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A CLASS OF CONDITIONAL LIMIT THEOREMS RELATED TO RUIN PROBLEM

By Michio Shimura

University of Tsukuba

For a random walk with mean zero and variance one, a conditional limit theorem is proved under conditions on the path until it for the first time becomes negative. This gives a generalization of a limit theorem for random walk conditioned to stay positive which was considered by Iglehart and others. It has an application to get a tail formula of the d.f. of the maximum for a stopped random walk.

1. Introduction and the main result. Let $\{X_n; n \geq 1\}$ be a sequence of i.i.d. random variables with $E(X_1) = 0$ and $E(X_1^2) = 1$. We define the *random walk* $\{S_n;\ n \geq 0\}$ by setting $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, $n \geq 1$. For $n \geq 1$, introduce the random process W_n starting at nonnegative x_n by $W_n(t) = n^{-1/2} S_{[nt]} + x_n$, $0 \leq t < +\infty$, where $[a]$ is the greatest integer not exceeding a. Next we set a *stopped process* $\mathring{W}_n(t) = W_n(t \wedge T_n)$, where $T_n = \inf\{t: W_n(t) < 0\}(\inf\{\phi\} = +\infty)$ and $a \wedge b = \min\{a, b\}$.

Let $\mathscr{D} = \mathscr{D}[0, +\infty)$ be the space of real valued, right-continuous functions on $[0, +\infty)$ having left limits, and define on it the topology J_1 (Stone [14]).

In this paper we will prove the following theorem.

Theorem 1. *Let* $\{A_n;\ n \geq 1\}$ *be a sequence of Borel subsets of* $\mathscr{D}$ *satisfying the conditions* (i) *and* (ii) *in Section* 2. *Suppose that the starting points* $\{x_n\}$ *satisfy* $x_n \geq 0$ *and* $x_n \to 0$ *as* $n \to \infty$. *Then the sequence of conditional probabilities* $\{P((T_n, \mathring{W}_n) \in \cdot \mid \mathring{W}_n \in A_n);\ n \geq 1\}$ *converges weakly as* $n \to \infty$ *in the product space* $[0, +\infty) \times \mathscr{D}$.

As will be given in Corollary to Theorem 1 in Section 3, *the limit law can be expressed in terms of the reflecting Brownian motion.*

By Theorem 1 we generalize a conditioned limit theorem posed by Iglehart [8]. Indeed, by setting $A_n = \{z: \inf_{0 \leq t \leq 1} z(t) \geq 0\}$, we can treat a "limit theorem for random walk conditioned to stay positive". (See also Bolthausen [3] and Durrett [4].) Other choices of conditioning sets are possible. For example, by taking $A_n = \{z: \sup_{0 \leq t < +\infty} z(t) > \rho\}$, $\rho > 0$, we get a theorem given by Green [7] and Pakes [12] without their "left-continuity" assumption.

In Section 2 we introduce the conditions (i) and (ii), and in Section 3 prove Theorem 1. In Section 4 we discuss two examples mentioned above, and as their consequence we get a tail formula of the d.f. of the maximum for a stopped random walk.

2. The conditions (i) and (ii). The former is easy:

(i) $$P(\mathring{W}_n \in A_n) > 0 \textit{ for all } n \geq 1.$$

In order to express the latter, we need some preparations. Let $\mathscr{C}$ be a subspace of $\mathscr{D}$ which consists of continuous functions. For $z \in \mathscr{C}$, $\mathbf{E}(z)$ is the set of finite open intervals (τ, υ) satisfying

(2.1) $$z(t) \geq z(\tau) \text{ for } 0 \leq t \leq \tau,\ z(t) > z(\tau) \text{ for } \tau < t < \upsilon \text{ and } z(\tau) = z(\upsilon).$$

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For $(\tau, v) \in \mathbf{E}(z)$ let

$$\alpha_{(\tau,v)}z(t) = z(\tau + t \wedge (v - \tau)) - z(\tau).$$

By d we stand for the metric on $\mathscr{D}$ which induces the topology J_1, and provide the set $\mathscr{D}$ with the other metric $\mathring{d}$ given by

$$\mathring{d}(z_1, z_2) = |T(z_1) - T(z_2)| + d(z_1(. \wedge T(z_1)), z_2(. \wedge T(z_2))), \tag{2.2}$$

where $T(z) = \inf\{t: z(t) < 0\}(\inf\{\phi\} = +\infty)$ and $+\infty - (+\infty) = 0$.

