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SOME RESULTS ON THE CLUSTER SET $C\left(\left\{\frac{S_n}{a_n}\right\}\right)$ AND THE LIL

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We investigate the cluster set $C(\{S_n/a_n\})$ under conditions necessary for the bounded law of the iterated logarithm, and obtain necessary and sufficient conditions for the LIL in spaces satisfying a certain comparison principle. In particular, these results settle some previously unanswered questions in the Hilbert space setting.

1. Introduction. Let B denote a real separable Banach space with topological dual B^* and norm $\|\cdot\|$. Throughout $X, X_1, X_2, \cdots$ are independent identically distributed B-valued random variables, and as usual $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. We write Lx to denote the function $\max(1, \log x)$ and we write L_2x to denote $L(Lx)$. The classical normalizing constants in the law of the iterated logarithm (LIL) are

$$a_n = \sqrt{2nL_2n}.$$

The set of all limit points of $\{x_n\}$ is denoted by $C(\{x_n\})$ and is called the cluster set of $\{x_n\}$.

The purpose of this paper is to prove some results regarding the cluster set $C(\{S_n/a_n\})$ and to examine the LIL in spaces satisfying a certain comparison principle. In particular, our results will settle some of the open problems mentioned in [8] and [12] when X takes values in a Hilbert space. Before stating our results precisely we provide some background and motivation for these matters.

In the infinite dimensional setting there are two forms of the LIL which are of interest. We say X satisfies the bounded LIL (and write $X \in$ BLIL) with respect to the classical normalizing constants if

$$\Lambda(X) \equiv \limsup_n \|S_n\|/a_n < \infty, \qquad \text{w.p.1.} \tag{1.1}$$

The second form of the LIL is the compact LIL. That is, we say X satisfies the compact LIL (and write $X \in$ CLIL) with respect to the classical normalizing constants if there exists a non-random compact set $D \subseteq B$ such that

$$d\left(\frac{S_n(\omega)}{a_n}, D\right) \to 0, \qquad \text{w.p.1,} \tag{1.2}$$

and

$$D = C\left(\left\{\frac{S_n(\omega)}{a_n}\right\}\right), \qquad \text{w.p.1.} \tag{1.3}$$

Here

$$d(x, A) = \inf_{y\in A}\|x - y\|$$

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and $C(\{S_n(\omega)/a_n\})$ denotes all limit points of the random sequence $(\{S_n(\omega)/a_n\})$. The set D in the CLIL is called the "limit set."

The fact that the limit set D in the compact LIL is non-random results from the observation that if $\{\alpha_n\}$ is any sequence of non-zero constants, then it is an easy consequence of the separability of B and the Hewitt-Savage zero-one law that with probability one we have

$$C\left(\left\{\frac{S_n}{\alpha_n}\right\}\right) = A$$

where A is a nonrandom set depending only on $\{\alpha_n\}$ and the law of X([13, Lemma 1].

Of course, A is necessarily closed, and if $\{\alpha_n\}$ is such that $\lim\sup_n \| S_n/\alpha_n \| = 0$, then

$$A = \{0\}.$$

If $\lim\sup_n \| S_n/\alpha_n \| > 0$, the nature of the cluster set is much less obvious, and in the case of the LIL with classical normalizing, this is precisely the situation we have provided $X \neq 0$.

Hence assume $X \in$ BLIL. Then for all $f \in B^*$ we have the real random variable $f(X) \in$ BLIL, so $Ef(X) = 0$ and $Ef^2(X) < \infty$ for all $f \in B^*$ are immediate necessary conditions for the BLIL (see, for example, [16, page 297]). For convenience we write $X \in WM_0^2$ if for all $f \in B^*$ we have $Ef(X) = 0$, $Ef^2(X) < \infty$. Further, if (1.1) holds, then there exists a constant Γ such that

$$P(\| X_n/a_n \| > \Gamma \text{ i.o.}) = 0.$$

Hence by the Borel-Cantelli lemma we have $\sum_n P(\| X \| > \Gamma a_n) < \infty$, and thus $E(\| X \|^2/L_2 \| X \|) < \infty$ is also a necessary condition for the LIL.

If $Ef(X) = 0$ for all $f \in B^*$ we define the covariance function of X to be

$$T(f, g) = E(f(X)g(X)), \qquad (f, g \in B^*).$$

It is immediate that the covariance function for X exists iff X is WM_0^2. Now the non-random set $A = C(\{S_n/a_n\})$ is closed since it is a cluster set, and if $X \in WM_0^2$, then it is known that there exists a canonical set K depending only on the covariance function of X such that

$$A \subseteq K \subseteq B$$

(see [8] or [11] for details). The set K is the unit ball of a Hilbert space determined by the covariance of X which we denote by $H_{\mathscr{L}(X)}$ and its properties are examined in [8] and [11]. It is known that K is compact iff T is weak-star sequentially continuous on $B^* \times B^*$.

The point to be emphasized, however, is that $X \in WM_0^2$ implies $A = C(\{S_n/a_n\}) \subseteq K$, and further, in case $X \in$ CLIL, that the limit set D is always K. Hence the cluster set is completely known in the CLIL, but if $X \in$ BLIL there are examples in [13] with $A = \phi$ and with $A = K$. Thus the following question is immediate. Is the cluster set A always ϕ or K? If the answer to this question is yes, it still remains to be decided when we have the cluster set empty and when it is K. A result in this direction is Theorem 3.1 of [8] which implies that the cluster set is K in many cases where the BLIL holds; however, this result has limited applicability. For example, if B is Hilbert space then necessary and sufficient conditions for both the BLIL and the CLIL were obtained in [8, Theorem 4.2], but in the BLIL situation the exact nature of the cluster set remained unknown. The results we present here imply that

$$C\left(\left\{\frac{S_n}{a_n}\right\}\right) = K$$

whenever the BLIL holds for Hilbert space random variables. More generally, the same holds for B a type 2 space, or in a general B if X satisfies some minimal conditions.

We also extend the CLIL and BLIL known for Hilbert space to a more general class of spaces containing certain smooth norm spaces. In fact, the results we present here actually sharpen the Hilbert space results, and give us a complete picture in that setting.

2. Statements of the main results. Our first theorem deals with the cluster set. We write X_τ to denote the truncation of X at level τ, i.e. $X_\tau = XI(\|X\| \le \tau)$, and $X \in \text{CLT}$ when X satisfies the central limit theorem in B, i.e.

$$\mathscr{L}\left(\frac{S_n}{\sqrt{n}}\right) \to_w \mathscr{L}(Z)$$

where Z is necessarily Gaussian. Throughout, K denotes the unit ball of the Hilbert space $H_{\mathscr{L}(X)}$ as in [8, Lemma 2.1].

THEOREM 1. *Let X be B-valued such that*

(2.1) $$X \text{ is } WM_0^2,$$

(2.2) $$X_\tau - E(X_\tau) \in \text{CLT} \quad \text{for all} \quad \tau > 0, \quad \text{and}$$

(2.3) $$\frac{S_n}{a_n} \to_{\text{prob}} 0.$$

Then

(2.4) $$C\left(\left\{\frac{S_n}{a_n}\right\}\right) = K, \qquad \text{w.p.1.}$$

M. Ledoux has recently informed us that it is possible to modify the proof of Theorem 1 so that (2.4) follows only from the assumptions (2.1) and (2.3) (see the appendix for details regarding Ledoux's improvement). We have retained our original proof because the techniques used have proved to be useful in other settings as well.

If B is a type 2 Banach space, Theorem 1 easily implies the following corollary.

COROLLARY 1. *Let B be of type* 2, *and assume X is B-valued such that* (2.1) *holds and*

(2.5) $$E(\|X\|^2/L_2\|X\|) < \infty.$$

Then

$$C\left(\left\{\frac{S_n}{a_n}\right\}\right) = K, \qquad \text{w.p.1.}$$

We also have

COROLLARY 2. *Let X be B-valued and such that*

(2.6) $$X \in \text{CLT}.$$

Then

$$C\left(\left\{\frac{S_n}{a_n}\right\}\right) = K, \qquad \text{w.p.1,}$$

and

(2.7) $$C\left(\left\{\frac{S_n}{\sqrt{n}}\right\}\right) = E, \qquad \text{w.p.1}$$

where E denotes the closure of $H_{\mathscr{L}(X)}$ *in B.*

REMARK. The set E in (2.7) is the support of the limiting Gaussian measure. This is well known.

The conditions (2.1) and (2.5) are known to be necessary and sufficient for the BLIL in Hilbert space, and our next result extends this equivalence to certain type 2 spaces. By Corollary 1 we see that in this setting the cluster set is K even for the BLIL. This contrasts with the example in [13] where X is c_0 valued and satisfies the BLIL, but has empty cluster set.

If X is B-valued, we say X is *pregaussian* if there exists a mean zero B-valued Gaussian random variable G with the covariance of G identical to that of X. We denote this by saying X is pregaussian with corresponding Gaussian random variable G.

For $\Lambda > 0$, $\delta > 0$, we let $\mathscr{G}_{\Lambda,\delta}$ denote the Borel functions $g: B \to [0, \infty)$ satisfying $g(x) = 0$ for $\|x\| \leq \Lambda$, $0 \leq g(x) \leq 1$ for $\Lambda \leq \|x\| \leq \Lambda + \delta$, and $1 \leq g(x) \leq c\|x\|^{\ell} + b$ for $\Lambda + \delta \leq \|x\|$ and constants ℓ, c, b possibly depending on g.

