2(X_j + a_n)^{-2}) = nEX_1^2(X_1 + a_n)^{-2} \\ &= 2n\int_0^1 tP(X_1(X_1+a_n)^{-1} > t)\,dt \\ &= 2n\int_0^1 tP(X_1 > (1-t)^{-1}a_n t)\,dt \\ &= 2na_n\int_0^\infty v(v+a_n)^{-3}H(v)\,dv \end{aligned} \tag{7}$$

where we have substituted $v = (1-t)^{-1}a_n t$. The first inequality in (2) now follows from (6), (7), (1) and (5). The first inequality in (3) is obtained by combining (7) with the following:

$$\begin{aligned} ER_n &= nE(X_n^2 S_n^{-2}) \\ &\ge nE(X_n^2 S_n^{-2} \mid S_{n-1} \le a_n)P(S_{n-1} \le a_n) \\ &\ge nE(X_1^2(X_1 + a_n)^{-2})(1 + o(1)). \quad \square \end{aligned} \tag{8}$$

The next result indicates the relationship between ER_n and H in cases when $EX_1 < \infty$. We note that the upper bound applies even when $EX_1 = \infty$ but it is not in general tight in that case, as can be seen by supposing $xH(x) \to \infty$.

THEOREM 1. *For any H,*

$$ER_n = O\left\{n^{-1}\int_0^n xH(x)\,dx\right\}. \tag{9}$$

If $EX_1 < \infty$,

$$ER_n \approx n^{-1}\int_0^n xH(x)\,dx. \tag{10}$$

PROOF. Consider any a such that $0 < a < \min(1, EX_1)$. An estimate due to Chernoff (1952) shows that

$$P\,(S_{n-1} \le an) = o(\rho^n) \tag{11}$$

where $0 < \rho < 1$. Therefore

$$\begin{aligned} P[S_n - M_n \le an] &= P[\cup_i\{S_n - X_i \le an\}] \\ &\le nP[S_{n-1} \le an] = o(n\rho^n) \end{aligned} \tag{12}$$

so that (1) holds with $a_n = an$. By (2),

$$ER_n \le 6a^{-2}n^{-1} \int_0^n vH(v)\, dv\{1 + o(1)\}$$

which proves (9). Similarly, if $EX_1 < a < \infty$, then $P[S_n > an] \to 0$ by the law of large numbers. Applying (3) with $a_n = an$,

$$ER_n \ge 2an^2 \int_0^n v(v + an)^{-3}H(v)\, dv(1 + o(1))$$

$$\ge 2an^2(n + an)^{-3} \int_0^n vH(v)\, dv(1 + o(1)),$$

which proves (10). □

We note that the upper bound (9) can also be obtained by a truncation argument like that used later in Theorem 4. For the rest of this section we give some simplifications and improvements in Theorem 1. We begin by showing that if H satisfies a moderate smoothness condition then we may replace the integrals in (9) and (10) by quantities which do not involve integration.

COROLLARY 1. *Suppose there exist numbers $a > 1$ and $m > a^{-2}$ such that $H(ax) \ge mH(x) > 0$ for all large x. Then $ER_n = O\{nH(n)\}$ and if $EX_1 < \infty$ then $ER_n \approx nH(n)$.*

PROOF. Since H is non-increasing, the hypotheses ensure that H is $R - O$ varying as in Seneta (1976, pages 92–94). By his Theorem A.2,

$$nH(n) \approx n^{-1} \int_0^n yH(y)\, dy. \tag{13}$$

The results now follow from (9) and (10). □

The next corollary provides an answer to Joffe's original question as cited in the introduction. It shows in particular that $ER_n = o\{(\ln n)^{-1}\}$ if $E\{X_1 \ln X_1\} < \infty$.