Let $\{W(t); t \geq 0\}$ be a *standard Brownian motion* starting at 0 with continuous sample paths. The condition (ii) is then as follows:

$$[\cup_{n=1}^{\infty}[\cap_{k=n}^{\infty} A_k]^{\circ}]^{-} \supseteqq \cap_{n=1}^{\infty}[\cup_{k=n}^{\infty} A_k]^{-}; \tag{ii.1}$$

where B^{-}(resp. B°) is the closure (resp. interior) of a set B with respect to the metric $\mathring{d}$. Let $A: = \cup_{n=1}^{\infty}[\cap_{k=n}^{\infty} A_k]^{\circ}$.

$$P(\mathbf{E}_{\partial A}(W) = \phi) = 1; \tag{ii.2}$$

$$P(\mathbf{E}_A(W) \textit{ is nonempty and has no finite limit point}) = 1; \tag{ii.3}$$

where $\partial A = A^{-} - A^{\circ}$ and $\mathbf{E}_*(z) = \{(\tau, v) \in \mathbf{E}(z): \alpha_{(\tau,v)}z \in *\}$.

REMARK 2.1. When $A_n \equiv B$, $A = B^{\circ}$ and (ii.1) is as follows:

$$[B^{\circ}]^{-} = B^{-}. \tag{ii.1'}$$

REMARK 2.2. Recall that $\{\bar{W}(t) := W(t) - \min_{0 \leq s \leq t} W(s);\ t \geq 0\}$ is the *reflecting Brownian motion* starting at 0 (Ito and McKean [10] 2.1). Then $\mathbf{E}(W)$ and $\alpha_{(\tau,v)}W$ above are the *set of intervals of excursions* and the *excursion process* at 0 of the $\bar{W}$ respectively (Ito [9]).

3. PROOF OF THEOREM 1. To simplify the situation, we prove it when the $x_n = 0$. The extension for general $\{x_n\}$ will be made easily.

First we change the conditional probabilities in Theorem 1 into unconditional ones. For $n \geq 1$ let $D_n (\subset \mathscr{D})$ be the set of functions being constant on each interval $[k/n, (k+1)/n)$, $k \geq 0$. For $z \in D_n$ set $\lambda_0 = 0$ and $\lambda_m = \inf\{t > \lambda_{m-1}: z(t) < z(\lambda_{m-1})\}$ (λ_m is undefined if so is λ_{m-1}, or if the $\{*\} = \phi$), $\mathbf{E}_n(z) = \{(\lambda_{m-1}, \lambda_m);\ m \geq 1\}$, and $\alpha_m z(t) = z(\lambda_{m-1} + t \wedge (\lambda_m - \lambda_{m-1})) - z(\lambda_{m-1})$ for $(\lambda_{m-1}, \lambda_m) \in \mathbf{E}_n(z)$. For $z \in D_n$ let

$$\Gamma_n(z) = (\lambda_{\hat{m}(n)} - \lambda_{\hat{m}(n)-1},\ \alpha_{\hat{m}(n)}z),$$

where $\hat{m}(n) = \min\{m: \alpha_m z \in A_n\}$. The following lemma is an extension of Bolthausen [3], Lemma 3.1.

LEMMA 3.1. *Assume the condition* (i). *Then $\Gamma_n(W_n)$ is defined almost surely, and moreover*

$$P((T_n, \mathring{W}_n) \in B \mid \mathring{W}_n \in A_n) = P(\Gamma_n(W_n) \in B) \tag{3.1}$$

for all measurable subsets B of $[0, +\infty) \times \mathscr{D}$.

Next we show a continuity property of the maps $\{\Gamma_n\}$. For $z \in \mathscr{C}$ let $\mathbf{E}'(z) = \{(\tau, v) \in \mathbf{E}(z): z(t) > z(\tau)$ for $0 \leq t < \tau$ if $\tau > 0$, and $\exists\ \{s_n\}$ such that $s_n \downarrow v$ and $z(s_n) < z(\tau)\}$. Consider the subsets $C_i (0 \leq i \leq 3)$ of $\mathscr{C}$; $C_1 = \{\mathbf{E}'(z) = \mathbf{E}(z)\}$, $C_2 = \{\mathbf{E}_{\partial A}(z) = \phi\}$, $C_3 = \{\mathbf{E}_A(z)$ is nonempty and has no finite limit point$\}$, and $C_0 = \cap_{i=1}^{3} C_i$. Since $P(W \in C_1) = 1$ by (3.4) in [3], and since $P(W \in C_i) = 1 (i = 1, 2)$ by (ii.2 and 3), we have

$$P(W \in C_0) = 1. \tag{3.2}$$

For $z \in C_0$ let $(\hat{\tau}, \hat{\upsilon})$ be the first element of $\mathbf{E}_A(z)$, and define

$$\Gamma(z) = (\hat{\upsilon} - \hat{\tau}, \alpha_{(\hat{\tau},\hat{\upsilon})} z).$$

It will not be difficult to prove the measurability of Γ, and moreover we get the following lemma.