For Theorem 2 we assume B satisfies the following comparison principle. That is, we say a Banach space B satisfies the *upper Gaussian comparison principle* if for all $\delta > 0$, all $\Lambda > 0$, and each sequence of independent, mean zero, pregaussian, bounded random variables $\{Y_j : j \geq 1\}$ with corresponding independent Gaussian random variables $\{G_j : j \geq 1\}$, there exists an $\alpha > 0$ and $g \in \mathscr{G}_{\Lambda,\delta}$ such that for all $\beta > 0$

$$E(g(\beta \textstyle\sum_{j=1}^n Y_j)) \leq E(g(\beta \sum_{j=1}^n G_j)) + C(\delta, \Lambda, \alpha) \sum_{j=1}^n E\|Y_j\|^{2+\alpha} (\beta^{2+\alpha}) \tag{2.8}$$

where $C(\delta, \Lambda, \alpha)$ is a finite constant independent of n and β, but possibly on $\{Y_j : j \geq 1\}$.

THEOREM 2. *Let B be a type 2 Banach space which satisfies the upper Gaussian comparison principle. Then, $X \in$ BLIL iff*

$$X \text{ is } WM_0^2 \tag{2.9}$$

and

$$E(\|X\|^2/L_2\|X\|) < \infty. \tag{2.10}$$

Further, if (2.9) *and* (2.10) *hold we have*

$$\limsup_n \|S_n/a_n\| = \sup_{x \in K} \|x\|, \qquad \text{w.p.1}, \tag{2.11}$$

and

$$C(\{S_n/a_n\}) = K, \qquad \text{w.p.1}. \tag{2.12}$$

In regard to the CLIL, an immediate corollary is the following.

COROLLARY 3. *Let B be a type 2 Banach space which satisfies the upper Gaussian comparison principle. Then, $X \in$ CLIL iff* (2.9) *and* (2.10) *hold, and K is compact.*

In order to apply Theorem 2 and Corollary 3 we need Banach spaces satisfying the upper Gaussian comparison principle. To classify spaces with this property seems to be difficult, but examples are readily available. For example, all type 2 Banach spaces which satisfy Condition (A) of [8] also satisfy the upper Gaussian comparison principle.

We recall that a Banach space satisfies condition (A) if: the norm on B is twice directionally differentiable and the second derivative of the norm, D_x^2, is such that

(a) $\sup_{\|x\|=1}\|D_x^2\| < \infty$, and
(b) D_x^2 is Lip(α) away from zero for some $\alpha > 0$.

For the definition of D_x^2 and the other terms used in (A) we refer the reader to [9] or to [10]. The reader should note, however, that the relevant definitions in [9] and [10] differ slightly in that in [10] the first derivative of the norm is assumed to be Lip(1) on the surface of the unit ball of B. Hence we say a Banach space B satisfies condition (A') if B

satisfies condition (A) and the first directional derivative of the norm is Lip(1) on the surface of the unit ball of B.

We now have

PROPOSITION 1. *If B is of type 2 and satisfies Condition* (A), *then B satisfies the upper Gaussian comparison principle. Further, if B satisfies Condition* (A′), *then B is also of type 2.*

REMARK. It is shown in [9, pages 83–86] that the L^p spaces $(2 \le p < \infty)$ satisfy Condition (A), but the same arguments also imply that they satisfy Condition (A′).

A close inspection of the proofs of Theorem 2 and Corollary 3 reveal that the space B involved need not satisfy the upper Gaussian comparison principle. What is actually required to make our proof work is that a comparison of the type in (4.13) is possible for Γ, all $\varepsilon > 0$, and some $\alpha > 0$. In some situations the particular random variable X, its related truncations $\{u_j : j \ge 1\}$ as defined in (4.1), and the corresponding Gaussian sequence $\{G_j : j \ge 1\}$ may satisfy (4.13) yet the space B does not have the upper Gaussian comparison principle. Nevertheless, in such cases X satisfies the BLIL if (2.9) and (2.10) hold, or the CLIL if (2.9), (2.10) hold and K is compact.

The condition K compact depends only on the covariance structure of X, and is equivalent to the covariance function $T(f, g) = E(f(X)g(X))$, $f, g \in B^*$, being weak-star sequentially continuous on $B^* \times B^*$. For example, K is compact whenever the covariance function of X is the same as that of a random variable Y satisfying $E\|Y\|^2 < \infty$. This is the situation if X is pregaussian, and hence Corollary 3 can be viewed as a variation of the theme of Theorem 4.1 of [8] which relates the CLT and the CLIL.

For Hilbert space valued random variables we have a complete blending of the BLIL and the CLIL into what simply might be called the LIL. The result is

COROLLARY 4. *Let X be Hilbert space valued. If* (2.9) *and* (2.10) *hold, then*

$$C\left(\left\{\frac{S_n}{a_n}\right\}\right) = K, \quad \text{w.p.1}, \tag{2.13}$$

and

$$\lim_n d\left(\frac{S_n}{a_n}, K\right) = 0, \quad \text{w.p.1} \tag{2.14}$$

where, of course $K = K_{\mathscr{L}(X)}$. Conversely, if (2.14) *holds for any bounded set K, then* (2.9) *and* (2.10) *hold.*

It might be worthwhile to mention that the methods of this paper are capable of producing a new proof of the Hartman-Wintner LIL by what is in essence a variant of Lindeberg's method. That is, by the methods of Theorem 2, one could first prove that if $EX = 0$ and $EX^2 = \sigma^2 < \infty$, then we have

$$\limsup_n \frac{S_n}{a_n} \le \sigma.$$

With this fact in hand, some elementary modifications of the proof of Theorem 1 would then yield

$$C\left(\left\{\frac{S_n}{a_n}\right\}\right) = [-\sigma, \sigma].$$

However, one could also considerably streamline the proof of Theorem 1 in this case, since $EX^2 < \infty$ makes the use of the Lévy decomposition unnecessary, and allows a direct application of Lemma 3.2 (see [2] for the details in this regard). The reader should also note that our proof depends on truncation levels not previously use, but which are, in some sense, the natural ones.

Finally, we mention that the complete blending of the LIL for Hilbert space random variables is obtainable because we have a direct vector proof of (2.14), without passing through a finite dimensional argument as in the proof of Corollary 3.

3. The proof of Theorem 1, Corollary 1, and Corollary 2.

We recall that if μ denotes a Borel probability measure on B such that μ is WM_0^2, then the map $S: B^* \to B$ defined by

$$Sf = \int_B xf(x)\, d\mu(x), \qquad (f \in B^*) \tag{3.1}$$

has many useful properties which have been examined in [8, Lemma 2.1]. Of course, S depends on μ, and when there are several measures involved in our arguments we write S_μ to denote the mapping attached to μ. In particular, we will have need of the Hilbert space $H_\mu \subseteq B$ with norm $\|\cdot\|_\mu$ as defined in [8, Lemma 2.1].

The following lemmas are useful for the proof of Theorem 1. The first one clarifies and slightly strengthens Lemma 2.2 in [7]. Lemma 3.2 follows easily from Lemma 3.1 and the Cameron-Martin translation formula for Gaussian measures as presented, for example, in [1].

LEMMA 3.1. *Let $X \in \mathrm{CLT}(\gamma)$ in B. Let U be a convex open set in B, and assume $0 < a_n$, $a_n^2/n \to \infty$, $a_n/n \to 0$. Then, for every $t > 0$*

$$\liminf_n n a_n^{-2} \log P(S_n/a_n \in U) \geq t^{-2} \log \gamma(tU). \tag{3.2}$$

PROOF. For each $\varepsilon > 0$ let

$$U_\varepsilon = \{y : d(y, U^c) > \varepsilon\}.$$

Then U_ε is open and convex. Now let $p_n = [n^2t^2/a_n^2]$, $q_n = [n/p_n]$, and $r_n = a_n/tq_n$. Then

$$\begin{aligned}(P(S_{p_n}/r_n \in tU_\varepsilon))^{q_n} &\leq P(S_{p_nq_n}/r_n \in tq_nU_\varepsilon)\\ &\leq P(S_{p_nq_n}/a_n \in U_\varepsilon).\end{aligned} \tag{3.3}$$

Since $S_n = S_{p_nq_n} + (S_n - S_{p_nq_n})$ we have

$$P(S_n/a_n \in U) \geq P(S_{p_nq_n}/a_n \in U_\varepsilon, \|S_{p_nq_n} - S_n\| \leq \varepsilon a_n), \tag{3.4}$$

and since $p_nq_n \sim n$, $X \in \mathrm{CLT}$, and $a_n/\sqrt{n} \to \infty$, we have

$$\lim_n P(\|S_{p_nq_n} - S_n\| \leq \varepsilon a_n) = 1. \tag{3.5}$$

Now $r_n \sim \sqrt{p_n}$, $q_n \sim a_n^2/nt^2$ and hence by the independence of $S_{p_nq_n}$ and $S_n - S_{p_nq_n}$, (3.5), (3.4), and then (3.3) we have

$$\begin{aligned}\liminf_n n a_n^{-2} \log P(S_n/a_n \in U) &\geq \liminf_n n a_n^{-2} \log P(S_{p_nq_n}/a_n \in U_\varepsilon)\\ &\geq \liminf_n n a_n^{-2} q_n \log P(S_{p_n}/r_n \in tU_\varepsilon)\\ &\geq t^{-2} \log \gamma(tU_\varepsilon).\end{aligned} \tag{3.6}$$

Letting $\varepsilon \downarrow 0$ we have $U_\varepsilon \nearrow U$, so (3.2) follows and the lemma is proved.