COROLLARY 2. *Let K be a function which is positive and non-decreasing for all large x and for which there exists constants $a > 1$ and $m > a^{-2}$ such that $K(ax) \le m^{-1}K(x)$ for all large x. If $E\{K(X_1)\} < \infty$, then $ER_n = o\{n/K(n)\}$.*

PROOF. Let C be sufficiently large that the conditions on K apply for all $x \ge C$. These conditions imply in particular that $K(x) = o(x^2)$ so that

$$\int_C^\infty x\{K(x)\}^{-1}\, dx = \infty.$$

Also,

$$K(x)H(x) \le \int_x^\infty K(y)F(dy) \to 0,$$

so that $H(x) = o[\{K(x)\}^{-1}]$. The last two statements together imply that

$$\int_0^n xH(x)\, dx = o\left\{\int_C^n x(K(x))^{-1}\, dx\right\}. \tag{14}$$

The result now follows from (9) and (14) by applying (13) to $\{K(x)\}^{-1}$. □

If we now put further smoothness restrictions on H than we did in Corollary 1, we can obtain a more precise result. Specifically we assume X_1 falls in the domain of attraction of a stable law with exponent b (notation: $X_1 \in \mathscr{D}(b)$). For $b < 2$, this amounts to assuming that $H(v)$ is regularly varying with exponent $-b$. Note that this implies H satisfies the hypotheses of Corollary 1. For $b = 2$, X_1 is in the domain of attraction of a normal law. A sufficient condition for $X_1 \in \mathscr{D}(2)$ is of course that $EX_1^2 < \infty$. Background material for these notions may be found in Feller (1971, VIII.8).

THEOREM 2. *Suppose $X_1 \in \mathscr{D}(b)$ and $\mu = EX_1 < \infty$. Then,*

$$ER_n \sim 2n^{-1}\mu^{-2} \int_0^n vH(v)\,dv \quad \textit{if} \quad b = 2 \tag{15}$$

$$ER_n \sim \mu^{-b}\Gamma(2-b)\Gamma(1+b)nH(n) \quad \textit{if} \quad 1 \le b < 2. \tag{16}$$

PROOF. We first assume $1 \le b < 2$. Fix $a < \mu$, let $a_n = an$, and let $\varepsilon > 0$. By Lemma 1 with the estimate (12) and by the substitution $v = nu$,

$$ER_n \le \left[2n^{-1}a^{-2} \int_0^{n\varepsilon} vH(v)\,dv + 2anH(n) \int_\varepsilon^\infty H(nu)\{H(n)\}^{-1}u(a+u)^{-3}\,du\right](1 + o(1)). \tag{17}$$

By the regular variation of H and by dominated convergence, the second integral in (17) converges as $n \to \infty$ to

$$\int_\varepsilon^\infty u^{1-b}(a+u)^{-3}\,du \le 0.5a^{-b-1}\Gamma(2-b)\Gamma(1+b). \tag{18}$$

By Theorem 1 on page 281 of Feller (1971), the first integral at (17) is

$$\int_0^{n\varepsilon} vH(v)\,dv \sim \varepsilon^{2-b}(2-b)^{-1}n^2H(n). \tag{19}$$

Since ε is arbitrary, (18) and (19) together give

$$ER_n \le a^{-b}\Gamma(2-b)\Gamma(1+b)nH(n)\{1 + o(1)\}. \tag{20}$$

On the other hand, if $EX_1 < a < \infty$ and $\varepsilon > 0$ is sufficiently small, similar calculations show that

$$\begin{aligned} ER_n &\ge 2\mu n^2 \int_{n\varepsilon}^\infty v(v+na)^{-3}H(v)\,dv\{1 + o(1)\} \\ &\ge 2\mu nH(n) \int_\varepsilon^\infty u^{1-b}(a+u)^{-3}\,du\{1 + o(1)\} \ge a^{-b-2}\mu^2 nH(n)\Gamma(2-b)\Gamma(1+b) \end{aligned} \tag{21}$$

for large n. The result follows by letting a increase to μ in (20) and decrease to μ in (21).

