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THE CENTRAL LIMIT THEOREM FOR STOCHASTIC INTEGRALS WITH RESPECT TO LÉVY PROCESSES[1]

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Let M be a symmetric independently scattered random measure on [0, 1] with control measure m which is uniformly in the domain of normal attraction of a stable measure of index $p \in (0, 2]$. Let f be a non-anticipating process with respect to $X(t) = M[0, t]$ if m is continuous, and a previsible process in general, satisfying $\int_0^1 E|f|^p\, dm < \infty$. Then the stochastic integral $\int_0^t f\, dM$ can be defined as a process in $D[0, 1]$ and is in the domain of normal attraction of a stable process of order p in $D[0, 1]$ in the sense of weak convergence of probability measures. If M is Gaussian and continuous in probability then the central limit theorem holds in $C[0, 1]$; in particular, Itô and diffusion processes satisfy the CLT. Our main tool is an upper bound for the weak L^p norm of $\sup_{0\le t\le 1} |\int_0^t f\, dM|$ in terms of the $L^p(P \times m)$ norm of f.

1. Introduction. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and $L_0(\Omega, \mathscr{F}, P)$ denote the measurable functions on $(\Omega, \mathscr{F}, P)$. Let $\mathscr{B}$ denote the Borel sets on [0, 1]. A random measure M is a mapping from $\mathscr{B}$ into $L_0(\Omega, \mathscr{F}, P)$ such that for disjoint sets $A_1, \cdots, A_n, \cdots \in \mathscr{B}$, $M(\cup_{i=1}^n A_i) = \sum_{i=1}^m M(A_i)$ a.s. and $\lim_{n\to\infty} M(\cup_{i=1}^n A_i) = M(A)$ in probability, where $A = \cup_{i=1}^\infty A_i$. We will consider random measures M satisfying the following conditions:

(1.1) M has independent symmetric increments, that is, for each $A \in \mathscr{B}$, $M(A)$ is symmetric and if $A_1, \cdots, A_n \in \mathscr{B}$ are disjoint then $M(A_1), \cdots, M(A_n)$ are independent;

(1.2) there exists a finite positive measure m on $([0, 1], \mathscr{B})$, called the control measure of M, such that for every $A \in \mathscr{B}$, $M(A)$ is in the domain of normal attraction of $(m(A))^{1/p}\theta$, where θ is a symmetric, stable random variable of index $0 < p < 2$, i.e. $E \exp(it\theta) = \exp(-|t|^p)$, $-\infty < t < \infty$;

(1.3) $\sup_{A\in\mathscr{B}} \Lambda_p(M(A)/(m(A))^{1/p}) \le c$ for some constant $c < \infty$.

Here we define, for $\xi \in L_0(\Omega, \mathscr{F}, P)$,

$$\Lambda_p(\xi) = (\sup_{\lambda>0} \lambda^p P\{|\xi| > \lambda\})^{1/p}. \tag{1.4}$$

Let $\mathscr{M}(M, m)$ denote a class of functions $\{f(t, \omega), t \in [0, 1]\}$ on $(\Omega, \mathscr{F}, P)$. For $f \in \mathscr{M}(M, m)$ we define the stochastic integral

$$F = F(t) = \int_0^t f\, dM, \quad t \in [0, 1] \tag{1.5}$$

where the relationship between M and m is given in (1.2) and (1.3). The class $\mathscr{M}(M, m)$ will be discussed in Section 3. Here we only remark that when m is continuous, $\mathscr{M}(M, m)$ is precisely the class of non-anticipating processes satisfying

$$\int_0^1 E|f|^p dm < \infty. \tag{1.6}$$

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We show that $F(t)$ has a version with almost all its sample paths in $D[0, 1]$ and satisfies the maximal inequality

$$\Lambda_p(\|F\|) \le c_p\left(\int_0^1 E|f|^p dm\right)^{1/p}, \tag{1.7}$$

where $\|F\| = \sup_{t\in[0,1]}|F(t)|$ and c_p is a constant depending only on p.

We can view F as a random variable with values in $D[0, 1]$ and ask whether F satisfies the central limit theorem (CLT) in $D[0, 1]$. Specifically, let $\{F_i\}$ denote independent copies of F. Define

$$S_n = n^{-1/p}\sum_{i=1}^n F_i \tag{1.8}$$

(recall $0 < p < 2$). Clearly S_n is a $D[0, 1]$ valued random variable and as such induces a measure, say μ_n, on $D[0, 1]$. We say that F satisfies the CLT in $D[0, 1]$ if the measures $\{\mu_n\}$ converge weakly to some measure, say τ, on $D[0, 1]$. It is well known that if such a measure exists it is a stable measure. The main result of this paper is that the stochastic integral F defined in (1.5) satisfies the CLT in $D[0, 1]$. The limiting measure τ can be characterized in terms of f and m.

All the questions above make sense when $p = 2$. In this case we consider random measures satisfying:

(1.9) M has independent symmetric increments;

(1.10) there exists a finite positive control measure m on $[0, 1]$ such that for every $A \in \mathscr{B}$

$$EM^2(A) = m(A).$$

The definition of the stochastic integral in this case is well known, as is the maximal inequality

$$\left(E\left\|\int_0^t f\,dM\right\|^2\right)^{1/2} \le 2\left(\int_0^1 E|f|^2 dm\right)^{1/2}. \tag{1.11}$$

The question of whether F satisfies the CLT in $D[0, 1]$ is still revelant and indeed we show that it does. Furthermore, when $X(t) = M[0, t]$ has a sample continuous version we show that F satisfies the CLT in $C[0, 1]$.

Our motivation for this work is to continue our study of domains of attraction of stable laws in $C[0, 1]$ and $D[0, 1]$. This problem is considered in [11], [14], [7] and [17] in the case when the limiting measure is Gaussian. The results in these papers do not imply the ones we present here, which use properties specific to stochastic integrals. The only work we know of which considers the case of non-Gaussian limit measures is [8] and [9]. This paper is related to [8] (in [8] we consider stochastic integrals in which the integrand is independent of the measure). Reference [9] contains an extension of the central limit theorem of [14] to the case of non-Gaussian limits; however this result, which applies in a more general setting (but only in $C[0, 1]$), does not imply all the results we give here. So far, there is no comprehensive theory of the CLT in $C[0, 1]$ and $D[0, 1]$. In the meantime, it seems worthwhile to consider these questions for special classes of processes.

In Section 2 we present some preliminary results. In Section 3 we define the stochastic integral in the case $0 < p < 2$. We also describe a rather large class of measures M that satisfy (1.1) – (1.3). The central limit theorem, $0 < p < 2$, is considered in Section 4. The case $p = 2$ is studied in Section 5, where we also give examples of the random measures satisfying (1.9) and (1.10). As an application we show that Itô and diffusion processes satisfy the CLT in $C[0, \infty)$.

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2. Notation and preliminaries. A real symmetric random variable ξ is in the domain of normal attraction (DNA) of a symmetric stable law ρ of index p if

$$\mathscr{L}(n^{-1/p}\textstyle\sum_{i=1}^{n}\xi_i) \to_w \rho \tag{2.1}$$

where $\{\xi_i\}$ are independent copies of ξ, $\mathscr{L}(\xi)$ is the probability law of ξ and $\to_w$ denotes weak convergence of probability measures. Domains of attraction are defined similarly in more abstract linear spaces, in particular $D[0, 1]$ and $C[0, 1]$ (and $D[0, \infty)$ and $C[0, \infty)$). Here we use standard notation: $C[0, 1]$ denotes the space of real valued continuous functions on $[0, 1]$ with the sup norm and $D[0, 1]$ denotes the real valued functions on $[0, 1]$ which are right continuous and have finite left limits at each $t \in [0, 1]$, equipped with the Skorohod topology ([4], Chapter 3). This topology is induced by a metric d_0, smaller than the sup norm, which makes $D[0, 1]$ a complete separable metric space. Addition is not continous in $D[0, 1]$ but it is measurable so that sums of $D[0, 1]$ valued random variables are also $D[0, 1]$ valued random variables.

At various times we will speak of the following, all of which are equivalent: a process $X(t)$ having a version with almost all its sample paths in $D[0, 1]$, a $D[0, 1]$ valued random variable, or a Borel probability measure on $D[0, 1]$ (denoted $\mathscr{L}(X)$ as usual). Indeed the correspondence

$$X(t) = M[0, t]$$

for M as given in (1.1)—(1.3) yields stochastic processes X on $[0, 1]$ satisfying the following analogous conditions to (1.1) – (1.3):

(2.2) X has independent symmetric increments;

(2.3) for every $0 \le s < t \le 1$, $X(t) - X(s)$ is in the domain of normal attraction of $(m(s, t])^{1/p}\theta$;

(2.4) $\sup_{0\le s<t\le 1} \Lambda_p((X(t) - X(s))/(m(s, t])^{1/p}) \le c$ for some constant $c < \infty$,

and it is well known that $X(t)$ has a version with sample paths in $D[0, 1]$. This is a classical result of P. Lévy (see e.g. [5], Chapter VIII, Theorem 7.2). Conversely, for every process X on $[0, 1]$ satisfying (2.2)—(2.4) there exists a random measure M satisfying (1.1)—(1.3), given by $M[0, t] = X(t)$, as is easy to prove using Lemma 2.1.