LEMMA 3.2. *Let $X \in \mathrm{CLT}(\gamma)$ in B, $b \in H_\gamma$, and $\{a_n\}$ be as in Lemma 3.1. Then, for* *every* $\varepsilon > 0$

$$\liminf_n n a_n^{-2} \log P(\| S_n/a_n - b \| < \varepsilon) \geq - \| b \|_\gamma^2/2. \tag{3.7}$$

PROOF. Let $U = \{x \in B: \| x - b \| < \varepsilon\}$. Then, by the Cameron-Martin formula (see, for example, [1]) we have

$$\gamma(tU) = \exp\left\{-\frac{1}{2} t^2 \| b \|_\gamma^2\right\} \int_{\varepsilon t V} \exp\{-t\tilde{b}\}\, d\gamma \tag{3.8}$$

where $V = \{x \in B: \| x \| < 1\}$ and $\tilde{b}$ is Gaussian with mean zero and variance $\| b \|_\gamma^2$. By Jensen's inequality we thus have

$$\gamma(tU) \geq \exp\{-\tfrac{1}{2} t^2 \| b \|_\gamma^2\}\, \gamma(\varepsilon t V), \tag{3.9}$$

and hence

$$t^{-2} \log \gamma(tU) \geq -\tfrac{1}{2} \| b \|_\gamma^2 + t^{-2} \log \gamma(\varepsilon t V). \tag{3.10}$$

As $t \to \infty$, $\gamma(\varepsilon t V) \to 1$, and hence (3.7) holds by applying (3.2).

PROOF OF THEOREM 1. Since $X \in WM_0^2$, the unit ball $K = K_{\mathscr{L}(X)}$ is defined and from Lemma 2.1 (v) [8] we have K closed in B. From the same lemma we also know that if $\mu = \mathscr{L}(X)$, then

$$\sigma(\mu) = \sup_{\| f \|_{B^*} \leq 1} \left(\int_B f^2(x)\, d\mu(x) \right)^{1/2} < \infty \tag{3.11}$$

and for $x \in H_\mu$ we have

$$\| x \| \leq \sigma(\mu)\, \| x \|_\mu. \tag{3.12}$$

Now $X \in WM_0^2$, and the argument used to establish (3.2) in [11], implies

$$C\left(\left\{\frac{S_n}{a_n}\right\}\right) \subseteq K, \qquad \text{w.p.1.} \tag{3.13}$$

Thus, since K is closed in B, we will have (2.4) if we show that a dense subset of K is in $C(\{S_n/a_n\})$ w.p.1. In view of (3.12) this will be accomplished by showing that for all $b \in S(B^*)$, $b = Sf$, $\| b \|_\mu < 1$ we have

$$\liminf_n \| S_n/a_n - b \| = 0 \qquad \text{w.p.1.} \tag{3.14}$$

To establish (3.14) we write X in terms of its Lévy decomposition at truncation level τ as developed in [3]; τ to be specified as sufficiently large later. That is, if ξ, η, U, V are independent random variables such that ξ, η are Bernoulli with $E(\xi) = E(\eta) = P(\| X \| \leq \tau)$, and

$$\mathscr{L}(U)(A) = P(\| X \| \leq \tau, X \in A)/P(\| X \| \leq \tau), \tag{3.15}$$

$$\mathscr{L}(V)(A) = P(\| X \| > \tau, X \in A)/P(\| X \| > \tau), \tag{3.16}$$

then

$$\mathscr{L}(X) = \mathscr{L}(\eta U + (1 - \xi) V + (\xi - \eta) U).$$

Further, we have $\mathscr{L}(\eta U) = \mathscr{L}(X_\tau)$, $\mathscr{L}((1 - \xi) V) = \mathscr{L}(X - X_\tau)$.

Now let $\{\eta_j, \xi_j, U_j, V_j\}$ be an independent system with ξ_j, η_j Bernoulli satisfying

$$E(\xi_j) = E(\eta_j) = P(\| X \| \leq \tau)$$

and $\{U_j\}$ and $\{V_j\}$ be identically distributed according to $\mathscr{L}(U)$ and $\mathscr{L}(V)$ as in (3.15) and (3.16), respectively. Then, setting $Y_j = \eta_j U_j$, $W_j = (1 - \xi_j) V_j$ and $Z_j = (\xi_j - \eta_j) U_j$ for $j \geq 1$ we have that $\{X_j : j \geq 1\}$ and $\{Y_j + W_j + Z_j : j \geq 1\}$ are both i.i.d. sequences with the same law. Hence (3.14) will hold if we prove

$$\liminf_n \left\| \sum_{j=1}^n (Y_j + W_j + Z_j)/a_n - b \right\| = 0 \qquad \text{w.p.1.} \tag{3.17}$$

Now fix $\varepsilon > 0$. The first step in the proof of (3.17) is to observe that $(\xi - \eta)U \in$ CLIL. To see this note that the sequence $\{(\xi_j - \eta_j)U_j : j \geq 1\}$ consists of independent, symmetric, bounded (by τ), identically distributed random variables. Further, we have

$$\sum_{j=1}^n (\xi_j - \eta_j)U_j/\sqrt{n} = \sum_{j=1}^n (\xi_j U_j - EX_\tau)/\sqrt{n} - \sum_{j=1}^n (\eta_j U_j - EX_\tau)/\sqrt{n}, \tag{3.18}$$

and, since $\mathscr{L}(\xi_j U_j) = \mathscr{L}(\eta_j U_j) = \mathscr{L}(X_\tau)$ with $X_\tau - EX_\tau \in$ CLT, we easily have

$$\{\mathscr{L}(\textstyle\sum_{j=1}^n (\xi_j - \eta_j)U_j/\sqrt{n}) : n \geq 1\}$$

tight in B. Since the finite dimensional distributions converge we see $(\xi - \eta)U \in$ CLT, and hence $(\xi - \eta)U \in$ CLIL by [8, Theorem 4.1].

Thus for $\mathscr{L}(Z) = \mathscr{L}(Z_1)$ we have

$$\limsup_n \left\| \sum_{j=1}^n Z_j/a_n \right\| = \sup_{x \in K_{\mathscr{L}(Z)}} \|x\| \qquad \text{w.p.1} \tag{3.19}$$

and since

$$\begin{aligned} \sup_{x \in K_{\mathscr{L}(Z)}} \|x\| &= \sup_{Ef^2(Z) \leq 1} \|S_{\mathscr{L}(Z)} f\| = \sup_{\|g\|_{B^*} \leq 1, Ef^2(Z) \leq 1} g(S_{\mathscr{L}(Z)} f) \\ &= \sup_{\|g\|_{B^*} \leq 1, Ef^2(z) \leq 1} E(f(Z)g(Z)) = \sup_{\|g\|_{B^*} \leq 1} (E(g^2(Z)))^{1/2} \\ &\leq (2P(\|X\| > \tau))^{1/2} \sigma(\mu) \end{aligned} \tag{3.20}$$

where $\sigma(\mu) < \infty$ by (3.11), we can make

$$\limsup_n \left\| \sum_{j=1}^n Z_j/a_n \right\| \leq \varepsilon \qquad \text{w.p.1.} \tag{3.21}$$

by taking $\tau \geq \tau_1(\varepsilon)$, say. The last inequality in (3.20) follows easily from the definition of Z_j.

The next step is to show

$$\sum_{j=1}^n (W_j + EX_\tau)/a_n \to_{\text{prob}} 0. \tag{3.22}$$

Now (3.22) is obvious since $S_n/a_n \to_{\text{prob}} 0$ by (2.3), $Z \in$ CLT by the previous argument, and $\mathscr{L}(\eta U - EX_\tau) = \mathscr{L}(X_\tau - EX_\tau)$ also satisfies the CLT by (2.2).

Now we approximate $b = Sf$, $f \in B^*$, where

$$\|b\|_\mu = \left(\int_B f^2(x)\, d\mu(x) \right)^{1/2} < 1.$$

That is, for $\tau > 0$ let $\mu_\tau = \mathscr{L}(X_\tau - EX_\tau)$. Then, there exists $\tau_2(\varepsilon)$ such that $\tau \geq \tau_2(\varepsilon)$ implies

$$\|S_{\mu_\tau} f - Sf\| \leq \varepsilon/4 \tag{3.23}$$

and

$$\|S_\mu f_\tau\|_{\mu_\tau} \leq \|Sf\|_\mu < 1. \tag{3.24}$$

Now (3.23) follows since

$$\begin{aligned} &\|S_{\mu_\tau} f - Sf\|_B \\ &\quad = \sup_{\|g\|_{B^*} \leq 1} |E(g(X_\tau - EX_\tau)f(X_\tau - EX_\tau)) - E(g(X)f(X))| \\ &\quad = \sup_{\|g\|_{B^*} \leq 1} |E[g(X_\tau)f(X_\tau) - g(X)f(X)] - g(EX_\tau)f(EX_\tau)| \\ &\quad = \sup_{\|g\|_{B^*} \leq 1} |E(g(X)f(X)I(\|X\| > \tau)) - E(g(X)I(\|X\| \leq \tau))E(f(X)I(\|X\| \leq \tau))| \\ &\quad \leq \sup_{\|g\|_{B^*} \leq 1} (E(g^2(X)))^{1/2} [\{E(f^2(X)I(\|X\| > \tau))\}^{1/2} + |E(f(X)I(\|X\| \leq \tau))|] \\ &\quad \to 0 \quad \text{as} \quad \tau \to \infty \end{aligned}$$

by applying (3.11) and (2.1). The proof of (3.24) follows similarly.