For the case $b = 2$, well known facts on domains of attraction (see Feller (1971), page 577) provide that $x^2H(x)\{\int_0^x v^2F(dv)\}^{-1} \to 0$ as $x \to \infty$. We deduce that

$$nH(n) = o\left\{n^{-1} \int_0^n vH(v)\,dv\right\}. \tag{22}$$

As in (17) with $a < \mu$ and $\varepsilon = 1$, we therefore obtain

$$ER_n \lesssim \left[2a^{-2}n^{-1}\int_0^n vH(v)\,dv + 2an\int_1^\infty H(nu)u(a+u)^{-3}\,du\right](1+o(1))$$

$$= \left[2a^{-2}n^{-1}\int_0^n vH(v)\,dv + 2anO(H(n))\right](1+o(1))$$

$$= 2a^{-2}n^{-1}\int_0^n vH(v)\,dv(1+o(1)).$$

By (22), $\int_0^x vH(v)\,dv$ is slowly varying. For $a > EX_1$ and $\varepsilon > 0$, we see from (3) that

$$ER_n \geq 2n^2 a\int_0^{n\varepsilon}\frac{v}{(\varepsilon n + an)^3}H(v)\,dv(1+o(1))$$
$$\geq 2n^{-1}a(\varepsilon + a)^{-3}\int_0^n vH(v)\,dv(1+o(1)).$$

These bounds on ER_n lead to (15). □

COROLLARY 3. *$ER_n \approx n^{-1}$ if and only if $EX_1^2 < \infty$. In this case, $ER_n \sim n^{-1}EX_1^2(EX_1)^{-2}$.*

PROOF. The "if" part and the second sentence follow from Theorem 2. If $EX_1 < \infty$, the "only if" part follows from (10). We delay the "only if" part with $EX_1 = \infty$ until Theorem 5. □

We have given an integral free expression for the asymptotic behaviour of ER_n when the conditions of Corollary 1 hold. When $EX_1^2 < \infty$, we know $ER_n \approx n^{-1}$. Part of the gap between these two cases is filled by the following result.

THEOREM 3. *Suppose $H(x) = x^{-2}\tau(x)$ where τ satisfies the conditions specified in Theorem 6 below. Then $ER_n \approx n(\ln n)H(n)$.*

We do not prove this result, but only remark that the proof is similar to that of Theorem 6.

3. The case when $\mathbf{EX_1 = \infty}$. It follows from Theorem 1 that if $EX_1 < \infty$ then $ER_n \to 0$. We begin this section with a more complete result.

THEOREM 4. *$ER_n \to 0$ iff $\int_0^x H(y)\,dy$ is slowly varying (or equivalently iff $\int_0^x y\,dF(y)$ is slowly varying).*

PROOF. If $ER_n \to 0$, then $S_n^{-1}\{\max(X_1, X_2, \cdots, X_n)\} \to 0$ in probability. It was shown by Breiman (1965) that this implies the slow variation of the integral. (This result is compatible with those of Logan et al (1973) and Darling (1952).)

Now suppose $\int_0^x H(y)\,dy$ is slowly varying. It is shown by Feller (1971, pages 236-237) that there exists constants $a_n > 0$ such that, for $\varepsilon > 0$,

$$P(|a_n^{-1}S_n - 1| > \varepsilon) \leq n\varepsilon^{-2}a_n^{-2}\int_0^{a_n} x^2\,dF(x) + nH(a_n) \to 0. \tag{23}$$

It follows that

$$ER_n \le 4a_n^{-2}E(T_n, \text{ all } X_i \le a_n \text{ and } S_n \ge \tfrac{1}{2}a_n) \tag{24}$$
$$+ P(\text{some } X_i > a_n) + P(S_n < \tfrac{1}{2}a_n)$$
$$\le 4a_n^{-2}n\int_0^{a_n} x^2\,dF(x) + nH(a_n) + P(S_n < \tfrac{1}{2}a_n) \to 0. \quad \square$$

The next result shows that if $EX_1 = \infty$, ER_n cannot converge to zero at anything but a slow rate.

THEOREM 5. *Suppose $EX_1 = \infty$. There is a slowly varying sequence b_n such that* $\limsup_{n\to\infty} b_n ER_n = \infty$.