Weak convergence of Borel probability measures on $D[0, 1]$ will also be denoted by $\to_w$ or w-lim. A $D[0, 1]$ valued random variable $S = S(t)$, $t \in [0, 1]$, is called stable of index p if all its finite dimensional distributions are stable of index p. A symmetric $D[0, 1]$ valued random variable X is in the domain of normal attraction of a symmetric stable process S if:

$$\begin{cases} S \text{ has a version with almost all its sample} \\ \text{paths in } D[0, 1], \text{ and } \mathscr{L}(n^{-1/p}\sum_{i=1}^{n} X_i) \to_w \mathscr{L}(S) \end{cases} \tag{2.5}$$

as probability measures on $D[0, 1]$, where X_i are independent copies of X. In a similar way one can define domain of attraction in $C[0, 1]$. If X is in the domain of normal attraction of S we will sometimes write $X \in \mathrm{DNA}(S)$.

It is well known that a real symmetric random variable ξ is in the domain of normal attraction of some stable law of index p, $0 < p < 2$, if and only if the sequence $\{nP[|\xi| > n^{1/p}]\}$ is convergent. Therefore it is natural to consider the function Λ_p defined in (1.4). The set $\{\xi \in L_0(\Omega, \mathscr{F}, P): \Lambda_p(\xi) < \infty\}$ is a linear space which can be equipped with a norm equivalent to Λ_p for which it is a Banach space for $p > 1$ and a Fréchet space for $p \le 1$. If we call this space $L_{p,\infty}$ then for every $q < p$ we have $L_p \subset L_{p,\infty} \subset L_q$ and the inclusions are continuous. (This follows by integration by parts).

The following lemma is generally known. A proof in a somewhat more general context is given in [8], Lemma 2.1.

2.1. LEMMA. *Let $\{\eta_k\}$ be a sequence of independent symmetric real valued random* *variables satisfying for some* $p \in (0, 2)$

$$\max_k \Lambda_p(\eta_k) = \alpha < \infty. \tag{2.6}$$

Then for any sequence of real numbers $\{a_k\}$ *we have*

$$\Lambda_p(\textstyle\sum_{k=1}^n a_k\eta_k) \le \alpha\left(\dfrac{4-p}{2-p}\right)^{1/p}(\textstyle\sum_{k=1}^n |a_k|^p)^{1/p}. \tag{2.7}$$

The next lemma, a slight modification of a result of Pisier ([18], part of Theorem 3.1), is a useful criterion for weak convergence. For a proof see [8], Lemma 2.2. (Here we present a version suitable for the problem at hand).

2.2. LEMMA. *Let* $\{X_n\}$ *and* $\{Y_n^m\}$, $n = 1, \cdots; m = 1, \cdots$, *be sequences of* $D[0, 1]$ *valued random variables such that*

$$\{\mathscr{L}(Y_n^m)\}_{n=1}^\infty \text{ is weakly convergent for each } m \text{ and} \tag{2.8}$$

$$\lim_{m\to\infty} \sup_n \Lambda_p(\|X_n - Y_n^m\|) = 0. \tag{2.9}$$

Then $\{\mathscr{L}(X_n)\}_{n=1}^\infty$ *converges weakly and*

$$w - \lim_{n\to\infty} \mathscr{L}(X_n) = w - \lim_{m\to\infty}\{w - \lim_{n\to\infty} \mathscr{L}(Y_n^m)\}. \tag{2.10}$$

Also an analogous statement holds for $C[0,1]$ valued random variables.

We use Billingsley [4] for results on the weak convergence of measures on $D[0,1]$. Following [4], for $X \in D[0,1]$, define

$$w_X(T) = \sup_{s,t\in T}|X(t) - X(s)|, \quad T \subset [0,1], \text{ and}$$

$$w_X'(\delta) = \inf_{\{t_i\}}[\max_i w_X[t_i, t_{i+1})], \quad \delta > 0,$$

where the infimum is taken over all the partitions $0 = t_0 < \cdots < t_r = 1$, $r < \infty$, of $[0,1]$ such that $t_i - t_{i-1} > \delta$, $i = 1, \cdots, r$.

The following weak convergence criterion is a combination of Theorems 15.3, 15.4 and the arguments on pages 133, 134, [4].

2.3. THEOREM. *Let* $\{X_n\}_{n=1}^\infty$ *be a sequence of* $D[0,1]$ *valued random variables such that*:

(1) *all the finite dimensional distributions of* $\{X_n\}$ *are weakly convergent (i.e., for any finite set* $\{t_1, \cdots, t_k\} \subset [0,1]$, *the k-dimensional random variables* $\{(X_n(t_1), \cdots, X_n(t_k))\}$ *are weakly convergent)*;

(2) *there exist* $\alpha > 1/2$, $\beta > 0$ *and a right continuous, non-decreasing function* F *on* $[0,1]$ *such that*:

$$P\{|X_n(t) - X_n(t_1)| \ge \lambda, |X_n(t_2) - X_n(t)| \ge \lambda\} \le \lambda^{-\beta}(F(t) - F(t_1))^\alpha(F(t_2) - F(t))^\alpha \tag{2.11}$$

for all $0 \le t_1 \le t \le t_2 \le 1$;

(3) *for every* $\varepsilon > 0$

$$\lim_{\delta\downarrow 0}\sup_n P\{w_{X_n}[1-\delta, 1) > \varepsilon\} = 0, \tag{2.12}$$

$$\lim_{\delta\downarrow 0}\sup_n P\{|X_n(\delta) - X_n(0)| > \varepsilon\} = 0. \tag{2.13}$$

Then $\{\mathscr{L}(X_n)\}_{n=1}^\infty$ *is a weakly convergent sequence of probability measures on* $D[0,1]$ *and the limit is determined by the limits of the finite dimensional distributions.*

PROOF. We will show that the conditions of Theorem 15.3 [4] are satisfied. Condition (ii), (15.7) of Theorem 15.3 [4], follows from condition (2) above by the argument on page

133, 134 [4]. Conditions (i) and (ii), (15.8) of Theorem 15.3 [4] follows from our condition (2.13) and from (ii), (15.7) of Theorem 15.3, as in the proof of Theorem 15.4, [4]. Finally, our (3) is (ii), (15.9) of Theorem 15.3, [4]. The fact that the limit is determined by the finite dimensional distributions is given in Theorem 15.1, [4]. □

The next lemma is technical. It also follows easily from the results in [4].

2.4. LEMMA. *Let $\{Z_n^i\}_{n=1}^{\infty}$, $i = 1, \cdots, r < \infty$, be r sequences of $D[0,1]$ valued random variables such that:*

(1) *there exists $0 = s_0 < s_1 < \cdots < s_r = 1$ such that for each i and all n, $Z_n^i(t) = 0$ if $t < s_{i-1}$ and $Z_n^i(t) = Z_n^i(s_i)$ if $t \geq s_i$;*
(2) *for each $i = 1, \cdots, r$, $\{\mathscr{L}(Z_n^i|_{[s_{i-1},s_i]})\}_{n=1}^{\infty}$ is tight as a sequence of measures on $D[s_{i-1}, s_i]$.*

Then $\{\mathscr{L}(\sum_{i=1}^r Z_n^i)\}_{n=1}^{\infty}$ is tight as a set of measures on $D[0,1]$.

PROOF. Since $\mathscr{L}(Z_n^i|_{[s_{i-1},s_i]})_{n=1}^{\infty}$ is tight for each $i = 1, \cdots, r$, we have by Theorem 15.2 [4] applied to each $\{Z_n^i\}_{n=1}^{\infty}$ that

$$P\{\sup_t|\textstyle\sum_{i=1}^r Z_n^i| > u\} \leq \textstyle\sum_{i=1}^r P\{\sup_t|Z_n^i| > u/r\} \to 0$$

uniformly in n as $u \to \infty$. Thus we see that $\sum_{i=1}^r Z_n^i$ satisfies (i) of Theorem 15.2, [4]. It remains to show that $\sum_{i=1}^r Z_n^i$ satisfies (ii) of Theorem 15.2, [4], i.e. that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that:

$$P\{w'_{\sum_{i=1}^r Z_n^i}(\delta) \geq \varepsilon\} < \varepsilon. \tag{2.14}$$

As above we apply Theorem 15.2, (ii) of [4] to each sequence $\{Z_n^i\}_{i=1}^{\infty}$, $i = 1, \cdots, r$ and get that for each $\varepsilon > 0$ there exist a $\delta > 0$ and a partition $\{t_j\}$ of [0,1] which contains the points $\{s_i\}$ and which satisfies $t_{j+1} - t_j > \delta$ for all j, such that

$$P\{\max_j \sup_{t_j \leq s \leq t < t_{j+1}}|Z_n^i(t) - Z_n^i(s)| \geq \varepsilon/r\} \leq \varepsilon/r.$$

This implies inequality (2.14), thus completing the proof. □

3. A maximal inequality and construction of the stochastic integral, $0 < p < 2$. We construct the stochastic integral in the standard way except that we use different inequalities. To every random M satisfying (1.1)—(1.3) we associate a family of increasing σ-algebras $\mathscr{F}_t \subset \mathscr{F}$, $t \in [0,1]$, as follows: $\mathscr{F}_t$ is the σ-algebra generated by the random variables $M(A)$, $A \in \mathscr{B} \cap [0, t]$ and by the sets of P-probability measure zero. A stochastic process $f(t, \omega)$, $t \in [0,1]$, $\omega \in \Omega$, is non-anticipating with respect to M if it is jointly measurable and if for each $t \in [0,1]$, the random variable $f(t, \omega)$ is $\mathscr{F}_t$-measurable.

A stochastic process is said to be simple non-anticipating if there exists a partition $0 = t_1 < \cdots < t_{r+1} = 1$ and random variables $\{\alpha_i\}$, $i = 0, \cdots, r$ such that α_0 is a.s. constant and α_i is $\mathscr{F}_{t_i}$-measurable, $i = 1, \cdots, r$, and

$$f(t, \omega) = \alpha_0(\omega)I_{\{0\}}(t) + \textstyle\sum_{i=1}^r \alpha_i(\omega)I_{(t_i,t_{i+1}]}(t), \tag{3.1}$$

where I_A indicates the characteristic function of the set A.