Now let $\tau \geq \max(\tau_1(\varepsilon), \tau_2(\varepsilon))$ and set

$$T_n = \sum_{j=1}^n (Y_j + W_j) = \sum_{j=1}^n (Y_j - EX_\tau) + \sum_{j=1}^n (W_j + EX_\tau). \tag{3.25}$$

Then, for all $n \geq n_0(\varepsilon)$, we have from (3.22) and the independence of the Y_j's and W_j's that

$$\begin{aligned} P(\| T_n/a_n - b \| \leq \varepsilon) &\geq P(\| \textstyle\sum_{j=1}^n (Y_j - EX_\tau)/a_n \\ &\quad - S_{\mu_\tau} f \| < \varepsilon/2, \| \textstyle\sum_{j=1}^n (W_j + EX_\tau)/a_n \| < \varepsilon/4) \\ &\geq \frac{1}{2} P(\| \textstyle\sum_{j=1}^n (Y_j - EX_\tau)/a_n - S_{\mu_\tau} f \| < \varepsilon/2). \end{aligned} \tag{3.26}$$

Since $\mathscr{L}(Y_j) = \mathscr{L}(X_\tau)$ and $X_\tau - E(X_\tau) \in \mathrm{CLT}(\gamma)$ by (2.2), where the covariance of γ is that of $X_\tau - EX_\tau$, we have from (3.7) that

$$\liminf_n n a_n^{-2} \log P(\| \textstyle\sum_{j=1}^n (Y_j - EX_\tau)/a_n - S_{\mu_\tau} f \| < \varepsilon/2) \geq - \| S_{\mu_\tau} f \|_{\mu_\tau}^2 /2 \tag{3.27}$$

for any sequence $\{a_n\}$ satisfying the condition of Lemma 3.1. In particular, when $a_n = \sqrt{2n L_2 n}$, then (3.27) implies that for $\delta > 0$, and all $n \geq n_0(\delta)$, we have

$$P(\| \textstyle\sum_{j=1}^n (Y_j - EX_\tau)/a_n - S_{\mu_\tau} f \| < \varepsilon/2) \geq \exp\{-(1 + \delta) \| S_{\mu_\tau} f \|_{\mu_\tau}^2 L_2 n\}. \tag{3.28}$$

Since (3.24) holds we can choose $\delta > 0$ such that

$$(1 + \delta) \| S_{\mu_\tau} f \|_{\mu_\tau} \leq 1 - \delta,$$

and hence for all $n \geq \max(n_0(\delta), n_0(\varepsilon))$ we have

$$P(\| T_n/a_n - b \| \leq \varepsilon) \geq \tfrac{1}{2}(Ln)^{-(1-\delta)}.$$

Thus

$$\textstyle\sum_{n \geq 1} P(\| T_n/a_n - b \| \leq \varepsilon)/n = \infty \tag{3.29}$$

for all $\varepsilon > 0$, and hence by the proofs of Lemma 5 and Lemma 4 of [13, pages 388–390] we have

$$\liminf_n \| T_n/a_n - b \| \leq 6\varepsilon \qquad \text{w.p.1.} \tag{3.30}$$

Now

$$\liminf_n \| S_n/a_n - b \| \leq \liminf_n \| T_n/a_n - b \| + \limsup_n \| \textstyle\sum_{j=1}^n Z_j/a_n \| \text{ w.p.1,} \tag{3.31}$$

so taking $\tau \geq \max(\tau_1(\varepsilon), \tau_2(\varepsilon))$ we have from (3.30) and (3.21) that

$$\liminf_n \| S_n/a_n - b \| \leq 7\varepsilon \qquad \text{w.p.1.} \tag{3.32}$$

Since $\varepsilon > 0$ was arbitrary (3.14) holds completing the proof of Theorem 1.

Proof of Corollary 1. If B is of type 2, then (2.2) holds and (2.3) follows from the conditions (2.1) and (2.5) by Proposition 7.2 of [8, page 747]. Hence Theorem 1 applies to yield $C(\{S_n/a_n\})$ as claimed.

A lemma which assists in the proof of Corollary 2 and is of some independent interest is presented next. A related result appears as Lemma 2 in [15].

Lemma 3.3. *Let $X_1, X_2, \cdots$ be independent mean-zero B-valued random variables. Let $\phi : [0, \infty) \to [0, \infty)$ be a non-decreasing convex function such that $\phi(0) = 0$ and such that $\phi(2t) \leq M \phi(t)$ for some $M > 0$, $t > 0$. Let q denote a continuous semi-norm on B, and* *set*

$$Y_j = X_j I(\|X_j\| \le \tau_j) \qquad (j \ge 1)$$

where $\{\tau_j : j \ge 1\}$ *is arbitrary. Then, for* $n \ge 1$,

$$E[\phi(q(\textstyle\sum_{j=1}^{n} (Y_j - E(Y_j))))] \le 4ME[\phi(q(S_n))]. \tag{3.33}$$

PROOF. Let $\{Y_j' : j \ge 1\}$ be an independent copy of $\{Y_j : j \ge 1\}$ and let $\{\varepsilon_j : j \ge 1\}$ be a sequence of independent random variables independent of both $\{Y_j\}$ and $\{Y_j'\}$ and such that $P(\varepsilon_j = \pm 1) = ½$. We write E_X to denote the expectation with respect to X. Then

$$\begin{aligned}
E[\phi(q(\textstyle\sum_{j=1}^{n} (Y_j - E(Y_j))))] &= E_Y[\phi(q(\textstyle\sum_{j=1}^{n} (Y_j - E(Y_j))))] \\
&\le E_Y E_{Y'}[\phi(q(\textstyle\sum_{j=1}^{n} (Y_j - Y_j')))] \\
&= E_Y E_{Y'} E_\varepsilon[\phi(q(\textstyle\sum_{j=1}^{n} \varepsilon_j (Y_j - Y_j')))] \\
&\le ME_Y E_\varepsilon[\phi(q(\textstyle\sum_{j=1}^{n} \varepsilon_j Y_j))] \text{ by the triangle inequality, the} \\
&\quad \text{convexity of } \phi, \text{ and the condition } \phi(2t) \le M\phi(t) \\
&\le 2ME[\phi(q(\textstyle\sum_{j=1}^{n} \varepsilon_j X_j))] \text{ since } P(q(\textstyle\sum_{j=1}^{n} \varepsilon_j Y_j) > t) \le \\
&\quad 2P(q(\textstyle\sum_{j=1}^{n} \varepsilon_j X_j) > t) \\
&\le 4ME(\phi(q(S_n)))
\end{aligned}$$

where the last inequality follows as in standard comparison principles (see, for example, [5, page 108]).

PROOF OF COROLLARY 2. If $X \in$ CLT, then (2.1) and (2.3) are immediate. Further, an application of the previous lemma to the semi-norm involving the distance to a suitably chosen finite dimensional subspace, along with elementary arguments, immediately implies $X_\tau - EX_\tau$ satisfies the CLT for all $\tau > 0$. Hence, again Theorem 1 applies to yield $C(\{S_n/a_n\}) = K$, and (2.7) follows from (2.12) of [13, page 380] completing the proof of Corollary 2.

4. The proof of Theorem 2 and Corollary 3. In order to prove Theorem 2 we use the following notation in which $\beta > 1$ is a parameter to be chosen in the proof.

Let $\{X_j : j \ge 1\}$ be independent copies of the mean zero random variable X of Theorem 2. Let $\beta > 1$, $n_0 = 1$, and put $n_k = [\beta^k]$ for $k \ge 1$, where $[\cdot]$ denotes the greatest integer function. Let $I(k) = \{n_k + 1, \cdots, n_{k+1}\}$ for $k \ge 0$, and set

$$\tau_k = 2n_{k+1} L_2 n_{k+1} = a^2_{n_{k+1}}.$$

Then, for $j \in I(k)$ let

$$\begin{aligned}
u_j &= X_j I(\| X_j \|^2 \le \tau_k) - E(X_j I(\| X_j \|^2 \le \tau_k)) \\
w_j &= X_j I(\tau_k < \| X_j \|^2) - E(X_j I(\tau_k < \| X_j \|^2)),
\end{aligned} \tag{4.1}$$

and set

$$\begin{aligned}
U_n &= \textstyle\sum_{j=1}^{n} u_j \\
W_n &= \textstyle\sum_{j=1}^{n} w_j.
\end{aligned} \tag{4.2}$$

PROOF OF THEOREM 2. If $X \in$ BLIL, then it is well known that (2.9) and (2.10) hold. On the other hand, if (2.9) and (2.10) hold, then B of type 2 allows us to apply Corollary 1 to obtain (2.12). Hence all that remains is to establish (2.11) under the conditions (2.9) and (2.10).

Let $\Gamma = \sup_{x\in K} \| x \|$ and fix $\varepsilon > 0$. Since we know (2.12) holds, then (2.11) holds if we prove

$$\limsup_n \| S_n/a_n \| \le \Gamma + \varepsilon \qquad \text{w.p.1.} \tag{4.3}$$

Now (4.3) will be established by showing that for all $\beta > 1$ we have

$$\lim_n \| W_n/a_n \| = 0 \qquad \text{w.p.1,} \tag{4.4}$$

and for $\beta > 1$, sufficiently close to 1, we have

$$\limsup_n \| U_n/a_n \| \le \Gamma + \varepsilon \qquad \text{w.p.1.} \tag{4.5}$$

LEMMA 4.1. *If* $E(\|X\|^2/L_2\|X\|) < \infty$, *then* (4.4) *holds for all* $\beta > 1$.