LEMMA 3.2. *If $z_n \in D_n$ and $z \in C_0$ such that $z_n \to z$ in $\mathscr{D}$, then we have $\Gamma_n(z_n) \to \Gamma(z)$ in $[0, +\infty) \times \mathscr{D}$.*

PROOF. First note that, since $z \in \mathscr{C}$, $z_n \to z$ in $\mathscr{D}$ is identical to the uniform convergence at every compact interval (u.c.c.), that is, $\lim_{n\to\infty} \sup_{0\le t\le K} | z_n(t) - z(t) | = 0$ for every $K > 0$ (Stone [14]). Hence the lemma follows from

$$\lambda_{\hat{m}(n)-1}(z_n) \to \hat{\tau}(z) \text{ and } \lambda_{\hat{m}(n)}(z_n) \to \hat{\upsilon}(z) \text{ as } n \to \infty. \tag{3.3}$$

To prove (3.3) first observe that we can choose $(\lambda_{m(n)-1}, \lambda_{m(n)}) \in \mathbf{E}_n(z_n)$ such that

$$\lambda_{m(n)-1} \to \hat{\tau}(z) \text{ and } \lambda_{m(n)} \to \hat{\upsilon}(z) \text{ as } n \to \infty. \tag{3.4}$$

We easily check this from the u.c.c. and $z \in C_1$.

Next show that

$$m(n) = \hat{m}(n) \text{ for almost all } n. \tag{3.5}$$

Since $\alpha_{(\hat{\tau},\hat{\upsilon})} z \in [\cap_{k=n_0}^{\infty} A_k]^{\circ}$ for some n_0 by the definition of A given in (ii), and since $\mathring{d}(\alpha_{m(n)} z_n, \alpha_{(\hat{\tau},\hat{\upsilon})} z) \to 0$, we have $\alpha_{m(n)} z_n \in \cap_{k=n_0}^{\infty} A_k$ for almost all n. Therefore we have $m(n) \ge \hat{m}(n)$ for almost all n. Now suppose that $m(n) > \hat{m}(n)$ would hold for infinitely many n, say for $\{n_\nu\}_{\nu\ge1}$. We may assume in addition that $\lambda_{\hat{m}(n_\nu)-1}(z_{n_\nu}) \to t_0$ and $\lambda_{\hat{m}(n_\nu)}(z_{n_\nu}) \to u_0$ as $\nu \to \infty$. This easily leads to a contradiction if we note $(t_0, u_0) \in \mathbf{E}(z)$, $\mathring{d}(\alpha_{\hat{m}(n_\nu)} z_{n_\nu}, \alpha_{(t_0,u_0)} z) \to 0$ and (ii.1).

By (3.4) and (3.5) we have (3.3), and hence the lemma.

Now we have proved Theorem 1. Indeed, recall the *Donsker's theorem: W_n converges weakly to W as $n \to \infty$ in $\mathscr{D}$* (Billingsley [1] and Stone [14]). Combining (3.2) and Lemma 3.2 with the Donsker's theorem, we can apply the continuous mapping theorem ([1], Theorem 5.5) to get

$$\Gamma_n(W_n) \text{ converges weakly to } \Gamma(W) \text{ as } n \to \infty \text{ in } [0, +\infty) \times \mathscr{D}. \tag{3.6}$$

Theorem 1 follows from Lemma 3.1 and (3.6).

COROLLARY TO THEOREM 1. *The limit random element of Theorem 1 is given by $\Gamma(W) = (\hat{\upsilon} - \hat{\tau}, \bar{W}(\hat{\tau} + .\wedge(\hat{\upsilon} - \hat{\tau})))$, where $\bar{W}$ is the reflecting Brownian motion in Remark 2.2 and $(\hat{\tau}, \hat{\upsilon})$ is the first element of $\mathbf{E}_A(W) = \{(\tau, \upsilon) \in E(W) : \bar{W}(\tau + .\wedge(\upsilon - \tau)) \in A\}$.*

4. Examples. Consider two specific conditioning sets $A_n = A'_\rho = \{z : \inf_{0\le t\le\rho} z(t) \ge 0\}$ and $A_n = A''_\rho = \{z : \sup_{0\le t<+\infty} z(t) > \rho\}$ for $\rho > 0$. Let $(\tau'_\rho, \upsilon'_\rho)$ (resp. $(\tau''_\rho, \upsilon''_\rho)$) be the first element of $\mathbf{E}_{A'_\rho}(W)$ (resp. $\mathbf{E}_{A''_\rho}(W)$). Then by Theorem 1 and its Corollary, we get the following result.