The random measure M is based on a finite positive measure m called the control measure of M. To each such M and m we will associate a class of stochastic processes denoted by $\mathscr{M}(M, m)$ as follows:

3.1. DEFINITION. A stochastic process $f(t, \omega)$, $t \in [0,1]$, belongs to the class $\mathscr{M}(M, m)$ if it is non-anticipating with respect to M and if there exists a sequence $\{f_n\}$ of simple non-anticipating processes such that:

$$\lim_{n\to\infty} \int_0^1 E|f_n - f|^p dm = 0,$$

where m and p are related to M by (1.2) and (1.3).

3.2. REMARK. If m is a continuous measure on [0,1] then $\mathscr{M}(M, m)$ is precisely the class of non-anticipating processes with respect to M such that:

$$\int_0^1 E|f|^p dm < \infty$$

(or equivalently, $f \in L^p(P \times m)$). The proof of this assertion follows that of [16], Lemma 4.4, by replacing 2 by p, Lebesgue measure by m, and the classical Lebesgue differentiation theorem by the following generalization: if $g \in L^1(m)$ then

$$\lim_{n\to\infty}(m(k - 1/n, t])^{-1} \int_{t-1/n}^{t} g\, dm = g(t) \quad \text{a.e.}$$

The last statement can be proved by classical arguments using a generalized Vitali covering theorem as given in [13] or [10]. However, if m has an atom at t and α is a bounded $\mathscr{F}_t$-measurable random variable, then $\alpha I_{\{t\}} \in L^p(P \times m)$ and is non-anticipating, but it may not belong to $\mathscr{M}(M, m)$. If m is any finite positive measure with $m\{0\} = 0$, then the class $\mathscr{M}(M, m)$ is still large enough to contain all the previsible processes in $L^p(P \times m)$. (A process is previsible if it is measurable with respect to the σ-algebra of $[0,1] \times \Omega$ generated by the non-anticipating left continuous processes).

For f simple non-anticipating, given by (3.1), the stochastic integral of f with respect to M is

$$\int_0^t f(t, \omega) M(dt, \omega) = \alpha_0(\omega) M(\{0\}) + \textstyle\sum_{i=1}^{k-1} \alpha_i(\omega) M((t_i, t_{i+1}])$$

(3.2) $$+ \alpha_k(\omega) M((t_k, t]),\ t \in (t_k, t_{k+1}], \qquad k = 1, \cdots, r,$$

$$\int_{\{0\}} f(t, \omega) M(dt, \omega) = \alpha_0(\omega) M(\{0\}).$$

We denote this process by $\int_0^t f\, dM$. We choose a separable version of M such that the sample paths of the process $X(t) = M[0, t]$ are a.s. in $D[0, 1]$. Then clearly $F = F(t) = \int_0^t f\, dM$ is a.s. in $D[0,1]$ and is non-anticipating with respect to M.

We now prove the maximal inequality given in (1.7) for simple functions.

3.3. LEMMA. *Let M be a random measure on* [0,1] *satisfying* (1.1) *and* (1.3), *and let f be a simple non-anticipating process as given in* (3.1). *Then, if* (1.6) *holds, so does* (1.7), *with* $c_p = c(1 + 8(2 - p)^{-1})^{1/p}$.

PROOF. The process F is separable (it is a.s. in $D[0,1]$). Therefore

(3.3) $$P\left\{\sup_{0\le t\le 1}\left|\int_0^t f\, dM\right| > u\right\} = \lim_{\{s_i\}\uparrow} P\left\{\max_i \left|\int_0^{s_i} f\, dM\right| > u\right\}$$

where $[s_i\}$ is a finite set of elements of [0,1] increasing to a dense set in [0,1]. We can assume that each finite set $\{s_i\}$ contains 0, 1 and the points t_j where f has jumps. This assumption implies that

$$\int_{s_i}^{s_{i+1}} f\, dM = \beta_i M(s_i, s_{i+1}]$$

where β_i, which is equal to some α_j, is $\mathscr{F}_{s_i}$-measurable, in particular independent of $M((s_i, s_{i+1}])$. Let $0 = s_0 = s_1 < s_2 < \cdots < s_{m+1} = 1$ be one of these sets $\{s_i\}$, set $\beta_0 = \alpha_0$, $\beta_i = \alpha_j$ if $(s_i, s_{i+1}] \subseteq (t_j, t_{j+1}]$, $i = 1, \cdots, m$, $M_0 = M(\{0\})$, and $M_i = M((s_i, s_{i+1}])$, $i = 1, \cdots, m$. Then

(3.4) $$\int_0^{s_\ell} f dM = \textstyle\sum_{i=0}^{\ell-1} \beta_i M_i, \qquad \ell = 1, \cdots, m + 1,$$

$$\int_0^1 E|f|^p\,dm = E|\beta_0|^p m\{0\} + \textstyle\sum_{i=1}^{m-1} E|\beta_i|^p m(s_i, s_{i+1}]. \tag{3.5}$$

By (3.3) and (3.4) it will be enough to show:

$$P\{\max_{0\le l\le m}|\textstyle\sum_{i=0}^{\ell}\beta_i M_i| > u\} \le c_p u^{-p}\displaystyle\int_0^1 E|f|^p dm, \qquad u>0. \tag{3.6}$$

Now, by intersecting the event on the left side of (3.6) with the set $\{\max_i|\beta_i M_i| > u\}$ and with its complement, we obtain

$$\begin{aligned} &P\{\max_{0\le\ell\le m}|\textstyle\sum_{i=0}^{\ell}\beta_i M_i| > u\} \\ &\le \textstyle\sum_{i=0}^{m} P\{|\beta_i M_i| > u\} + P\{\max_{0\le\ell\le m}|\textstyle\sum_{i=0}^{\ell}\beta_i M_i I_{[|\beta_i M_i|\le u]}| > u\}, \qquad u>0. \end{aligned} \tag{3.7}$$

Since β_i is independent of M_i, Fubini's theorem together with (1.3) and (3.5) give

$$\textstyle\sum_{i=0}^{m} P\{|\beta_i M_i| > u\} \le c^p u^{-p}\displaystyle\int_0^1 E|f|^p dm, \quad u>0. \tag{3.8}$$

Consider now the following family of random variables and associated σ-algebras:

$$\{\textstyle\sum_{i=0}^{\ell}\beta_i M_i I_{[|\beta_i M_i|\le u]}, \mathscr{F}_{s_{\ell+1}}\}_{\ell=0}^{m}, \tag{3.9}$$

where u is fixed. M_ℓ is symmetric and independent of $\mathscr{F}_{s_\ell}$, and β_ℓ is $\mathscr{F}_{s_\ell}$-measurable; therefore it follows that

$$E(\beta_\ell M_\ell I_{[|\beta_\ell M_\ell|\le u]}|\mathscr{F}_{s_\ell}) = 0.$$

This shows that the family (3.9) is a martingale. Furthermore, since the set $\{s_i\}$ has $m+1$ elements, we have

$$\sup_{0\le\ell\le m}|\textstyle\sum_{i=0}^{\ell}\beta_i M_i I_{[|\beta_i M_i|\le u]}| \le (m+1)u.$$

Therefore (3.9) is a square integrable martingale, and the standard maximal inequality (see e.g. [16], Theorem 2.4) gives

$$P\{\max_{0\le\ell\le m}|\textstyle\sum_{i=0}^{\ell}\beta_i M_i I_{[|\beta_i M_i|\le u]}| > u\} \le 4u^{-2}E(\textstyle\sum_{i=0}^{m}\beta_i M_i I_{[|\beta_i M_i|\le u]})^2. \tag{3.10}$$

For $i<j$, M_j is independent of β_i, M_i and β_j, so that by symmetry,

$$E\beta_i M_i I_{[|\beta_i M_i|\le u]}\beta_j M_j I_{[|\beta_j M_j|\le u]} = 0.$$

Hence, inequality (3.10) becomes

$$P\{\max_{0\le\ell\le m}|\textstyle\sum_{i=0}^{\ell}\beta_i M_i I_{[|\beta_i M_i|\le u]}| > u\} \le 4u^{-2}\textstyle\sum_{i=0}^{m} E\beta_i^2 M_i^2 I_{[|\beta_i M_i|\le u]}. \tag{3.11}$$

Note that β_i and M_i are independent. Let β_i be defined on the probability space $(\Omega_1, \mathscr{F}_1, P_1)$ and M_i be defined on the probability space $(\Omega_2, \mathscr{F}_2, P_2)$. Then

$$\begin{aligned} E\beta_i^2 M_i^2 I_{[|\beta_i M_i|\le u]} &= \int_{\Omega_1}\int_{\Omega_2}\beta_i^2(\omega_1)M_i^2(\omega_2)I_{[|\beta_i(\omega_1)M_i(\omega_2)|\le u]}(\omega_1,\omega_2)\,P_2(d\omega_2)\,P_1(d\omega_1) \\ &= \int_{\{\beta_i\ne 0\}}\beta_i^2(\omega_1)\int_{\Omega_2} M_i^2(\omega_2)I_{[|M_i(\omega_2)|\le u/|\beta_i(\omega_1)|]}(\omega_1,\omega_2)\,P_2(d\omega_2)\,P_1(d\omega_1). \end{aligned} \tag{3.12}$$