PROOF. Since $E(\|X\|^2/L_2\|X\|) < \infty$ it follows from the Borel-Cantelli lemma that w_j as defined in (4.1) is with probability one eventually given by

$$w_j = -E(X_j I(\tau_k < \| X_j \|^2)).$$

Hence (4.4) will follow if we show that

$$\lim_r \textstyle\sum_{k=1}^r \sum_{j\in I(k)} E(\| X_j \| I(\tau_k < \| X_j \|^2))/\sqrt{n_r L_2 n_r} = 0.$$

Now

$$\begin{aligned}
&\textstyle\sum_{k=1}^r \sum_{j\in I(k)} E(\| X_j \| I(\tau_k < \| X_j \|^2))/\sqrt{n_r L_2 n_r} \\
&\qquad \le \textstyle\sum_{k=1}^r (n_{k+1} - n_k) E(\| X \| I(\tau_k < \| X \|^2))/\sqrt{n_r L_2 n_r} \\
&\qquad \le C \textstyle\sum_{k=1}^r \dfrac{(n_{k+1} - n_k)}{\sqrt{n_r L_2 n_r}} E\left(\dfrac{\| X \|^2}{L_2 \| X \|} I(\tau_k < \| X \|^2)\right) \dfrac{L_2 \tau_k}{\tau_k^{1/2}}
\end{aligned}$$

for some constant $C < \infty$. Further,

$$\lim_k E\left(\frac{\| X \|^2}{L_2 \| X \|} I(\tau_k < \| X \|^2)\right) = 0,$$

and since

$$\textstyle\sum_{k=1}^r (n_{k+1} - n_k) L_2 \tau_k/\tau_k^{1/2} = O(\sqrt{L_2 n_r} \sum_{k=1}^r \beta^{k/2})$$

we have our result and (4.4) holds.

LEMMA 4.2. *Let* B *be a type* 2 *Banach space which satisfies the upper Gaussian comparison principle. If* X *satisfies* (2.9) *and* (2.10), *then for every* $\varepsilon > 0$ *there exists* $\beta > 1$ *such that* (4.5) *holds.*

PROOF. The proof of (4.5) is obtained by showing that there exists $\beta > 1$, sufficiently close to one, such that

$$\limsup_r \max_{n\in I(r)} \| U_n/a_n \| \le \Gamma + \varepsilon \qquad \text{w.p.1.} \tag{4.6}$$

Indeed since $a_n \nearrow$ we have

$$\max_{n\in I(r)} \| U_n/a_n \| \le \frac{a_{n_{r+1}}}{a_{n_r}} \max_{n\in I(r)} \| U_n/a_{n_{r+1}} \| \tag{4.7}$$

and since $a_{n_{r+1}}/a_{n_r} \sim \sqrt{\beta}$ (4.7) implies

$$\limsup_r \max_{n\in I(r)} \| U_n/a_n \| \le \sqrt{\beta} \limsup_r \max_{n\in I(r)} \| U_n/a_{n_{r+1}} \|. \tag{4.8}$$

Now

$$\sup_{n\in I(r)} P(\| U_{n_{r+1}} - U_n \| > \varepsilon a_{n_{r+1}}/4) \le \left(\frac{4}{\varepsilon}\right) \sup_{n\in I(r)} E \| U_{n_{r+1}} - U_n \|/a_{n_{r+1}}$$

$$\le \left(\frac{4}{\varepsilon}\right) E \| U_{n_{r+1}} - U_{n_r} \|/a_{n_{r+1}}$$

$$\le \left(\frac{4}{\varepsilon}\right) \frac{[A(n_{r+1}-n_r)]^{1/2}}{a_{n_{r+1}}} (E \| XI(\| X \|^2 \le \tau_r) - E(XI(\| X \|^2 \le \tau_r))\|^2)^{1/2}$$

(4.9) where A is the constant in the type 2 inequality

$$\le \left(\frac{16A^{1/2}}{\varepsilon}\right)\left\{E\left(\frac{\| X \|^2}{L_2 \| X \|} I(\| X \|^2 \le \tau_r) \frac{L_2 \| X \|}{L_2 \tau_r}\right)\right\}^{1/2}$$

for large r

$\to 0$ as $r \to \infty$.

Hence for sufficiently large r a standard argument shows

(4.10) $$P(\max_{n\in I(r)} \| U_n \|/a_{n_{r+1}} > \Gamma + \varepsilon/2) \le 2P(\| U_{n_{r+1}} \| > (\Gamma + \varepsilon/4)a_{n_{r+1}}).$$

The next step of our proof is to show

(4.11) $$\textstyle\sum_r P(\| U_{n_{r+1}} \| > (\Gamma + \varepsilon/4)a_{n_{r+1}}) < \infty.$$

Once (4.11) is established, (4.10), the Borel-Cantelli lemma, and (4.8) imply

(4.12) $$\limsup_r \max_{n\in I(r)} \| U_n \|/a_n \le \sqrt{\beta}\,(\Gamma + \varepsilon/2) \quad \text{w.p.l.}$$

Taking $\beta > 1$, sufficiently close to one, (4.12) implies (4.6) which completes the proof of (4.5). Hence we must establish (4.11).

Since $\varepsilon > 0$, $\Gamma > 0$ are fixed, and B satisfies the upper Gaussian comparison principle, we choose $g \in \mathscr{G}_{\Gamma+\varepsilon/8,\varepsilon/8}$ such that (2.8) holds for the sequence $\{u_j : j \ge 1\}$ with corresponding Gaussian random variables $\{G_j : j \ge 1\}$. Of course, the $\{u_j : j \ge 1\}$ are pregaussian since they are bounded and B is a type 2 space. Hence

(4.13) $$P(\| U_{n_r} \|/a_{n_r} > \Gamma + \varepsilon/4) \le E(g(\textstyle\sum_{j=1}^{n_r} u_j/a_{n_r}))$$

$$\le E(g(\textstyle\sum_{j=1}^{n_r} G_j/a_{n_r})) + C\left(\varepsilon/8, \Gamma + \frac{\varepsilon}{8}, \alpha\right) \sum_{j=1}^{n_r} \frac{E \| u_j \|^{2+\alpha}}{a_{n_r}^{2+\alpha}}.$$

Now, choosing $b \ge 1$, we have

$$E(g(\textstyle\sum_{j=1}^{n_r} G_j/a_{n_r})) \le E\left(I\left(\| \sum_{j=1}^{n_r} G_j/a_{n_r} \| > \Gamma + \frac{\varepsilon}{8}\right)(c \| \sum_{j=1}^{n_r} G_j/a_{n_r} \|^{\ell} + b)\right)$$

(4.14) $$\le P(\| \textstyle\sum_{j=1}^{n_r} G_j/a_{n_r} \| > \Gamma + \varepsilon/8)^{1-1/p}$$

$$\cdot \{E([c \| \textstyle\sum_{j=1}^{n_r} G_j/a_{n_r} \|^{\ell} + b]^p)\}^{1/p},$$

so (4.11) holds if there exists p sufficiently large such that

(4.15) $$\textstyle\sum_r P\left(\| \sum_{j=1}^{n_r} G_j/a_{n_r} \| > \Gamma + \frac{\varepsilon}{8}\right)^{1-1/p} < \infty,$$

(4.16) $$\sup_r E([c \| \textstyle\sum_{j=1}^{n_r} G_j/a_{n_r} \|^{\ell} + b]^p) < \infty,$$

and

(4.17) $$\textstyle\sum_r \sum_{j=1}^{n_r} E \| u_j \|^{2+\alpha}/a_{n_r}^{2+\alpha} < \infty.$$

To establish (4.17) we note that if $\psi(x)$ denotes the logarithm of x to the base β, then we have

$$\begin{aligned}\sum_{r\geq 1}\sum_{j=1}^{n_r} E\|u_j\|^{2+\alpha}/a_{n_r}^{2+\alpha} &\leq \sum_{r\geq 1} 2^{1+\alpha} n_r E(\|X\|^{2+\alpha} I(\|X\|^2 \leq a_{n_r}^2))/(2n_r L_2 n_r)^{1+\alpha/2} \\ &\leq 2^{\alpha/2}\sum_{r\geq 1}\sum_{k=1}^{a_{n_r}^2} \frac{k^{1+\alpha/2}P(k-1<\|X\|^2\leq k)}{n_r^{\alpha/2}(L_2 n_r)^{1+\alpha/2}} \\ &\leq C\sum_{k=1}^{\infty}\sum_{r\geq\psi(k/4L_2k)-1} \frac{k^{1+\alpha/2}P(k-1<\|X\|^2\leq k)}{n_r^{\alpha/2}(L_2 n_r)^{1+\alpha/2}} \\ &\leq C\sum_{k=1}^{\infty} \frac{k^{1+\alpha/2}P(k-1<\|X\|^2\leq k)}{\left(\frac{k}{L_2 k}\right)^{\alpha/2}\left(L_2\left(\frac{k}{L_2 k}\right)\right)^{1+\alpha/2}} \\ &\leq C\sum_{k=1}^{\infty}\frac{k}{L_2 k}P(k-1<\|X\|^2\leq k) \\ &\leq CE\left(\frac{\|X\|^2}{L_2\|X\|}\right)<\infty\end{aligned}$$

where C is a positive finite constant which possibly changes from line to line. Hence (4.17) holds and we turn to the proof of (4.16).

Since there is a constant M, depending only on B and q, such that

$$E\|G\|^q \leq M(E\|G\|^2)^{q/2} \tag{4.19}$$

for all B-valued mean zero Gaussian random variables G, (4.16) holds if we establish

$$\sup_r E\left\|\sum_{j=1}^{n_r} G_j/a_{n_r}\right\|^2 < \infty. \tag{4.20}$$

Now, if $\{u_{j,k}: k\geq 1\}$ are independent copies of u_j, we have by the results in [4] that

$$E\|G_j\|^2 = \lim_n E\|(u_{j,1}+\cdots+u_{j,n})/\sqrt{n}\|^2 \leq AE\|u_j\|^2 \tag{4.21}$$

where A is the bounding constant in the type 2 inequality for B. Combining (4.21) and the type 2 inequality, we obtain from (4.1) that

$$\begin{aligned}\sup_r E\left\|\sum_{j=1}^{n_r} G_j/a_{n_r}\right\|^2 &\leq 2A\sup_r\sum_{j=1}^{n_r}\frac{E(\|X\|^2 I(\|X\|\leq\tau_{r-1}))}{2n_r L_2 n_r} \\ &\leq A\sup_r E\left(\frac{\|X\|^2}{L_2 n_r}I(\|X\|\leq\tau_{r-1})\right)\leq 2AE\left(\frac{\|X\|^2}{L_2\|X\|}\right)<\infty.\end{aligned} \tag{4.22}$$

Hence (4.16) holds for all $p>1$, so the proof will be complete once we establish (4.15) for some $p>1$ sufficiently large.