PROOF. By Theorem 4, we may assume $\mu(x) \equiv \int_0^x y\,dF(y)$ is slowly varying. By Feller (1971, page 236), we may choose constants a_n such that

$$P(S_n \ge 2a_n) \to 0 \tag{25}$$

and such that, as a function of n, a_n is an inverse of the regularly varying function $s(\mu(s))^{-1}$. The result of Seneta (1976, page 21) shows that a_n is itself regularly varying with exponent 1. Using the hypothesis $EX_1 = \infty$, an integral comparison test yields

$$\textstyle\sum_{n=1}^{\infty} (\ln n)^{-2}H(n(\ln n)^{-2}) = \infty$$

so that $H(n(\ln n)^{-2}) > n^{-1}$ for infinitely many n. Thus

$$P(T_n > n^2(\ln n)^{-4}) \ge P(\max(X_1, X_2, \cdots, X_n) > n(\ln n)^{-2})$$
$$= 1 - [F(n(\ln n)^{-2})]^n \not\to 0.$$

Combining this with (25), we see that there exists $C > 0$ such that

$$ER_n \ge E(R_n, S_n < 2a_n, T_n > n^2(\ln n)^{-4})$$
$$\ge (2a_n)^{-2}n^2(\ln n)^{-4}[P(T_n > n^2(\ln n)^{-4}) - P(S_n \ge 2a_n)]$$
$$\ge Cn^2a_n^{-2}(\ln n)^{-4}$$

for infinitely many n. The result follows with $b_n = a_n^2 n^{-2}(\ln n)^5$. $\square$

We have shown that, if the integral in Theorem 4 is slowly varying and if $EX_1 = \infty$, then ER_n converges slowly to zero. In the next theorem, we put some smoothness conditions on H and then obtain a precise rate.

THEOREM 6. *Suppose $H(x) = x^{-1}\tau(x) > 0$ for all $x > 0$ where τ satisfies*

$$m \le \frac{\tau(x^\lambda)}{\tau(x)} \le M \tag{26}$$

for all $\lambda \in [1, a]$ and $x \ge A$ where m, M, a and A are constants satisfying

$$0 < a^{-1} < m < 1 < M < \infty.$$

Then $ER_n \approx (\ln n)^{-1}$.

PROOF. First observe that $K(x) \equiv \tau(e^x)$ is $R - 0$ varying and the hypotheses of Theorem A.2 of Seneta (1976, page 94) are met. For $y \ge x \ge \ln A$ we therefore have

$$m\left[\frac{\ln x}{\ln y}\right]^{\beta}\tau(x) \le \tau(y) \le M\left[\frac{\ln y}{\ln x}\right]^{\alpha}\tau(x) \tag{27}$$

where $\alpha = (\ln M)(\ln a)^{-1}$ and $\beta = -(\ln m)(\ln a)^{-1} < 1$. It is obvious from (27) that

$$\tau(x) = O((\ln x)^{\alpha}) \tag{28}$$

and

$$(\ln x)^{-\beta} = O(\tau(x)). \tag{29}$$

By Seneta's theorem we also have

$$\int_A^x H(y)\,dy = \int_A^x y^{-1}\tau(y)\,dy = \int_{\ln A}^{\ln x} \tau(e^t)\,dt \approx \tau(x)\ln x. \tag{30}$$

For any $q > 0$ and for $0 < p < q(q+1)^{-1}$,

$$\begin{aligned}\int_0^x y^q H(y)\,dy &\le \int_0^{x^p} y^q\,dy + \int_{x^p}^x y^{q-1}\tau(y)\,dy\\ &\le (q+1)^{-1}x^{(q+1)p} + q^{-1}x^q \sup\{\tau(y): x^p \le y \le x\}\\ &\le q^{-1}x^q\tau(x)m^{-1}p^{-\beta}(1+o(1)),\end{aligned}$$

where we have used (29) and (27). A similar lower bound shows that in fact for any $q > 0$

$$\int_0^x y^q H(y)\,dy \approx x^q\tau(x). \tag{31}$$

Define $b_n = n\tau(n)\ln n$. Then

$$n(\ln n)^{\alpha+2} \ge b_n \ge C_1 n(\ln n)^{1-\beta} \tag{32}$$

for large n. Thus,

$$(\ln(\delta b_n))(\ln n)^{-1} \to 1 \tag{33}$$

as $n \to \infty$ for any $\delta > 0$. It follows from (27) that

$$\frac{\tau(\delta b_n)}{\tau(n)} \ge m\left[\frac{\ln n}{\ln(\delta b_n)}\right]^\beta \ge \frac{m}{2} \tag{34}$$