The interior integral in the expression immediately above is simply

$$\int_0^{u/|\beta_i(\omega_1)|} t^2\,dP_2(|M_i|\le t) \le 2\int_0^{u/|\beta_i(\omega_1)|} tP_2(|M_i|>t)\,dt$$

$$\leq 2c^p m(s_i, s_{i+1}] \int_0^{u/|\beta_i(\omega_1)|} t^{1-p}\, dp$$

$$= 2c^p m(s_i, s_{i+1}](2-p)^{-1} u^{2-p}/|\beta_i(\omega_1)|^{2-p},$$

where we use integration by parts and (1.3). Thus the left side of (3.12) is bounded from above by

$$2c^p(2-p)^{-1}u^{2-p}E|\beta_i|^p m(s_i, s_{i+1}].$$

Using this and (3.11) we get

$$P\{\max_{0\leq \ell \leq m} |\textstyle\sum_{i=0}^{\ell} \beta_i M_i I_{[|\beta_i M_i|\leq u]}| > u\} \leq 8c^p(2-p)^{-1}u^{-p} \displaystyle\int_0^1 E|f|^p\, dm, \quad u>0. \tag{3.13}$$

Finally, we get (3.6) by using (3.8) and (3.13) in (3.7). □

Having established Lemma 3.3 we can proceed to define the stochastic integral by standard arguments. If $\{f_n\}$ is a Cauchy sequence in $L^p(P \times m)$ of simple non-anticipating processes then by Lemma 3.3 and the Borel-Cantelli lemma there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that the sequence of processes

$$\left\{\int_0^t f_{n_k}\, dM\right\}_{k=1}^{\infty}$$

converges uniformly in $t \in [0,1]$ a.s. Also, clearly, all such subsequences that converge have the same limit. This limiting process is obviously adapted to $\{\mathscr{F}_t\}$, has almost all its sample paths in $D[0,1]$ and satisfies (1.7). A consequence of this is that if $f_n \to 0$ in $L^p(P \times m)$ then for every subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{\int_0^t f_{n_k}\, dM\}$ converges uniformly a.s., the limit is zero.

3.4 Definition. Let M be a random measure satisfying (1.1) and (1.3) and let $f \in \mathscr{M}(M, m)$. Let $\{f_n\}$ be a sequence of simple non-anticipating functions such that $f_n \to f$ in $L^p(P \times m)$ and such that $\{\int_0^t f_n\, dM\}$ converges uniformly a.s. Then the stochastic integral of f with respect to M, denoted as in (1.5), is defined as the limit

$$F = F(t) = \int_0^t f\, dM = \lim_{n\to\infty} \int_0^t f_n\, dM \text{ a.s.}, \qquad t \in [0,1].$$

We sum up the previous remarks in the following theorem.

3.5 Theorem. *Let M be a random measure satisfying* (1.1) *and* (1.3) *and let $f \in \mathscr{M}(M, m)$. Then the process $\int_0^t f\, dM$ is adapted to $\{\mathscr{F}_t\}$, has a version with almost all its sample paths in* $D[0,1]$ *and satisfies the maximal inequality* (1.7) (*where c_p is given in Lemma* 3.3).

The next lemma will be useful when we consider the central limit theorem.

3.6 Lemma. *Let $\{M^j, f^j\}_{j=1}^{\infty}$ be independent copies of (M, f) where M is a random measure satisfying* (1.1) *and* (1.3), *and $f \in \mathscr{M}(M, m)$. Then for every $n \in N$,*

$$\Lambda_p\left(\left\| n^{-1/p} \textstyle\sum_{j=1}^{n} \displaystyle\int_0^t f^j\, dM^j \right\|\right) \leq c_p'\left(\int_0^1 E|f|^p\, dm\right)^{1/p}, \tag{3.14}$$

where $c_p' = c_p[(4-p)/(2-p)]^{1/p}$.

Proof. It is enough to prove (3.14) for simple non-anticipating processes. The proof is identical to that of Lemma 3.3 if one takes into account the following two observations.

If (β_i^j, M_i^j) are independent copies of $(\beta_i\, M_i)$ in Lemma 3.3, then (2.7) and (1.3) give

$$\sum_{i=0}^m P\{|\, n^{-1/p} \sum_{j=1}^n \beta_i^j M_i^j \,| > u\} \le c^p(4-p)(2-p)^{-1}u^{-p}\int_0^1 E\,|\, f\,|^p\, dm,$$

$$u > 0, \quad n = 1, \cdots.$$

(This inequality plays the role of inequality (3.8) in this new situation). The second observation is that, assuming that the random measures M^j (together with the processes f^j) are defined on a product probability space, and letting $\{\mathscr{F}_i^j\}$ be the associated σ-algebras, the family of random variables

$$\{\textstyle\sum_{i=0}^{\ell} n^{-1/p}(\sum_{j=1}^n \beta_i^j M_i^j I_{[|\beta_i^j M_i^j| \le u]})\}_{\ell=0}^m$$

is a martingale with respect to the family of σ-algebras

$$\{\mathscr{F}^1_{s_{\ell+1}} \otimes \cdots \otimes \mathscr{F}^n_{s_{\ell+1}}\}_{\ell=0}^m.$$

This fact allows for an inequality similar to (3.10) for the truncated sums, and the proof of (3.14) now follows as in Lemma 3.3. □

We will now give some examples of random measures M that satisfy (1.1)—(1.3). Of course one obvious example is that M is itself a stable measure of index p with control measure m. However, the class we are concerned with is larger than this.

3.7. Example. Let μ be a symmetric Lévy measure on $\mathbb{R}$, i.e. $\mu\{0\} = 0$ and $\int \min(1, x^2)\, d\mu(x) < \infty$. Asume that

$$\sup_t t^p \mu[-t, t]^c < \infty \tag{3.15}$$

and

$$\lim_{t\to\infty} t^p \mu[-t, t]^c = \alpha, \tag{3.16}$$

where $\alpha = 2p^{-1}(\int_{-\infty}^{\infty} (1-\cos u)\,|\, u\,|^{-1-p}\, du)^{-1}$. Let m be a finite positive measure on [0, 1] and let M be a random measure with independent increments given by

$$E\exp(itM(A)) = \exp\left(m(A)\int_{-\infty}^{\infty} (\cos tx - 1)\, d\mu(x)\right), \qquad -\infty < t < \infty, \quad A \in \mathscr{B}. \tag{3.17}$$

Such a measure exists by the Kolmogorov Extension Theorem, and satisfies (1.1)-(1.3).

Proof. M satisfies (1.1) by definition. To see that M satisfies (1.2) let M^i be independent copies of M and let $A \in \mathscr{B}$ satisfy $m(A) \ne 0$. The random variable $n^{-1/p}\sum_{i=1}^n M^i(A)$ has characteristic function

$$\exp\left(nm(A)\int_{-\infty}^{\infty} (\cos tx - 1)\, d\mu_n(x)\right)$$

where

$$\mu_n(B) = \mu(n^{1/p}B), \qquad B \in \mathscr{B}.$$

By (3.16),

$$nm(A)\mu_n[-\delta, \delta]^c = nm(A)\mu[-\delta n^{1/p}, \delta n^{1/p}]^c \to \alpha\delta^{-p}m(A). \tag{3.18}$$

By (3.15)

$$n\int_{|x|\le\delta} x^2\, d\mu_n(x) \le c\delta^{2-p}$$

for some constant c (by integration by parts). Thus

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} n \int_{|x|\le\delta} x^2\, d\mu_n(x) = 0. \tag{3.19}$$

Using (3.18) and (3.19) in [2], Chapter 2, Theorem 5.5, we see that $M(A)$ is in the domain of normal attraction of $(m(A))^{1/p}\theta$, i.e. M satisfies (1.2). (Note that the case $m(A) = 0$ is trivial).

In order to show that M satisfies (1.3), consider the random measures M_k, $k = 1, \cdots$, given by characteristic functions

$$E \exp(itM_k(A)) = \exp\left[m(A) \int_{[-k^{-1},k^{-1}]^c} (\cos tx - 1)\, d\mu(x) \right], \qquad -\infty < t < \infty, \qquad A \in \mathscr{B}.$$

Then, for every $A \in \mathscr{B}$ such that $m(A) \neq 0$,

$$\mathscr{L}(M_k(A)/(m(A))^{1/p}) \to_w \mathscr{L}(M(A)/(m(A))^{1/p}) \tag{3.20}$$

(use characteristic functions). So, it will be enough to show that

$$\Lambda_p(M_k(A)/(m(A))^{1/p}) \le c \tag{3.21}$$

for some constant $c < \infty$ independent of A and k. Letting $\mu_k = \mu|_{[-k^{-1},k^{-1}]^c}$, we see that the law of $M_k(A)$ is given by the probability measure

$$e^{-m(A)\mu_k(\mathbb{R})} \textstyle\sum_{n=0}^{\infty} (m(A))^n \mu_k^n/n!$$

where $\mu_k^n = \mu_k * \overset{(n)}{\cdots} * \mu_k$. Then, for any $t > 0$,

$$\begin{aligned} t^p P\{|M_k(A)| > t(m(A))^{1/p}\} \\ = t^p e^{-m(A)\mu_k(\mathbb{R})} \textstyle\sum_{n=1}^{\infty} (m(A))^n \mu_k^n\{|x| > t(m(A))^{1/p}\}/n!. \end{aligned} \tag{3.22}$$

If $\{\eta_i\}$ are independent random variables with law $\mu_k/\mu_k(\mathbb{R})$ then by (3.15) there is a constant $r < \infty$ independent of k such that

$$u^p P\{|\eta_i| > u\} \le r/\mu_k(\mathbb{R}), \quad u > 0.$$

Hence, by Lemma 2.1,

$$u^p \mu_k^n\{|x| > u\} = u^p(\mu_k(\mathbb{R}))^n P\{|\textstyle\sum_{i=1}^{n}\eta_i| > u\} \le r'n(\mu_k(\mathbb{R}))^{n-1}, \qquad u > 0$$

where $r' = r((4-p)/(2-p))$. Using this in (3.22) we get that the left side of (3.22) is dominated by

$$e^{-m(A)\mu_k(\mathbb{R})} \textstyle\sum_{n=1}^{\infty} r'(m(A)\mu_k(\mathbb{R}))^{n-1}/(n-1)! = r'.$$

From this, (3.21) follows with $c = (r')^{1/p}$. □

3.8. REMARK. Let M be a random measure satisfying

$$E \exp(itM(A)) = \exp\left[-\tau(A)t^2/2 + \sigma(A) \int_{-\infty}^{\infty} (\cos tx - 1)\, d\mu(x) \right], \quad -\infty < t < \infty, \quad A \in \mathscr{B}, \tag{3.22}$$

where τ and σ are finite positive measures on [0,1] and μ is a symmetric Lévy measure. In order that M satisfy (1.2) it is necessary that $\tau = 0$, $\sigma = m$ and that μ satisfy Condition (3.16). Condition (3.15) seems to be the right condition in order to handle small values of $m(A)$ in (1.3), and it may even be necessary for (1.3) to hold if m has an infinite support. These remarks follow from well known facts on infinitely divisible measures and Theorem 5.5, Chapter 2 in [2].