EXAMPLE 4.1. *Let $\mathring{W}$ be the stopped process starting at $x_n \ge 0$ with $x_n \to 0$ as $n \to \infty$. Then*

$$((T_n, \mathring{W}_n) | \mathring{W}_n \in A'_\rho) \Rightarrow (\upsilon'_\rho - \tau'_\rho, \bar{W}(\tau'_\rho + .\wedge(\upsilon'_\rho - \tau'_\rho))) \tag{4.1}$$

and

$$((T_n, \mathring{W}_n) | \mathring{W}_n \in A''_\rho) \Rightarrow (\upsilon''_\rho - \tau''_\rho, \bar{W}(\tau''_\rho + .\wedge(\upsilon''_\rho - \tau''_\rho))) \tag{4.2}$$

as $n \to \infty$ in $[0, +\infty) \times \mathscr{D}$, where $\Rightarrow$ denotes the weak convergence.

REMARK 4.2. The formula (4.1) is an extension of a theorem given by Bolthausen [3]. To prove Example 4.1, we need only to check the following lemma.

LEMMA 4.3. *Both* A'_ρ *and* A''_ρ *satisfy the conditions* (i) *and* (ii).

Clearly both satisfy the conditions (i), (ii.1′) and (ii.3). Next observe that $\partial A'_\rho = \{z : T(z) = \rho\}$, where $T(z) = \inf\{t : z(t) < 0\}$, and $\partial A''_\rho = \{z : \sup_{0 \le t < +\infty} z(t) = \rho\}$. Then (ii.2) is easily shown to hold by using some elementary properties of the reflecting Brownian motion, and we omit the detail here.

5. The maximum of a stopped random walk. Set $N(y) = \inf\{n : S_n + y < 0\}$ and $M(y) = \sup\{S_n + y : 0 \le n < N(y)\}$ for $y \ge 0$. When $\{S_n\}$ is a left-continuous random walk, that is, $P(S_1 \in \{-1, 0, 1, \cdots\}) = 1$, Lindvall [11] gave a tail formula of the d.f. of $M(y)$. In this section we prove it without the "left-continuity" assumption through the formulas (4.1) and (4.2).

Let $V(x)$ be the *renewal function* generated by the d.f. $H(x) = P(-S_{N(0)} \le x) : V(x) = \sum_{n=0}^{\infty} H^{n*}(x)$ (Feller [6] Chapter 11).

THEOREM 2. (i) *Let* $\{y_n;\ n \ge 1\}$ *be a sequence of nonnegative numbers such that* $n^{-1/2} y_n \to 0$ *as* $n \to \infty$. *Then, for every* $\xi > 0$,

$$P(M(y_n) > n^{1/2}\xi) \sim (\pi/2)^{1/2}\xi^{-1}P(N(y_n) > n) \text{ as } n \to \infty.$$

(ii) *Let* $y_n = y \ge 0$, *then at continuity points* y *of* $V(.)$ (*including* $y = 0$),

$$P(M(y) > r) \sim 2^{-1/2}CV(y)r^{-1} \text{ as } r \to +\infty,$$

where $C = \exp \sum_{k=1}^{\infty} k^{-1}\{2^{-1} - P(S_k < 0)\}$.

Before proving Theorem 2, we give the following five lemmas.

LEMMA 5.1 (Bingham [2]). *At continuity points* y *of* $V(.)$ (*including* $y = 0$), $P(N(y) > r) \sim CV(y)(\pi r)^{-1/2}$ *as* $r \to +\infty$.

Let $p_{1,\rho}(\xi) = P(\max_{\tau'_\rho \le t \le \nu'_\rho} \bar{W}(t) \le \xi)$ and $p_{2,\rho}(\xi) = P(\nu''_\rho - \tau''_\rho \le \xi)$. Since both A'_ρ and A''_ρ satisfy (ii.2) (Lemma 4.3), we easily conclude the following lemma.