for sufficiently large n and that

$$\tau(\delta b_n) \approx \tau(n). \tag{35}$$

Now define $X_i' = X_i$ if $X_i \le b_n$, 0 otherwise. Let $S_n' = \sum_{i=1}^n X_i'$ and let

$$a_n = ES_n' = n\int_0^{b_n} x\,dF(x) = -nb_nH(b_n) + n\int_0^{b_n} H(x)\,dx. \tag{36}$$

The last integral $n\int_0^{b_n} H(x)\,dx \approx n\tau(b_n)\ln b_n \approx n\tau(n)\ln n = b_n$ by (30), (33), and (35) and similarly $nb_nH(b_n) \approx n\tau(n)$. Thus,

$$a_n \approx b_n. \tag{37}$$

The next step is to estimate the rate of growth of S_n by a truncation argument and Chebyshev's inequality (see also Feller (1971), pages 236–237). For any $\varepsilon > 0$,

$$\begin{aligned}P[|S_n - a_n| > \varepsilon a_n] &\le P[|S_n' - a_n| > \varepsilon a_n] + nH(b_n)\\ &\le \varepsilon^{-2}a_n^{-2}n\int_0^{b_n} x^2\,dF(x) + nH(b_n)\\ &\le 2\varepsilon^{-2}a_n^{-2}n\int_0^{b_n} xH(x)\,dx + nH(b_n)\\ &\le C_2(a_n^{-2}nb_n\tau(b_n)) + nb_n^{-1}\tau(b_n)\\ &\le C_3nb_n^{-1}\tau(n) = C_3(\ln n)^{-1},\end{aligned} \tag{38}$$

for n sufficiently large, where we have used (31), (37), and (35). A similar truncation at b_n combined with the argument at (24) yields the upper bound

$$ER_n = O(\ln n)^{-1}). \tag{39}$$

Next, choose $\delta \in \left(0, \frac{m}{4C_3}\right)$ where C_3 is given by (38) with $\varepsilon = 1$. By (35),

$$nH(\delta b_n) = \frac{n}{\delta n \tau(n) \ln n} \tau(\delta b_n) \to 0 \tag{40}$$

as $n \to \infty$. On the other hand, by (34),

$$nH(\delta b_n) \geq \frac{m}{2\delta \ln n} \tag{41}$$

for large n. By Bonferroni's inequality, (40) and (41),

$$\begin{aligned} P[\max(X_1, X_2, \cdots, X_n) > \delta b_n] &\geq nH(\delta b_n) - \binom{n}{2} (H(\delta b_n))^2 \\ &\geq \tfrac{1}{2}\, nH(\delta b_n) \geq \frac{m}{4\delta \ln n} \end{aligned} \tag{42}$$

for large n. By (38) with $\varepsilon = 1$, (42) and (37),

$$\begin{aligned} ER_n &\geq \frac{\delta^2 b_n^2}{4a_n^2} P[T_n > \delta^2 b_n^2, S_n \leq 2a_n] \geq C_4\{P[T_n > \delta^2 b_n^2] - P[S_n > 2a_n]\} \\ &\geq C_4\{P[\max(X_1, \cdots, X_n) > \delta b_n] - C_3(\ln n)^{-1}\} \\ &\geq C_4\left\{\frac{m}{4\delta \ln n} - \frac{C_3}{\ln n}\right\} = C_4\left(\frac{m}{4\delta} - C_3\right)(\ln n)^{-1} \end{aligned} \tag{43}$$

for sufficiently large n. The theorem now follows from (39) and (43). □

REMARK. It is obvious that if $EX_1 < \infty$ then $\liminf xH(x)\ln x = 0$. We see from (29) that this is not true here; thus the hypotheses of the theorem imply $EX_1 = \infty$. By (27) the hypotheses also imply that $\int_0^x H(y)\, dy$ is slowly varying.