3.9. EXAMPLE. The next example is simpler. Let $m = \sum_{i=1}^{\infty} m_i i \delta_{t_i}$, where m_i are positive numbers and δ_{t_i} point masses at t_i, be a finite discrete measure on [0,1]. Let $\{\eta_i\}$ be symmetric independent identically distributed random variables in the domain of normal attraction of θ. Then the random measure

$$M(A) = \sum_{t_i \in A} m_i^{1/p} \eta_i \tag{3.24}$$

satisfies (1.1)—(1.3).

Finally, the next example gives the form of some stochastic integrals.

3.10. EXAMPLE. (1) Let M be a random measure of the type considered in 3.8, with $p \in (0,1)$. Then the process $X(t) = M[0,t]$ has almost all its sample paths of bounded variation (as $\int_{|x| \le 1} |x| \, d\mu(x) < \infty$) and pure jump. In this case the stochastic integral can be computed pathwise and we have that for every continuous function f on [0,1],

$$\int_0^t f(M[0,s)) \, dM(s) = \sum_{0 \le s \le t} f(M[0,s)) M\{s\}.$$

(2) If M is as in Remark 3.8 but $p \in [1, 2)$, then we can use a generalization of Itô's formula to compute integrals with respect to M ([3], Theorem 5.8 and Remark 5.9). In this case $X(t)$ is still pure jump (but not of bounded variation) and Itô's formula in [3] gives that for any C^1 function f on [0,1],

$$\int_0^t f(M[0,s)) \, dM(s) = \int_{M\{0\}}^{M[0,t]} f(s) \, ds - \sum_{0 \le s \le t} \left[\int_{M[0,s)}^{M[0,s]} f(u) \, du - f(M(0,s)) M\{s\} \right].$$

4. Central limit theorem in $D[0,1]$, $0 < p < 2$. The first step in proving the central limit theorem in $D[0,1]$ for the stochastic integral $\{F(t), t \in [0,1]\}$ defined in (1.5) and in Definition 3.4 is to show that the finite dimensional distributions of $\{\int_0^t f \, dM, t \in [0,1]\}$, for f simple non-anticipating, satisfy the central limit theorem. This is done in the next two lemmas.

4.1. LEMMA. *Let M be a random measure satisfying* (1.1) *and* (1.2) *and let f be a simple non-anticipating process. Then the random variable $\int_0^1 f \, dM$ is in the domain of normal attraction of $(\int_0^1 E|f|^p \, dm)^{1/p}\theta$, where θ is given in* (1.2).

PROOF. It is well known that random variables ξ_i are in the domain of normal attraction of symmetric stable random variables of index p if and only if

$$\lim_{t \to \infty} t^p P\{\xi_i > t\} = c_i < \infty \tag{4.1}$$

and (4.1) holds also for $-\xi_i$. Let ξ_1 and α_1 be independent of ξ_2. Let $\xi_1, -\xi_1, \xi_2$ and $-\xi_2$ satisfy (4.1) for $i = 1, 2$, where c_1 and c_2 need not be equal, and assume $E|\alpha_1|^p < \infty$. Then we will show that

$$\lim_{t \to \infty} t^p P\{\xi_1 + \alpha_1 \xi_2 > t\} = c_1 + c_2 E|\alpha_1|^p \tag{4.2}$$

and that (4.2) holds also for $-(\xi_1 + \alpha_1 \xi_2)$. Once we have (4.2) the proof of this lemma follows by iteration (since f is simple). The proof of (4.2) is a generalization of an argument of Feller ([6], page 278). For one direction of the inequality we use that for $0 < \varepsilon < 1$ and all $t > 0$,

$$t^p P\{\xi_1 + \alpha_1 \xi_2 > t\} \le t^p P\{\xi_1 > t(1-\varepsilon)\} + t^p P\{\alpha_1 \xi_2 > t(1-\varepsilon)\} + t^p P\{\xi_1 > t\varepsilon, \alpha_1 \xi_2 > t\varepsilon\}.$$

The limit of the first term at the right side as $t \to \infty$ is $c_1/(1-\varepsilon)^p$ by hypothesis. By independence of ξ_2 and α_1 we can also easily conclude that the limit of the second term is $c_2E|\alpha_1|^p/(1-\varepsilon)^p$. As for the third, we note that since $t^pP_2\{\alpha_1\xi_2>t\varepsilon\}\le c\varepsilon^p|\alpha_1|^p$ for some $c<\infty$ and all t, the random variables $\{I_{[\xi_1>t\varepsilon]}t^pP_2\{\alpha_1\xi_2>t\varepsilon\},\ t>0\}$ are dominated by an integrable random variable and therefore, by the dominated convergence theorem,

$$\lim_{t\to\infty}t^pP\{\xi_1>t\varepsilon,\ \alpha_1\xi_2>t\varepsilon\}=\lim_{t\to\infty}E_1I_{[\xi_1>t\varepsilon]}t^pP_2\{\alpha_1\xi_2>t\varepsilon\}=0.$$

(As usual we assume here that ξ_1, α_1, ξ_2 are defined in a product probability space and E_i, P_i denote expectation and probability, respectively, in the ith factor space, $i=1,2$). Collecting these results and letting $\varepsilon\to 0$ we get that

$$\limsup_{t\to\infty}t^pP\{\xi_1+\alpha_1\xi_2>t\}\le c_1+c_2\,E|\alpha_1|^p,$$

which is half of (4.2). For the reverse inequality we use the fact that

$$t^pP\{\xi_1+\alpha_1\xi_2>t\}\ge t^pP\{\xi_1>t(1+\varepsilon),\ |\alpha_1\xi_2|<\varepsilon t\}+t^pP\{\alpha_1\xi_2>t(1+\varepsilon),\ |\xi_1|<\varepsilon t\}.$$

The first term at the right side equals

$$E_1t^pI_{[\xi_1>t(1+\varepsilon)]}-E_1I_{[\xi_1>t(1+\varepsilon)]}t^pP_2\{|\alpha_1\xi_2|>\varepsilon t\},$$

which obviously tends to $c_1/(1+\varepsilon)^p$ as $t\to\infty$ by dominated convergence. The same type of argument shows that the limit of the second term is $c_2E|\alpha_1|^p/(1+\varepsilon)^p$. The second half of (4.2) is now obtained by letting $\varepsilon\to 0$. Clearly (4.2) holds also for $-(\xi_1+\alpha_1\xi_2)$. □

Consider the process $\int_0^t f\,dM$ where f and M are as in Lemma 4.1. The finite dimensional distributions of this process are determined by the laws of the random variables

$$\textstyle\sum_{j=1}^k a_j\int_0^{t_j} f\,dM=\int_0^1 [\sum_{j=1}^k a_jfI_{[t\le t_j]}]\,dM,$$

$\{a_j\}_{j=1}^k\subset\mathbb{R}$, $\{t_j\}_{j=1}^k\subset[0,1]$, $k=1,\cdots$. By Lemma 4.1 this random variable is in the domain of normal attraction of the stable random variable

$$\left(\int_0^1 E|\textstyle\sum_{j=1}^k a_jfI_{[t\le t_j]}|^p\,dm\right)^{1/p}\theta. \tag{4.3}$$

We will now exhibit a stable stochastic process of index p, $\{S(t),\ t\in[0,1]\}$, such that

$$\mathcal{L}(\textstyle\sum_{j=1}^k a_jS_{t_j})=\mathcal{L}\left(\left(\int_0^1 E|\sum_{j=1}^k a_jfI_{[t\le t_j]}|^p\,dm\right)^{1/p}\theta\right).$$

Let $(\Omega',\mathcal{F}',P')$ be an independent copy of $(\Omega,\mathcal{F},P)$, let

$$(V,\textstyle\sum,\nu)=([0,1]\times\Omega',\mathcal{B}\times\mathcal{F}',m\times P')$$

and let $\bar M$ be a symmetric stable measure of index p on $\sum$, with independent increments and with control measure ν. That is, if we let A, $A_i\in\sum$, then $\bar M$ satisfies (1.1), and instead of (1.2) we have more precisely, $\mathcal{L}(\bar M(A))=\mathcal{L}((\nu(A))^{1/p}\theta)$. ((1.3) is satisfied automatically). For the same process f as in (1.1) we consider $f(u,\omega')$ as a deterministic function on V and define

$$S=S_t=S(t,\omega)=\int_V I_{[u\le t]}(u)f(u,\omega')\bar M(d(u,\omega'),\omega), \tag{4.4}$$

where $d(u,\omega')$ indicates that we are integrating on the product space V. Note that $I_{[u\le t]}f(u,\omega')$ is a deterministic function relative to $(\Omega,\mathcal{F},P)$ and (4.4) is a Wiener type integral, not an Itô type integral as we have been considering. Consequently it is much easier to define. The definition of (4.4) proceeds in the standard fashion from Lemma 2.1. For further details see [8], Section 3. From the definition of (4.4) it is easy to show that

$$(4.5)\quad \begin{aligned} &E\exp\{iu\textstyle\sum_{j=1}^k a_jS_{t_j}\}\\ &\quad= \exp\left\{-|u|^p\displaystyle\int_V |\textstyle\sum_{j=1}^k a_jI_{[u\le t_j]}(u)f(u,\omega')|^p\,dm(u)\,dP(\omega')\right\}, \qquad -\infty<u<\infty.\end{aligned}$$

In view of (4.3) this gives us:

4.2. LEMMA. *Let M and f be as in Lemma* 4.1. *Then the finite dimensional distributions of the process $\int_0^t f\,dM$ are in the domain of normal attraction of the symmetric stable process S_t defined in* (4.4).