LEMMA 5.2. *For every positive* ρ, $p_{i,\rho}(.)(i = 1, 2)$ *is a continuous* d.f.

It follows from (4.1) and (4.2) (take $x_n = n^{-1/2}y_n$ here) that

$$(\sup_{0 \le t \le T_n} \mathring{W}_n(t) \mid \mathring{W}_n \in A'_\rho) \Rightarrow \max_{\tau'_\rho \le t \le \nu'_\rho} \bar{W}(t) \tag{5.1}$$

$$(T_n \mid \mathring{W}_n \in A''_\rho) \Rightarrow \nu''_\rho - \tau''_\rho \tag{5.2}$$

as $n \to \infty$ in $\mathscr{R}^1$. Therefore, by Lemma 5.2, we have

LEMMA 5.3. *For every* $\rho > 0$ *and* $-\infty < \xi < +\infty$,

$$P(M(y_n) \le n^{1/2}\xi \mid N(y_n) > n\rho) \to p_{1,\rho}(\xi) \tag{5.1′}$$

and

$$P(N(y_n) \le n\xi \mid M(y_n) > n^{1/2}\rho) \to p_{2,\rho}(\xi) \text{ as } n \to \infty. \tag{5.2′}$$

REMARK 5.4. When random walk is left-continuous and the $y_n = 0$, the conditioned limit theorem (5.2) was considered by Green [7] and Pakes [12].

Set $p_{i,1} = p_i$. The following lemma is an easy consequence of the *space-time renormal-*

ization property of the reflecting Brownian motion: Both $\bar{W}(r.)$ and $r^{1/2}\bar{W}(.)$ $(r > 0)$ are identical in law.

LEMMA 5.5. *For every $\rho > 0$ and $-\infty < \xi < +\infty$, $p_{1,\rho}(\xi) = p_1(\rho^{-1/2}\xi)$ and $p_{2,\rho}(\xi) = p_2(\rho^{-2}\xi)$.*

The Laplace-Stieltjes transform of $p_2(.)$ is given by Green [7]: For $\theta > 0$

$$\int_0^{\infty} \exp(-\theta\xi)\, dp_2(\xi) = (2\theta)^{1/2}\exp\{-(2\theta)^{1/2}\}/\sinh\{(2\theta)^{1/2}\}. \tag{5.3}$$

Apply a Tauberian theorem (Feller [6] Chapter 13) on (5.3) in getting the following:

LEMMA 5.6. $1 - p_2(\xi) = P(v_1'' - \tau_1'' > \xi) \sim (2/\pi\xi)^{1/2}$ *as* $\xi \to +\infty$.

PROOF OF THEOREM 2. By Lemmas 5.3 and 5.5, we have, for every $\xi > 0$ and $\rho > 0$,

$$P(M(y_n) > n^{1/2}\xi) \sim K(\xi, \rho)P(N(y_n) > n\rho) \quad \text{as} \quad n \to \infty, \tag{5.4}$$

where $K(\xi, \rho) = \{1 - p_1(\rho^{-1/2}\xi)\}/\{1 - p_2(\xi^{-2}\rho)\}$. Observe (5.4) when $y_n = y \geq 0$ a continuity point of $V(.)$. Then by Lemma 5.1,

$$\lim_{n\to\infty} n^{1/2}P(M(y) > n^{1/2}\, \xi) = \pi^{-1/2}\, CV(y)\, \rho^{-1/2}K(\xi, \rho). \tag{5.5}$$

Note that the left-hand side of (5.5) is independent of ρ, and hence we have

$$\rho^{-1/2}K(\xi, \rho) = K(\xi) \text{ a positive function independent of } \rho. \tag{5.6}$$

Let $\rho \to +\infty$ in (5.6), and use Lemma 5.6 and $p_1(+0) = 0$ to determine $K(\xi) = (\pi/2)^{1/2}\xi^{-1}$. This completes the proof.

REMARK 5.7. By the proof given above, we get a dual formula between p_1 and p_2:

$$1 - p_2(\rho) = (2/\pi\rho)^{1/2}\{1 - p_1(\rho^{-1/2})\} \text{ for } \rho > 0.$$

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Note added in proof. After submitting his revision, the author was informed by Professor Greenwood (U.B.C.) of her recent joint works with E. Perkins, in which they considered "Limit theorems of random walk conditioned to stay between square root boundaries".

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Institute of Mathematics
University of Tsukuba
Sakura-mura Niihari-gun
Ibaraki 305
Japan