We note that (38) gives us the following rate of convergence result for a generalized weak law of large numbers.

COROLLARY 4. *Let H satisfy the hypotheses of Theorem* 6 *and define a_n by* (36). *For* $\varepsilon > 0$,

$$P[|a_n^{-1}S_n - 1| > \varepsilon] = O((\ln n)^{-1}).$$

4. Further ramifications and examples. Theorem 2 implies $nER_n \to 1 + \alpha^{-1}$ when X_1 has a Gamma $(\alpha, 1)$ distribution. Simulations (when $\alpha = 0.5$) indicate that the convergence occurs very rapidly. We also simulated the value of R_n for variables having densities of the type $f(x) = c(1 + x)^{-b-1}$ for $x > 0$, $f(x) = 0$ otherwise. For example, when $b = 1.5$, the asymptotic behaviour should be given by (16), but graphs indicated a highly erratic behaviour. This results from the fact that $R_n/ER_n \to 0$ in probability in this case. In other words, the asymptotic behaviour of samples (and hence sample means) of R_n differs profoundly from that of ER_n. This can be shown whenever $H(x)$ is regularly varying with exponent $-b$, $1 < b < 2$, by first demonstrating that

$$nH(d_n^{1/2}) \to 0 \tag{44}$$

where $d_n = n^3H(n) \approx n^2ER_n \approx S_n^2ER_n$ and then verifying by truncating each X_i at $d_n^{1/2}$ that (44) implies $d_n^{-1}T_n \to 0$ in probability. Thus, under these conditions, the rate of convergence apparent in (16) represents for a single sequence (or average of a fixed number of sequences) an empirically unverifiable phenomenon.

Many of the foregoing results do not depend heavily on the independence of the random variables X_1 $X_2, \cdots$. Assume now that the random variables are identically distributed but possibly dependent. Take $a_n = an$ in Theorem 1. In order to draw the conclusion $ER_n = O\{n^{-1} \int_0^n vH(v)\, dv\}$ we may replace $o(\cdot)$ in the condition (1) of Lemma 1 by $O(\cdot)$. It is sufficient, then, that $P\{S_n < a_n\} = O(n^{-1})$. This may be seen to hold under a variety of dependence assumptions using only Chebyshev's inequality, for example when the sequence of random variables is pairwise independent, or when second moments are finite and the random variables are orthogonal. Similarly, the condition of Lemma 1(b) may be replaced in the case $a_n = an$, $a > EX_1$ by the condition lim sup $P\{S_n > an\} < 1$. This is also an easy consequence of Chebyshev's inequality in many cases.

Consider the class of examples obtained by letting $H(x) = x^{-b}(\ln x)^r$ for large x where $b \geq 0$ and $-\infty < r < \infty$ (with $r < 0$ if $b = 0$). If $b > 2$ or if $b = 2$ and $r < -1$, then $EX_1^2 < \infty$ and $ER_n \sim n^{-1}\mu^{-2}EX_1^2$ by Corollary 3. By Theorem 1, $ER_n \approx n^{-1}(\ln \ln n)$ if $b = 2$ and $r = -1$ and $ER_n \approx n^{-1}(\ln n)^{r+1}$ if $b = 2$ and $r > -1$. If $1 < b < 2$ or if $r < -1$ and $b = 1$, then $ER_n \approx n^{1-b}(\ln n)^r$. If $b = 1$ and $r = -1$, we obtain the answer from the proof of Theorem 6. Let $b_n = n \ln \ln n$ and calculate directly that $a_n \approx b_n$. Then deduce that $ER_n \approx \{(\ln n)(\ln \ln n)\}^{-1}$. If $b = 1$ and $r > -1$, Theorem 6 implies that $ER_n \approx (\ln n)^{-1}$. Finally, if $b < 1$, it can be shown that $ER_n \approx 1$ by the results of Logan et al (1973) and Darling (1952).

Acknowledgment. The authors are grateful to Professors H. Kesten, Y. Ogura and S. I. Resnick for pointing out references 7, 8 and 1, respectively.

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