4.3. REMARK. The process $S(t,\omega)$ defined in (4.4) for a simple function f can be extended to all $f\in\mathscr{M}(M,m)$. Moreover, if $f_n\to f$ in $L^p(P\times m)$ and $S_n(t,\omega)$ and $S(t,\omega)$ are the corresponding processes given in (4.4), then the finite dimensional distributions of $S_n(t,\omega)$ converge weakly to the corresponding finite dimensional distributions of $S(t,\omega)$. This follows immediately from (4.5).

Define

$$(4.6)\quad \begin{cases} G_0(t) = M\{0\}I_{[0,1]}(t)\\ G_i(t) = M(t_i,t]I_{(t_i,t_{i+1}]}(t) + M(t_i,t_{i+1}]I_{(t_{i+1},1]}(t), \qquad i=1,\cdots,r,\end{cases}$$

where M satisfies (1.1)—(1.3). Then the stochastic integral defined in (3.2) can be written as

$$(4.7)\quad \int_0^t f\,dM = \textstyle\sum_{i=0}^r \alpha_iG_i(t).$$

We first consider the central limit theorem in $D[0,1]$ for $\alpha_iG_i(t)$, $E|\alpha_i|^p<\infty$; in fact we start with α_i bounded a.s. It is clear that $\alpha_0G_0(t)$ satisfies the central limit theorem (Lemma 4.1), thus we consider α_iG_i, $i\ne 0$.

4.4. LEMMA. *Let G_i be any of the processes defined by* (4.6) *and let α_i be independent of G_i and such that $E|\alpha_i|^p<\infty$. Let $\{\alpha_i^k, G_i^k\}$, $k=1,\cdots,$ be independent copies of (α_i,G_i). Then the sequence of probability measures on $D[t_i,t_{i+1}]$,*

$$(4.8)\quad \{\mathscr{L}(n^{-1/p}\textstyle\sum_{k=1}^n \alpha_i^kG_i^k|_{[t_i,t_{i+1}]})\}_{n=1}^\infty$$

is weakly convergent (*to the stable process determined by Lemma* 4.2).

PROOF. Let us first assume that α_i is bounded a.s. We use Theorem 2.3 with $[0,1]$ replaced by $[t_i,t_{i+1}]$. Theorem 2.3 (1) is satisfied by Lemma 4.2. We now check condition (2). By (4.6) $G_i|_{[t_i,t_{i+1}]} = M(t_i,t]I_{[t_i,t_{i+1}]}(t)$, $t\in[t_i,t_{i+1}]$ (i.e. $G_i(t_i)=0$). Let $\{M^k\}$ denote independent copies of M, independent of $\{\alpha_i^k\}$. Denote by E_α and P_M expectation with respect to $\{\alpha_i^k\}$ and probability measure with respect to $\{M^k\}$ respectively. Then by Lemma 2.1 and (1.3) we get that for $t_i\le u_1\le u\le u_2\le t_{i+1}$ and $\lambda>0$,

$$(4.9)\quad \begin{aligned} &P\{|n^{-1/p}\textstyle\sum_{k=1}^n \alpha_i^kM^k(u_1,u]|\ge\lambda,\ |n^{-1/p}\sum_{k=1}^n\alpha_i^kM^k(u,u_2]|\ge\lambda\}\\ &\quad= E_\alpha\{P_M[|n^{-1/p}\textstyle\sum_{k=1}^n\alpha_i^kM^k(u_1,u]|\ge\lambda]P_M[|n^{-1/p}\sum_{k=1}^n\alpha_i^kM^k(u,u_2]|\ge\lambda]\}\\ &\quad\le \lambda^{-2p}K_pm(u_1,u]m(u,u_2]E[n^{-1}\textstyle\sum_{k=1}^n|\alpha_i^k|^p]^2.\end{aligned}$$

Thus, since α_i is bounded, we see that the processes in (4.8) satisfy Theorem 2.3, (2), with $\alpha=1$, $\beta=2p$ and $F(t)=K'_pm(0,t]$, where K'_p depends only on α_i and p and is constant with respect to n.

Next we check condition (3) in Theorem 2.3. Let $\tilde M^k$ denote the random measure $\tilde M^k(A) = M^k(A) - M^k(\{t_{i+1}\})\delta_{t_{i+1}}(A)$, $A\in\mathscr{B}$. Let w_n be the modulus w_{X_n} defined in Section 2 for the processes in (4.8) (i.e. $X_n = n^{-1/p}\sum_{k=1}^n\alpha_i^kG_i^k|_{[t_i,t_{i+1}]}$).

Then we have

$$w_n[t_{i+1}-\delta, t_{i+1}) \le 2 \sup_{t_{i+1}-\delta\le t\le t_{i+1}} | n^{-1/p} \sum_{k=1}^n \alpha_i^k \tilde{M}^k(t_i, t] |.$$

Therefore, since $\tilde{M}^k$ satisfy (1.1)–(1.3) with control measure $\tilde{m} = m - m(\{t_{i+1}\})\delta_{t_{i+1}}$, Lemma 3.6 gives

$$P\{w_n[t_{i+1}-\delta, t_{i+1}) > \varepsilon\} \le 2^p \varepsilon^{-p}(c_p')^p m[t_{i+1}-\delta, t_{i+1}) E|\alpha_i|^p,$$

and this tends to zero uniformly in n as $\delta \to 0$. This proves Condition (2.12). As for (2.13), it is easy to see in a similar way that for $\delta < t_{i+1} - t_i$,

$$P\{| n^{-1/p} \sum_{k=1}^n \alpha_i^k M^k(t_i, t_i+\delta) | > \varepsilon\} \le \varepsilon^{-p} K_p m(t_i, t_i+\delta] E[n^{-1} \sum_{k=1}^n |\alpha_i^k|^p]$$

which tends to zero uniformly as $\delta \to 0$ (the α_i^k are bounded). By Theorem 2.3, this lemma is proved in the case of bounded α_i.

For general α_i (i.e. $E|\alpha_i|^p < \infty$), set $\alpha_{iN} = \alpha_i I_{[|\alpha_i|\le N]}$. By Lemma 3.6 we have

(4.10) $$\Lambda_p(\sup_{t\in[t_i,t_{i+1}]} | n^{-1/p} \sum_{k=1}^n (\alpha_i^k - \alpha_{iN}^k) G_i^k(t) |) \le c_p'(E|\alpha_i - \alpha_{iN}|^p m[t_i, t_{i+1}])^{1/p}.$$

We now use Lemma 2.2. We have already shown that $\{\mathscr{L}(n^{-1/p} \sum_{k=1}^n \alpha_{iN}^k G_i^k|_{[t_i,t_{i+1}]})\}_{n=1}^\infty$ is weakly convergent in $D[t_i, t_{i+1}]$. This gives (2.8) in Lemma 2.2, and (4.10) immediately gives (2.9). Thus, the weak convergence of the sequence (4.8) follows from Lemma 2.2. □

We can now prove the central limit theorem for the stochastic integrals of Definition 3.4.

4.5. THEOREM. *Let M be a random measure satisfying* (1.1)—(1.3) *for some finite positive measure m and $p \in (0,2)$. Let $f \in \mathscr{M}(M, m)$ and let S_t be the stable process of index p defined by* (4.4). *Then S_t has a version with almost all its sample paths in $D[0, 1]$ and the process $\int_0^t f\, dM$ is in the domain of normal attraction of S_t, i.e. if $\{f^k, M^k\}$ are independent copies of (f, M) then*

(4.11) $$w - \lim_{n\to\infty} \mathscr{L}\left(n^{-1/p} \sum_{k=1}^n \int_0^t f^k\, dM^k\right) = \mathscr{L}(S_t)$$

as probability measures on $D[0, 1]$.

PROOF. In view of Lemma 4.4 it follows from Lemma 2.4 that for every simple non-anticipating process f, if $\{f^k, M^k\}$ are independent copies of (f, M) then

$$\left\{\mathscr{L}\left(n^{-1/p} \sum_{k=1}^n \int_0^t f^k\, dM^k\right)\right\}_{n=1}^\infty$$

is a tight sequence of probability measures on $D[0, 1]$. Therefore if S is the process defined in (4.4) for this f, m and p then by Lemma 4.2 we have that

(4.12) $$\int_0^t f\, dM \in \mathrm{DNA}(S)$$

as $D[0, 1]$-valued random variables (we choose a version of S with sample paths in $D[0, 1]$).