In order to prove (4.15) we need a remarkable result of C. Borell [6] which allows us to estimate the Gaussian probabilities in (4.15) with an accuracy we were unable to achieve by other methods in spaces other than Hilbert space. Indeed, for the Hilbert space situation we can estimate the probabilities in (4.15) directly. The details for this special case are included in a remark following the proof.

Let

$$\Phi(\alpha)=\int_{-\infty}^{\alpha}\frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du.$$

LEMMA 4.3. (*The inequality of C. Borell* [6, *Theorem* 3.1] *slightly modified*). *Let B be a separable Banach space, G a mean zero B-valued Gaussian random variable, q a continuous semi-norm on B, and* $\Gamma_G(q)=\sup_{x\in K_{\mathcal{L}(G)}} q(x)$. *Let $t_0>0$, $\alpha>0$ be fixed, and let*

$$\lambda(G)=\lambda_{q,t_0,\alpha}(G)=\inf\{\lambda>0: P(q(G)/\lambda<t_0)\geq\Phi(\alpha)\}. \tag{4.23}$$

Then for all $t \geq t_0$ *and all* $\rho > \lambda(G)$,

$$P(q(G) \geq \rho t) \leq 1 - \Phi(\alpha + \Gamma_G(q)^{-1}\rho(t - t_0)). \tag{4.24}$$

PROOF OF LEMMA 4.3. If $\rho > 0$ and $\nu = \mathscr{L}(G/\rho)$ we first observe that

$$\Gamma_\nu(q) \equiv \sup_{x\in K_\nu} q(x) = \Gamma_G(q)\rho^{-1}.$$

Now let $A = \{x : q(x) < t_0\}$. Then $\rho > \lambda(G)$ implies

$$P(G/\rho \in A) \geq \Phi(\alpha),$$

and hence by [6, Theorem 3.1] we have for all $\tau \geq 0$ that

$$P(G/\rho \in A + \tau K_\nu) \geq \Phi(\alpha + \tau). \tag{4.25}$$

Now

$$\tau K_\nu \subseteq \tau\{x : q(x) \leq \Gamma_\nu(q)\} = \tau\Gamma_G(q)\rho^{-1}\{x : q(x) \leq 1\},$$

and hence

$$\{G/\rho \notin A + \tau K_\nu\} \supseteq \{q(G/\rho) \geq t_0 + \tau\Gamma_G(q)\rho^{-1}\}. \tag{4.26}$$

Combining (4.25) and (4.26) we thus have for $t \geq t_0$, $\rho > \lambda(G)$, and $\tau = \Gamma_G(q)^{-1}\rho(t - t_0)$ that

$$P(q(G) \geq \rho t) = P(q(G/\rho) \geq t_0 + (t - t_0)) \leq 1 - \Phi(\alpha + \Gamma_G(q)^{-1}\rho(t - t_0)).$$

Thus the lemma is proved and we return to the proof of (4.15).

To estimate the probabilities

$$P(\| \textstyle\sum_{j=1}^{n_r} G_j/a_{n_r} \| > \Gamma + \varepsilon/8)$$

we set $G = \sum_{j=1}^{n_r} G_j/\sqrt{n_r}$ and let $q(x) = \| x \|$. Then, by the construction of $H_{\mathscr{L}(G)}$ we have

$$\begin{aligned}
\Gamma_G &= \sup_{x\in K_{\mathscr{L}(G)}} \| x \| = \sup_{\|f\|_{B^*}\leq 1}(E(f^2(G)))^{1/2} \\
&= \sup_{\|f\|_{B^*}\leq 1}\left(\textstyle\sum_{j=1}^{n_r} E(f^2(G_j))/n_r\right)^{1/2} \\
&= \sup_{\|f\|_{B^*}\leq 1}\left(\textstyle\sum_{j=1}^{n_r} E(f^2(u_j))/n_r\right)^{1/2} \\
&\leq \sup_{\|f\|_{B^*}\leq 1}(E(f^2(X)))^{1/2} \quad \text{from} \quad (4.1) \\
&= \Gamma.
\end{aligned} \tag{4.27}$$

Further, the calculation in (4.22) easily implies

$$\lim_r E \| \textstyle\sum_{j=1}^{n_r} G_j/a_{n_r} \|^2 = 0,$$

and hence, for $t_0 > 0$, $\alpha > 0$ fixed, an easy application of Markov's inequality implies

$$\lambda(G) = \lambda(\textstyle\sum_{j=1}^{n_r} G_j/\sqrt{n_r}) = o(\sqrt{L_2 n_r}) \quad \text{as} \quad r \to \infty.$$

Thus for any $\theta > 0$, no matter how small,

$$\lambda(G) \leq \theta\sqrt{2L_2n_r}\left(\Gamma + \frac{\varepsilon}{8}\right)$$

for all r sufficiently large. Hence by applying (4.24) to $G = \sum_{j=1}^{n_r} G_j/\sqrt{n_r}$ and recalling (4.27) we have, for any fixed θ and $r \geq r_0(\theta)$, that for $t \geq t_0$

$$\begin{aligned}
P\left(\frac{\| G \|}{\theta\sqrt{2L_2n_r}\left(\Gamma + \frac{\varepsilon}{8}\right)} > t\right) &\leq 1 - \Phi\left(\alpha + \Gamma_G(q)^{-1}\theta\sqrt{2L_2n_r}\left(\Gamma + \frac{\varepsilon}{8}\right)(t - t_0)\right) \\
&\leq 1 - \Phi\left(\alpha + \Gamma^{-1}\theta\sqrt{2L_2n_r}\left(\Gamma + \frac{\varepsilon}{8}\right)(t - t_0)\right).
\end{aligned} \tag{4.28}$$

Now choose $\theta > 0$, sufficiently small, so that if $t = 1/\theta$ then $t > t_0$ and

$$\Gamma^{-1}\theta\left(\Gamma + \frac{\varepsilon}{8}\right)\left(\frac{1}{\theta} - t_0\right) > 1 + \delta$$

for some $\delta > 0$. For this choice of θ and $t = 1/\theta$ fixed, we now have from (4.28) that for all r sufficiently large

$$(4.29)\quad P\left(\| G \| > \sqrt{2L_2n_r}\left(\Gamma + \frac{\varepsilon}{8}\right)\right) \le 1 - \Phi((1+\delta)\sqrt{2L_2n_r}) \le \exp\left\{-\left(1 + \frac{\delta}{2}\right)L_2n_r\right\}.$$

Taking p in (4.15) such that $(1 - 1/p)(1 + \delta/2) > 1 + \delta/4$, and recalling $G = \sum_{j=1}^{n_r} G_j/\sqrt{n_r}$ in (4.29), we easily have from (4.29) that (4.15) holds.

Thus Lemma 4.2 is proved, and Theorem 2 follows.

The special case of H-valued $\{G_j : j \ge 1\}$. For Hilbert space valued random variables we can estimate the probabilities in (4.15) directly since

$$(4.30)\quad P(\| G \| > t) \le E(e^{\alpha\|G\|^2})e^{-\alpha t^2},$$

and in the Hilbert space case a direct computation implies

$$(4.31)\quad E(e^{\alpha\|G\|^2}) = \prod_{j=1}^{\infty} E(e^{\alpha f_j^2(G)}) = \prod_{j=1}^{\infty}(1 - 2\alpha E(f_j^2(G)))^{-1/2}$$

as G can be expanded such that $G = \sum_{j\ge1} f_j(G)f_j$ where $\{f_j : j \ge 1\}$ is an orthonormal sequence in H and $\{f_j(G) : j \ge 1\}$ are independent mean zero Gaussian random variables. Now $E\| G \|^2 = \sum_{j\ge1} E(f_j^2(G)) < \infty$, so from (4.31) we see $E(e^{\alpha\|G\|^2}) < \infty$ for all

$$\alpha < (2 \sup_{\|f\|_{B^*}\le1} E(f^2(G)))^{-1}.$$

Hence if $\delta > 0$ is given we choose $\alpha = (1 - \delta)\Lambda^{-1}$ where $\Lambda = 2\sup_{\|f\|_{B^*}\le1} E(f^2(G))$. Then

$$(4.32)\quad 2\alpha E(f_j^2(G)) \le 1 - \delta$$

and, since

$$(4.33)\quad 1 - x \ge e^{-x/\delta}$$

for $0 \le x \le 1 - \delta$, we have by combining (4.31), (4.32), and (4.33) that

$$(4.34)\quad E(e^{\alpha\|G\|^2}) \le \prod_{j=1}^{\infty} e^{\alpha E(f_j^2(G))/\delta} = e^{\alpha E\|G\|^2/\delta}$$

when $\alpha = (1 - \delta)\Lambda^{-1}$. Combining (4.30) and (4.34) we obtain

$$(4.35)\quad P(\| G \| > t) \le \exp\left\{\frac{(1-\delta)}{\delta\Lambda}E\| G \|^2 - \frac{(1-\delta)}{\Lambda}t^2\right\}.$$

If $G = \sum_{j=1}^{n_r} G_j/\sqrt{n_r}$ as in (4.15) and $t = (\Gamma + \varepsilon/8)\sqrt{2L_2n_r}$, then $\Gamma = (\Lambda/2)^{1/2}$ and as before $E\| G \|^2 = o(L_2n_r)$. Hence, if $\delta > 0$ is chosen such that $(1 - \delta)(\Gamma + \varepsilon/8)^2/2\Gamma^2 > 1/2$, (4.35) implies there exists $\eta > 0$ such that for all r sufficiently large

$$(4.36)\quad P\left(\left\| \sum_{j=1}^{n_r} G_j/\sqrt{n_r} \right\| > \left(\Gamma + \frac{\varepsilon}{8}\right)\sqrt{2L_2n_r}\right) \le \exp\{-(1+\eta)L_2n_r\}.$$

The estimate in (4.36) is exactly what we need to verify (4.15), so the Hilbert space case stands independently of C. Borell's result.