Now let $f \in \mathscr{M}(M, m)$ and $\{f_i\}_{i=1}^\infty$ be simple non-anticipating functions converging to f in $L^p(P \times m)$. Denote by S_i and S the corresponding processes defined by (4.4) with f_i and f. Then, analogous to (4.12) we have

(4.13) $$\int_0^t f_i\, dM \in \mathrm{DNA}(S_i),\ i = 1, \cdots.$$

If $\{f_i^k, f^k, M^k\}$ are independent copies of $\{f_i, f, M\}$ then Lemma 3.6 gives

$$\lim_{i\to\infty}\sup_n\Lambda_p\left(\left\|n^{-1/p}\textstyle\sum_{k=1}^n \displaystyle\int_0^t (f_i^k - f^k)\, dM^k\right\|\right) \le \lim_{i\to\infty} c_p'\left(\int_0^t E\,|f_i - f|^p\, dm\right)^{1/p} = 0. \tag{4.14}$$

We use (4.13) and (4.14) in Lemma 2.2 with

$$Y_n(t) = n^{-1/p}\textstyle\sum_{k=1}^n \displaystyle\int_0^t f^k\, dM^k, \qquad t \in [0, 1]$$

and

$$Y_n^i(t) = n^{-1/p}\textstyle\sum_{k=1}^n \displaystyle\int_0^t f_i^k\, dM^k, \qquad t \in [0, 1]$$

to see that $\{\mathscr{L}(X_n)\}_{n=1}^\infty$ converges weakly and that its limit is $w - \lim_{i\to\infty}\mathscr{L}(S_i)$. By Remark 4.3 this limit is $\mathscr{L}(S)$. This completes the proof of the theorem. □

4.6. REMARK. All these results make sense in $D[0, \infty)$. Let m be a positive locally finite Borel measure on $[0, \infty)$ such that M and m satisfy (1.1) and (1.2) for some $p \in (0, 2)$ and all sets in $\mathscr{B}$ (the Borel σ-algebra of $[0, \infty)$). Instead of (1.3) we assume that for each $T < \infty$ there exists a constant c_T such that

$$\sup_{A\in\mathscr{B}\cap[0,T]}\Lambda(M(A)/(m(A))^{1/p}) \le c_T.$$

For any $T < \infty$, define $\mathscr{M}^T(M, m)$ by replacing [0, 1] in Definition 3.1 by $[0, T]$, and let $\mathscr{M}^\infty(M, m) = \bigcap_T \mathscr{M}^T(M, m)$. The process S_t can be defined on $[0, \infty)$ for all $f \in \mathscr{M}^\infty(M, m)$ as in (4.4). Thus if M and m are given as above and $f \in \mathscr{M}^\infty(M, m)$, it follows from Theorem 4.5 and Theorem 3 in [15] that $\int_0^t f\, dM$ belongs to the domain of normal attraction of S_t as random variables with values in $D[0, \infty)$. Note that $D[0, \infty)$ is the set of all real valued functions on $[0, \infty)$ which are right continuous and have left limits at all $t \in [0, \infty)$, equipped with a topology that extends the Skorokhod topology on $D[0, T]$ in a natural way (see e.g. [15] and the references there, in particular C. Stone's work where $D[0, \infty)$ was introduced and weak convergence on this space was first considered).

4.7. REMARK. When $0 < p < 1$ and the random measure M is given by

$$E\exp(itM(A)) = \exp\left[m(A)\int_0^\infty (e^{itx} - 1)\, d\mu(x)\right], \qquad -\infty < t < \infty, \qquad A \in \mathscr{B}, \tag{4.15}$$

where $t^p\mu\{x > t\} \to k$ and $\sup_{t>0} t^p\mu\{x > t\} < \infty$, then we can considerably simplify the definition of the stochastic integral $\int_0^t f\, dM$, $f \in \mathscr{M}(M, m)$ and the proof that it satisfies the CLT in $D[0, 1]$. First of all, if $\{\eta_k\}$ are independent and satisfy (2.6), then (2.7) can be strengthened to

$$\Lambda_p(\textstyle\sum_{k=1}^n |a_k\eta_k|) \le \alpha\left(\dfrac{2-p}{1-p}\right)^{1/p}(\textstyle\sum_{k=1}^n |a_k|^p)^{1/p}, \tag{4.16}$$

using, basically, the same proof as that of Lemma 2.1. Also, we note that $M(A)$ as defined in (4.15), is nonnegative for all $A \in \mathscr{B}$. The proof of Example 3.7 shows that M satisfies (1.3), and (4.16) readily gives

$$\Lambda_p\left(\left\|\int_0^t f\, dM\right\|\right) \le \Lambda_p\left(\left\|\int_0^t |f|\, dM\right\|\right) = \Lambda_p\left(\int_0^1 |f|\, dM\right) \le \alpha\left(\frac{2-p}{1-p}\right)^{1/p}\left(\int_0^1 E|f|^p\, dm\right)^{1/p} \tag{4.17}$$

for f simple non-anticipating. This inequality extends to all $f \in \mathscr{M}(M, m)$ and defines the stochastic integral. The proof of the CLT follows as in the general case. The absence of symmetry introduces no centering problem: the centering functions in the CLT can be taken to be zero (see e.g. [6], page 546).

5. Central limit theorem in $D[0, 1]$ and $C[0, 1]$, $p = 2$. Let M be a random measure satisfying (1.9) and (1.10). The family of functions $f \in \mathscr{M}(M, m)$ is defined exactly as in Definition 3.1, but with $p = 2$. For f simple nonanticipating we already have

$$E\left(\int_0^t f\,dM\right)^2 = \int_0^t E|f|^2\,dm, \qquad t \in [0, 1], \tag{5.1}$$

and since the stochastic integral is a square integrable martingale with right continuous trajectories, it satisfies (1.11) ([16], Theorem 3.2). Thus the stochastic integral can be defined for general $f \in \mathscr{M}(M, m)$ as in the case $p \in (0, 2)$ and the resulting process is a square integrable martingale with respect to $\{\mathscr{F}_t\}$ which satisfies (5.1) and (1.11) and has a version with almost all of its sample paths in $D[0, 1]$.

We now show that F satisfies the central limit thorem in $D[0, 1]$. The identity (5.1) implies that the finite dimensional distributions of $\int_0^t f\,dM$ belong to the domain of normal attraction of the Gaussian process

$$S = S_t = S(t, \omega) = \int_V I_{[u \le t]}(u) f(u, \omega')\bar{M}(d(u, \omega'), \omega) \tag{5.2}$$

where everything is as in (4.4) except that $\mathscr{L}(\bar{M}(A)) = \mathscr{L}((\nu(A))^{1/2}g)$, where g is a normal random variable with mean zero and variance one. Now note that if one defines $\Lambda_2(\xi)$ exactly as in (1.4) then it follows from Chebyshev's inequality and (1.11) that

$$\Lambda_2\left(\left\|\int_0^t f\,dM\right\|\right) \le 2\left(\int_0^1 E|f|^2\,dm\right)^{1/2}.$$

Thus, Lemma 4.4 and Theorem 4.5 are valid also for $p = 2$. We get:

5.1. THEOREM. *Let M be a random measure satisfying* (1.9) *and* (1.10) *for some finite positive measure m. Let $f \in \mathscr{M}(M, m)$, $p = 2$, and let S_t be the Gaussian process defined in* (5.2). *Then S_t has a version with almost all its sample paths in* $D[0,1]$ *and the process $\int_0^t f\,dM$ is in the domain of normal attraction of S_t, i.e. if $\{f^k, M^k\}$ are independent copies of (f, M) then*

$$w - \lim_{n\to\infty} \mathscr{L}\left(n^{-1/2} \textstyle\sum_{k=1}^n \displaystyle\int_0^t f^k\,dM^k\right) = \mathscr{L}(S_t) \tag{5.3}$$

as probability measures on $D[0, 1]$.

5.2. REMARK. Remark 4.6 on the CLT in $D[0, \infty)$ applies also in this case.

A well known result of P. Lévy is that a stochastic process with independent increments which is continuous in probability is sample continuous if and only if it is Gaussian. If m is continuous, (5.1) shows that M is continuous in probability and therefore if $X(t) = M[0, t]$ is sample continuous, then M must be a Gaussian measure and this is the only example of a random measure M satisfying (1.1)–(1.3) or (1.9)–(1.10) such that $M[0, t]$ is sample continuous. In this case, by construction, the process $\int_0^t f\,dM$ is also sample continuous. We will now show that it satisfies the CLT in $C[0, 1]$.

5.3. THEOREM. *Let M be a random measure satisfying* (1.9) *and* (1.10) *for some finite positive continuous measure m on* $[0, 1]$ *such that* $X(t) = M[0, t]$ *is a sample continuous*

(Gaussian) process. Let $f \in \mathscr{M}(M, m)$, $p = 2$, and let S_t be the Gaussian process defined in (5.2). *Then S_t has a version with almost all its sample paths in $C[0, 1]$ and the process $\int_0^t f\, dM$ is in the domain of normal attraction of S_t, i.e. if $\{f^k, M^k\}$ are independent copies of (f, M) then*

$$w - \lim_{n\to\infty} \mathscr{L}\left(n^{-1/2} \sum_{k=1}^n \int_0^t f^k\, dM^k\right) = \mathscr{L}(S_t) \tag{5.4}$$

as probability measures on $C[0, 1]$.