PROOF OF COROLLARY 3. If $X \in$ CLIL, then (2.9) and (2.10) hold as they are necessary conditions even for the BLIL. Further, by the proof of Corollary 3.1 of [11] we have the limit set equal to K, and hence K is compact.

Now assume (2.9) and (2.10) hold with K compact. Let Π_N and Q_N be the maps of Lemma 2.1 of [8] defined with respect to the probability $\mathscr{L}(X)$. Now fix $\varepsilon > 0$ and choose

N sufficiently large so that

$$\sup_{x \in K} \| Q_N x \| \le \varepsilon. \tag{4.37}$$

Such an N exists since K is compact, $Q_N K$ is decreasing, and $\cap_N Q_N K = \{0\}$. Thus by applying Theorem 2 to the random variables $\{Q_N X_j : j \ge 1\}$, and noting that $K_{\mathscr{L}(Q_N X)} = Q_N K_{\mathscr{L}(X)} = Q_N K$, we have

$$\limsup_n \left\| Q_N\left(\frac{S_n}{a_n}\right) \right\| = \sup_{x \in Q_N K} \| x \| \le \varepsilon \qquad \text{w.p.1.} \tag{4.38}$$

Now the random variables $\{\Pi_N X_j : j \ge 1\}$ are finite dimensional, and since all norms on the range of Π_N are equivalent and $\Pi_N K = K_{\mathscr{L}(\Pi_N X)}$ with $\Pi_N K \subseteq K$, we have by applying Corollary 4 that

$$\limsup_n d\left(\Pi_N\left(\frac{S_n}{a_n}\right), K\right) = 0 \qquad \text{w.p.1.} \tag{4.39}$$

Of course, a number of other sources could also be quoted to establish (4.39).

Since $\varepsilon > 0$ is arbitrary (4.38) and (4.39) combine to imply

$$\limsup_n d\left(\frac{S_n}{a_n}, K\right) = 0 \qquad \text{w.p.1.} \tag{4.40}$$

By Theorem 1 we have

$$C\left(\left\{\frac{S_n}{a_n}\right\}\right) = K \qquad \text{w.p.1,} \tag{4.41}$$

and hence by (4.40) and (4.41) with K compact we have $X \in \text{CLIL}$. Thus Corollary 3 is proved.

5. Proof of Proposition 1. Since Lemma 4.1 of [10, page 268] asserts that the Lip(1) assumption of Condition (A′) implies B is of type 2, it suffices to prove that if B satisfies Condition (A) and is of type 2, then B has the upper Gaussian comparison principle.

To prove the upper Gaussian comparison principle we let the class $\mathscr{G}_{\Lambda,\delta}(\Lambda > 0, \delta > 0)$ consists of the single function

$$g(x) = \phi(\| x \|), \tag{5.1}$$

where $\phi(t)$ is three times continuously differentiable on $(0, \infty)$ and such that

$$\phi(t) = \begin{cases} 0 & 0 \le t \le \Lambda \\ \text{increasing} & \Lambda < t < \Lambda + \delta \\ 1 & t \ge \Lambda + \delta. \end{cases} \tag{5.2}$$

Letting $\{Y_j : j \ge 1\}$ be independent, B-valued, mean zero, bounded (and hence pregaussian) random variables, with corresponding independent Gaussian random variables $\{G_j : j \ge 1\}$, we have

$$g(\textstyle\sum_{j=1}^n Y_j) - g(\sum_{j=1}^n G_j) = \sum_{k=1}^n A_k \tag{5.3}$$

where

$$A_k = g(B_k + Y_k) - g(B_k + G_k)$$

and

$$B_k = Y_1 + \cdots + Y_{k-1} + G_{k+1} + \cdots + G_n.$$

Then, by minor modifications of the argument on pages 73–78 of [9], we have a finite constant $C(\delta, \Lambda, \alpha)$, which is uniform in n, such that

$$| E(g(\textstyle\sum_{j=1}^n Y_j)) - E(g(\sum_{j=1}^n G_j))| \le C(\delta, \Lambda, \alpha) \sum_{j=1}^n [E \| Y_j \|^{2+\alpha} + E \| G_j \|^{2+\alpha}] \tag{5.4}$$

where $\alpha > 0$ is such that D_x^2 is Lip(α) away from zero. That is, to obtain (5.4) one proceeds as in [9] except that now the covariance structure varies and to verify the third equation of (2.12) of [9] one can argue through the Hilbert Space $H_{\mathscr{L}(Y_k)}$. However, since we know the CLT holds for each Y_k, it follows that

$$E(\phi(Y_k, Y_k)) = E(\phi(G_k, G_k)) \tag{5.5}$$

for every bounded symmetric bilinear from ϕ on B, and hence (2.12) of [9] also follows immediately from this observation.

Our next step is to obtain the bound on the right hand side of (5.4) in terms of $E\|Y_j\|^{2+\alpha}$, rather than one involving $E\|G_j\|^{2+\alpha}$. To do this we first observe that B is of type 2, and hence, if $\{Y_{j,k}: k \geq 1\}$ are independent copies of Y_j, we have by the results of [4] that

$$E\|G_j\|^2 = \lim_n E\|\textstyle\sum_{k=1}^n Y_{j,k}/\sqrt{n}\|^2 \leq AE\|Y_j\|^2. \tag{5.6}$$

The constant A in (5.6) is the bounding constant in the type 2 inequality for B. Further, there is a constant M, depending only on B and α, such that

$$E\|G\|^{2+\alpha} \leq M(E\|G\|^2)^{1+\alpha/2} \tag{5.7}$$

for all B valued mean zero Gaussian random variables G. Hence, by combining (5.4), (5.6), and (5.7), along with the increasing nature of the L^p-norms, we have that there is a finite constant, again call it $C(\delta, \Lambda, \alpha)$, such that for all $n \geq 1$

$$|E(g(\textstyle\sum_{j=1}^n Y_j)) - E(g(\sum_{j=1}^n G_j))| \leq C(\delta, \Lambda, \alpha) \sum_{j=1}^n E\|Y_j\|^{2+\alpha}. \tag{5.8}$$

We also have the constant $C(\delta, \Lambda, \alpha)$ independent of the sequences $\{Y_j\}$ and $\{G_j\}$ in (5.8). Hence more than (2.8) holds, and we have that B satisfies the upper Gaussian comparison principle.

6. Proof of Corollary 4. Since (2.9) and (2.10) are known to be necessary for the BLIL, it suffices to prove (2.13) and (2.14) when (2.9) and (2.10) hold. Of course, (2.13) follows immediately from Corollary 1 since Hilbert space is type 2.

To prove (2.14) we use the following lemma.

LEMMA 6.1. *If X takes values in a real separable Hilbert space H, X is WM_0^2, and S is the covariance operator of X, then S is a bounded, symmetric, non-negative operator, and*

$$S^{1/2}(V) = K \tag{6.1}$$

where $V = \{x \in H: \|x\| = \langle x, x\rangle^{1/2} \leq 1\}$ and, of course, $K = K_\mu$ where $\mu = \mathscr{L}(X)$.

PROOF. Recall that the covariance operator S is defined by the relation

$$\langle Sf, g\rangle = \int_H \langle f, x\rangle\langle g, x\rangle\, d\mu(x) \qquad (f, g \in H).$$

If we identify H and H^* as usual, then the covariance operator is the same as the operator $S = S_\mu$ of Lemma 2.1 of [8] and part (ii) of that lemma thus implies S is bounded since we have $X \in WM_0^2$. S is obviously symmetric, non-negative, and hence has a unique bounded, symmetric, non-negative square root $S^{1/2}$.

To prove (6.1) we first observe that

$$\begin{aligned}(Sf, Sg)_\mu &= \int_H \langle f, x\rangle\langle g, x\rangle\, d\mu(x) = \langle Sf, g\rangle \\ &= \langle S^{1/2}f, S^{1/2}g\rangle, \quad (f, g \in H)\end{aligned} \tag{6.2}$$

and hence

$$Sf \in K \quad \text{iff} \quad S^{1/2}f \in V. \tag{6.3}$$

Now let N denote the null space and M the closure of the range of $S^{1/2}$. Since $S^{1/2}$ is symmetric we have $M = N^{\perp}$ and hence

$$H = M \oplus N. \tag{6.4}$$

If $f \in V$ we have $f = m + n$ where $m \in M$, $n \in N$. Hence $\langle m, n \rangle = 0$ and $\|f\|^2 = \|m\|^2 + \|n\|^2 \leq 1$. Choose $m_k \in M$, $\|m_k - m\| \to 0$, $\|m_k\| \leq 1$, $m_k = S^{1/2}f_k$. Then $S^{1/2}$ continuous implies

$$\lim_k \| S^{1/2}m_k - S^{1/2}m \| = 0. \tag{6.5}$$

Furthermore,

$$S^{1/2}m_k = S^{1/2}(S^{1/2}f_k) = Sf_k \in K \tag{6.6}$$

since (6.3) holds and $m_k = S^{1/2}f_k \in V$. Now K is closed in H, as well as in H_μ, so

$$S^{1/2}m = \lim_k S^{1/2}m_k \in K.$$

Therefore

$$S^{1/2}f = S^{1/2}(m + n) = S^{1/2}m \in K$$

for all $f \in V$. Thus $S^{1/2}V \subseteq K$.

If $x \in K$, then there exists g_k such that $Sg_k \in K$ and $\lim_k \|Sg_k - x\|_\mu = 0$. Thus $\{Sg_k : k \geq 1\}$ is Cauchy in H_μ, and from (6.2) we thus have $\{S^{1/2}g_k : k \geq 1\}$ is Cauchy in H. Let $f = \lim_k S^{1/2}g_k$. Then $f \in V$ and $S^{1/2}f = \lim_k S^{1/2}(S^{1/2}g_k) = x$ as $S^{1/2}$ is continuous and $\|x\| \leq c\|x\|_\mu$ for all $x \in H_\mu$. Thus $K \subseteq S^{1/2}V$, and Lemma 6.1 is proved.

To finish the proof of Corollary 4 we fix $\varepsilon > 0$. Then, if I is the identity map on H, we have that $(S^{1/2} + \varepsilon I)^{-1}$ exists and is a bounded operator on H (recall the spectrum of $S^{1/2}$ is a subset of $[0, \infty)$). Hence let

$$q(x) = \|(S^{1/2} + \varepsilon I)^{-1}(x)\| = \langle (S^{1/2} + \varepsilon I)^{-1}(x), (S^{1/2} + \varepsilon I)^{-1}(x) \rangle^{1/2}, \tag{6.7}$$

so q is obtained from an inner product on H.

Thus H is a Hilbert space with the inner product norm q, and as a result (H, q) is a type 2 space with property (A). Hence Proposition 1 implies (H, q) has the upper Gaussian comparison principle, and Theorem 2 then implies

$$\limsup_n q(S_n/a_n) = \sup_{x \in K} q(x). \tag{6.8}$$

Now Lemma 6.1 implies

$$\begin{aligned} \sup_{x \in K} q(x) &= \sup_{x \in K} \|(S^{1/2} + \varepsilon I)^{-1}x\| \\ &= \sup_{y \in V} \|(S^{1/2} + \varepsilon I)^{-1}S^{1/2}y\| = ||| (S^{1/2} + \varepsilon I)^{-1}S^{1/2} ||| \end{aligned} \tag{6.9}$$

where $||| \cdot |||$ denotes the operator norm or $L(H, H)$. To estimate $||| (S^{1/2} + \varepsilon I)^{-1}S^{1/2} |||$ we let $U = S^{1/2} + \varepsilon I$ and notice that

$$\begin{aligned} \langle x, x \rangle &= \langle U^{-1}U(x), U^{-1}U(x) \rangle \\ &= \langle U^{-1}S^{1/2}(x), U^{-1}S^{1/2}(x) \rangle + 2\varepsilon \langle U^{-1}S^{1/2}(x), U^{-1}(x) \rangle \\ &\quad + \varepsilon^2 \langle U^{-1}(x), U^{-1}(x) \rangle \\ &\geq \langle U^{-1}S^{1/2}(x), U^{-1}S^{1/2}(x) \rangle \end{aligned} \tag{6.10}$$

since $U^{-1}S^{1/2} = S^{1/2}U^{-1}$ (note $S^{1/2} = U - \varepsilon I$) and hence $\langle U^{-1}S^{1/2}(x), U^{-1}(x) \rangle =$

$\langle S^{1/2}U^{-1}(x),\ U^{-1}(x)\rangle \geq 0$ as $S^{1/2}$ is a non-negative operator on H. Thus (6.8), (6.9) and (6.10) combine to imply

$$\limsup_n q\left(\frac{S_n}{a_n}\right) \leq 1 \qquad \text{w.p.l.} \tag{6.11}$$

Now

$$\{x : q(x) \leq 1\} \subseteq K + \varepsilon V, \tag{6.12}$$

since $q(x) \leq 1$ iff $(S^{1/2} + \varepsilon I)^{-1}(x) = y$ for some $y \in V$, and by Lemma 6.1 $S^{1/2}(V) = K$. Thus by combining (6.11) and (6.12) we have

$$\limsup_n d\left(\frac{S_n}{a_n}, K\right) \leq \varepsilon \qquad \text{w.p.1.}$$

where

$$d(x, K) = \inf_{y \in K} \| x - y \|.$$

Since $\varepsilon > 0$ was arbitrary, we have (2.14) and Corollary 4 is proved.

APPENDIX

ELIMINATION OF ASSUMPTION (2.2) IN THEOREM 1

The following argument was communicated to us by M. Ledoux. Step (i), which seemed to be obvious, required some additional details. We will assume that only (2.1) and (2.3) are satisfied and will point out the necessary modifications in the proof of Theorem 1.

(i) Assumption (2.3) implies

$$X_\tau - EX_\tau \in \text{CLIL} \quad \text{for each} \quad \tau > 0. \tag{A.1}$$

Sketch of proof: (2.3) implies $\sup_n nP\{\|X\| > \varepsilon a_n\} < \infty$ for every $\varepsilon > 0$ (see e.g. [3], Theorem 2.2).

From this we easily see that $E\|X\|^p < \infty$ for $0 \leq p < 2$ and hence by (2.1) that $EX = 0$. Using Lemma 3.3 with $\phi(t) = t$ we see (A.1) holds (for example, see Theorem 4 of [12]) provided

$$\lim_n E\| S_n \| / a_n = 0. \tag{A.2}$$

Since $EX = 0$ it is easy to see that it suffices to show (A.2) under the assumption X is symmetric and X satisfies (2.1) and (2.3).

To prove (A.2) under these assumptions we consider

$$S_{nk} = \textstyle\sum_{j=1}^{n} [S_{kj} - S_{k(j-1)}]$$

where $S_0 = 0$. Then for every $\delta > 0$

$$P(\max_{1 \leq j \leq n} \| S_{kj} - S_{k(j-1)} \| \geq \delta) \leq 2P(\| S_{nk} \| > \delta)$$

and hence

$$nP(\| S_k \| > \delta) \leq -\log(1 - 2P(\| S_{nk} \| > \delta)).$$

From (2.3) we obtain $k_0(\varepsilon)$ such that for $k \geq k_0(\varepsilon)$ and all $n \geq 1$ we have

$$P(\| S_{nk} \| > \varepsilon a_{nk}) \leq 1/4,$$

and hence from the above there exists an absolute constant C, independent of ε, such that

$$P(\| S_k \| > \varepsilon a_{nk}) \leq C/n$$

for $n \geq 1$, $k \geq k_0(\varepsilon)$. Thus for $n \geq 1$, $k \geq k_0(\varepsilon)$

$$P\left(\| S_k/a_k \| > \varepsilon \sqrt{\frac{nL_2nk}{L_2k}} \right) \le C/n,$$

and since

$$\sqrt{\frac{nL_2nk}{L_2k}} \le n^{3/4}$$

for all $n \ge e^e$, $k \ge e^e$, we see

$$P(\| S_k/a_k \| > \varepsilon n^{3/4}) \le C/n$$

for $n \ge e^e$, $k \ge k_0(\varepsilon)$.

(A.2) now follows in a standard way. Hence also $(\xi - \eta)\mathrm{U} \in$ CLIL and the same steps of the proof of Theorem 1 give (3.21).

(ii) Assumption (2.3) and (A.1) imply (3.22) and by Ottaviani's inequality we have:

$$\text{(A.3)} \qquad \sup_{2^k \le n < 2^{k+1}} \| \textstyle\sum_{j=1}^n (W_j + EX_\tau)/a_n \| \to_P 0 \text{ as } k \to \infty.$$

(iii) (3.26) is replaced by: for $n \ge n_0(\varepsilon)$, taking into account (A.3),

$$P(\inf_{2^k \le n < 2^{k+1}} \| T_n/a_n - b \| \le \varepsilon)$$

$$\ge P(\inf_{2^k \le n < 2^{k+1}} \| \textstyle\sum_{j=1}^n (Y_j - EX_\tau)/a_n - S_{\mu_\tau} f \| \le \varepsilon/2)$$

$$\cdot P(\sup_{2^k \le n < 2^{k+1}} \| \textstyle\sum_{j=1}^n (W_j + EX_\tau)/a_n \| < \varepsilon/4)$$

$$\text{(A.4)} \qquad \ge (1/2)\, P(\inf_{2^k \le n < 2^{k+1}} \| \textstyle\sum_{j=1}^n (Y_j - EX_\tau)/a_n - S_{\mu_\tau} f \| \le \varepsilon/2).$$

(iv) Since $X_\tau - EX_\tau \in$ CLIL and $\mathscr{L}(Y_j) = \mathscr{L}(X_\tau), S_{\mu_\tau} f \in C(\{\sum_{j=1}^n (Y_j - EX_\tau)/a_n\})$ a.s. and by the Borel-Cantelli lemma we have

$$\textstyle\sum_k P\{\inf_{2^k \le n < 2^{k+1}} \| \sum_{j=1}^n (Y_j - EX_\tau)/a_n - S_{\mu_\tau} f \| \le \varepsilon/2\} = \infty,$$

which implies by (A.4)

$$\textstyle\sum_k P\{\inf_{2^k \le n < 2^{k+1}} \| T_n/a_n - b \| \le \varepsilon\} = \infty.$$

The proof of Lemma 4 of [13] implies now

$$\liminf \| T_n/a_n - b \| \le 6\varepsilon \qquad \text{w.p.1.}$$

The steps in Theorem 1 complete the proof.

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