PROOF. We have already established weak convergence of the finite dimensional distributions. We can continue to trace all the steps of the proof of $D[0, 1]$ convergence making the necessary substitutions. We already remarked that Lemma 2.2 is also valid for $C[0, 1]$. Lemma 2.4 is not needed in this case because finite sums of tight sequences of $C[0, 1]$ valued random variables are obviously tight. In fact we only need to prove the counterpart of Lemma 4.4 in the case of bounded α_i, i.e. that the processes $\alpha_i G_i(t)$ defined in (4.5), where α_i is a bounded random variable, satisfy the central limit theorem. Now clearly the processes G_i satisfy the CLT because in the case we are now considering they are continuous Gaussian processes. The following lemma shows that $\alpha_i G_i$ satisfies the CLT. It is fairly well known but we will include a proof for lack of a suitable reference.

5.4. LEMMA. *Let $\{X_i\}$ be independent identically distributed random variables with values in a separable Banach space B. Let $\{a_i\}$ be bounded independent identically distributed random variables independent of $\{X_i\}$. Suppose there exists a Gaussian measure γ such that*

$$w - \lim_{n\to\infty} \mathscr{L}(n^{-1/2} \textstyle\sum_{i=1}^n X_i) = \gamma. \tag{5.5}$$

Then there exists a Gaussian measure γ' such that

$$w - \lim_{n\to\infty} \mathscr{L}(n^{-1/2} \textstyle\sum_{i=1}^n a_i X_i) = \gamma'. \tag{5.6}$$

PROOF. By Theorem 1.3 in [18], (5.5) holds if and only if for each $\varepsilon > 0$ there exists a sequence of independent identically distributed random variables $\{Y_{i,\varepsilon}\}$ taking values in a finite dimensional subspace of B such that $EY_{i,\varepsilon} = 0$, $E\| Y_{i,\varepsilon}\|^2 < \infty$ and

$$\sup_n E| n^{-1/2} \textstyle\sum_{i=1}^n (X_\iota - Y_{i,\varepsilon}) \| < \varepsilon.$$

The $Y_{i,\varepsilon}$ may be chosen independent of the a_i. Then by [12], Corollary 4.2,

$$\sup_n E| n^{-1/2} \textstyle\sum_{i=1}^n (a_i X_i - a_i Y_{\iota,\varepsilon}) \| \le (\text{ess sup } a_i)\varepsilon.$$

Since $Ea_\iota Y_{\iota,\varepsilon} = 0$ and $E\| a_i Y_{\iota,\varepsilon}\|^2 < \infty$, Theorem 1.3, [18], implies 5.6. (Note that [12], Corollary 4.2, requires $EX_i = 0$, but it is well known that (5.5) implies this; in the present situation where $X_1 = G_\iota$ we do not even need this additional argument as G_i is symmetric). □

With Lemma 5.4, the proof of Theorem 5.3 is completed. (Note that tightness of the sequence in (5.4) together with the fact that the finite dimensional distributions of this sequence converge weakly to the corresponding finite dimensional distributions of S_t prove both that S_t is sample continuous and that (5.4) holds). □

5.5. REMARK. Theorem 5.4 holds, with appropriate modifications, for $C[0, \infty)$, the space of continuous functions on $[0, \infty)$ with the topology of uniform convergence on compact sets. As in the case of $D[0, \infty)$, $\mathscr{L}(X_n) \to_w \mathscr{L}(X)$ in $C[0, \infty)$ if and only if $\mathscr{L}(X_n|_{[0,T]}) \to_w \mathscr{L}(X|_{[0,T]})$ in $C[0, T]$ for all $0 < T < \infty$ ([19], Theorem 5). Therefore, if a random measure satisfies (1.9) and (1.10) for every Borel set $A \subset [0, \infty)$ and for some locally finite positive measure m, and if $f \in \mathscr{M}^\infty(M, m)$ (see Remark 4.6) then (5.4) holds as a weak limit of probability measures on $C[0, \infty)$.

5.6. EXAMPLE. We apply Theorem 5.3 to Itô processes. Let $W = (W_t, \mathscr{F}_t)$, $t \in [0, \infty)$, be a Wiener process defined on the probability space $(\Omega, \mathscr{F}, P)$, where $\mathscr{F}_t$ is the σ-algebra generated by $\{W_s, s \leq t\}$ and by the sets of P-measure zero. A process ξ_t, $t \in [0, \infty)$, is called an Itô process with respect to W if

$$\xi_t(\omega) = \xi_0 + \int_0^t a(s, \omega)\, ds + \int_0^t b(s, \omega) W(ds, \omega), \qquad 0 \leq t < \infty, \tag{5.7}$$

where a and b are non-anticipating processes, ξ_0 is a real valued random variable and the last term is the integral of b with respect to the random measure generated by W. The conditions

$$E\xi_0^2 < \infty, \qquad \sup_{t \leq T} Ea^2(t, \omega) < \infty, \qquad \int_0^t Eb^2(t, \omega)\, dt < \infty, \qquad 0 < T < \infty, \tag{5.8}$$

imply that the process

$$\eta_t = \xi_t - E\xi_t = \xi_t - E\xi_0 - E\int_0^t a(s, \omega)\, ds, \qquad 0 \leq t < \infty, \tag{5.9}$$

satisfies the central limit theorem in $C[0, \infty)$ (i.e. it belongs to the domain of normal attraction of a Gaussian law in $C[0, \infty)$).

PROOF. Since $E\xi_t^2 < \infty$ and $E\xi_t = E\xi_0 + E\int_0^t a(s, \omega)\, ds$, the finite dimensional distributions of the process $\{\eta_t, 0 \leq t < \infty\}$ defined in (5.9) satisfy the central limit theorem. Therefore it is enough to check that each one of the three terms

$$\xi_0 - E\xi_0, \quad \int_0^t a(s, \omega)\, ds - E\int_0^t a(s, \omega)\, ds \quad \text{and} \quad \int_0^t b(s, \omega) W(ds, \omega),$$

satisfies the CLT in $C[0, \infty)$. $\xi_0 - E\xi_0$ obviously does and $\int_0^t b(s, \omega) W(ds, \omega)$ does also by Theorem 5.4 and Remark 5.5. As for

$$\int_0^t a(s, \omega)\, ds - E\int_0^t a(s, \omega)\, ds, \tag{5.10}$$

let us note that for every $0 \leq u < v \leq T$, $0 \leq T < \infty$,

$$\begin{aligned} E\left[\int_u^v a(s, \omega)\, ds - E\int_u^v a(s, \omega)\, ds\right]^2 &\leq E\left[\int_u^v a(s, \omega)\, ds\right]^2 \\ &\leq E\left[\left(\int_u^v a^2(s, \omega)\, ds\right)^{1/2} (v - u)^{1/2}\right]^2 \\ &\leq (v - u)^2 \sup_{t \leq T} Ea^2(t, \omega). \end{aligned}$$

Since (5.10) satisfies the Kolmogorov conditions it satisfies the CLT on $C[0, T]$ for every $T < \infty$ ([1]). Therefore it satisfies the CLT on $[0, \infty)$. □

We can apply the previous remarks to diffusion processes, i.e. to solutions of stochastic differential equations of the form

$$\begin{cases} d\xi_t = a(t, \xi_t)\, dt + b(t, \xi_t)\, dW_t, & t \in [0, \infty) \\ \xi_0 = \eta \end{cases} \tag{5.11}$$

where $\eta \in \mathbb{R}$ and the functions $a(t, x)$, $b(t, x)$, $t \in [0, \infty)$ and $x \in (-\infty, \infty)$, are jointly measurable and satisfy the following Lipschitz and growth conditions:

$$\begin{cases} [a(t,x)-a(t,y)]^2+[b(t,x)-b(t,y)]^2 \le L(x-y)^2 \\ a^2(t,y)+b^2(t,y) \le L(1+y^2), \qquad x,y\in\mathbb{R} \end{cases} \tag{5.12}$$

for some constant $L<\infty$. A classical result of Itô states that under these conditions the stochastic differential equation (5.11) has a unique solution, which is square integrable (see e.g. [16], page 136–137), and satisfies

$$\xi_t = \eta + \int_0^t a(s,\xi_s)\,ds + \int_0^t b(s,\xi_s)\,W(ds), \qquad 0\le t<\infty.$$

Note that then $c_T = \sup_{t\le T} E\xi_t^2 < \infty$. Therefore, by (5.12),

$$Ea^2(t,\xi_t) + Eb^2(t,\xi_t) \le L(1+c_T), \quad t\le T.$$

So, a and b satisfy condition (5.8). It follows then that $\xi_t - \eta - E\int_0^t a(s,\xi_s)\,ds$ (i.e.. $\xi_t - E\xi_t$) satisfies the CLT in $C[0,\infty)$. (In fact conditions (5.12) can be replaced by the less restrictive conditions (4.110) and (4.111) in [16] page 128).

Finally, we give some examples of random measures satisfying (1.9) and (1.10).

5.7. EXAMPLE. If M has the characteristic function

$$E\exp(itM(A)) = \exp\left[-m(A)\sigma^2t^2/2 + m(A)\int_{-\infty}^{\infty}(\cos tx - 1)\,d\mu(x)\right],$$

$$-\infty<t<\infty, \qquad A\in\mathscr{B},$$

where $\sigma^2<\infty$ and μ is a symmetric measure such that

$$\int_{-\infty}^{\infty} x^2\,d\mu(x) < \infty,$$

then some constant times M satisfies conditions (1.9) and (1.10). In fact,

$$EM^2(A) = \left[\sigma^2 + \int_{-\infty}^{\infty} x^2\,d\mu(x)\right] m(A).$$

5.8. EXAMPLE. If M is given by (3.24) with $p=2$ and the variables η_i symmetric, independent, identically distributed and such that $E\eta_i^2 = 1$, then M satisfies (1.9) and (1.10).

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