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THE BEHAVIOR OF ASYMMETRIC CAUCHY PROCESSES FOR LARGE TIME

BY WILLIAM E. PRUITT[1] AND S. JAMES TAYLOR[1]

University of Minnesota and University of Liverpool

This paper develops precise estimates for the potential kernel, capacities of large intervals, and the probabilities of hitting large intervals for the asymmetric Cauchy processes. These are then applied to study three problems concerning the sample paths: (i) the rate of escape of $|X_t|$ as $t \to \infty$; (ii) the sizes of the large holes in the range of the process; (iii) the asymptotic behavior of the Lebesgue measure of that part of the range of the process that is in a large interval.

1. Introduction. The stable processes in $\mathbb{R}^1$ have been extensively studied over the years. Typically some property has been found for Brownian motion but then very similar methods usually suffice to obtain the analogous property for the strictly stable processes. A simple fact that is very useful here is the scaling property: for any $r > 0$, $r^{-1/\alpha}X_{rt}$ is another version of the process X_t. However, the asymmetric Cauchy processes do not satisfy this simple scaling property and this makes their analysis much more difficult. It also leads to some rather surprising results which we will obtain in the present paper.

The Cauchy processes in $\mathbb{R}^1$ have stationary independent increments and characteristic function

$$Ee^{iuX_t} = e^{-t|u|(1+ih\,\mathrm{sgn}\,u\log|u|)}, \tag{1.1}$$

where h is a parameter in $[-2/\pi, 2/\pi]$. It is possible to introduce two additional parameters by multiplying the process by a scale factor and adding a linear drift term; an examination of the characteristic function shows that it is possible to remove the drift term by making a simultaneous change of scale in both time and space. Thus the general asymmetric Cauchy process Y_t may be expressed in terms of X_t satisfying (1.1) as $Y_t = cX_{rt}$ for appropriate positive c and r. The parameter h is the same for X_t and Y_t. (Note that this is not possible in the symmetric case $h = 0$.) Now it is easy to deduce the properties of Y_t from those of X_t so we shall omit these two additional parameters.

It follows from (1.1) that X_t has a density $p(t, x)$ which is infinitely differentiable in x. The fact that for $r > 0$, $t > 0$, rX_t and $X_{rt} - htr\log r$ have the same distribution follows easily from (1.1) and leads to

$$p(t, x) = p(1, xt^{-1} - h\log t)t^{-1}. \tag{1.2}$$

We will still refer to (1.2) as the scaling property. Even though it is more involved than in the strictly stable case, it does still allow us to obtain information about the entire family of densities $p(t, x)$ from just $p(1, x)$.

The analytic work begins with the determination of the asymptotic behavior of the potential kernel

$$g(x) = \int_0^\infty p(t, x)\,dt \tag{1.3}$$

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as $x \to \pm\infty$. The leading term here was obtained by Port and Stone (1969) but we require much sharper bounds for our results. Once these are available, we apply the potential theory of Hunt since all of his conditions for the process to be nice are satisfied. We must estimate the capacity of large intervals and even the manner in which the capacitary distribution is spread over such an interval. These in turn lead to estimates of the probability of hitting such an interval. One of the unusual features of the present work is the very precise knowledge that is required of the potential kernel and the capacitary distributions. These estimates on the potential kernel, capacities, and hitting probabilities are given in Sections 3, 4, and 5.

The sample path properties are obtained in the final three sections. Section 6 deals with the rate of escape problem; in Theorem 6.1, we have an integral test for determining the lower envelope for $|X_t|$. The results in Sections 7 and 8 are more unusual. They include an integral test which determines which large intervals $[x, x\varphi(x)]$ are missed by the sample path and the very surprising result that the Lebesgue measure of that part of the range of the sample path contained in $[0, a]$, normalized by $a/\log a$, converges in distribution to a geometric distribution. These last two results were announced in [12].

The problem that first attracted us to these processes was posed by Kesten (1976), namely, to determine whether the range of the process is nowhere dense. This problem turned out to be very difficult and the solution took a long time. The answer is that the range is in fact nowhere dense with probability one except in the completely asymmetric case where it is a countable union of intervals. In both cases the range has infinite Lebesgue measure. For comparison, note that the range of the symmetric Cauchy process has zero Lebesgue measure but is dense in $\mathbb{R}^1$. This result and some related results about the local behavior of the sample paths will appear in [13]. The analysis in this case starts with determining very accurately the local behavior of the potential kernel and then goes on to estimating capacitary distributions and hitting probabilities of small intervals. This part of the analysis is similar to that of the present paper, but the probabilistic arguments that follow are much more complex in [13] and a much deeper understanding of the evolution of the sample path is required.

2. Preliminaries. We consider the Cauchy processes in $\mathbb{R}^1$. They have stationary independent increments and characteristic function as in (1.1). The Lévy measure for this process is given by

$$\nu(dx) = \begin{cases} \dfrac{2}{\pi} p x^{-2}\, dx, & x > 0, \\[2ex] \dfrac{2}{\pi} q x^{-2}\, dx, & x < 0, \end{cases}$$

where $p \geq 0$, $q \geq 0$, $p + q = 1$. The parameter h in (1.1) is then $(p - q)2/\pi$. *If* $p = q$, then $h = 0$ and we have the symmetric Cauchy process which is strictly stable; its properties have been studied extensively along with those of the other strictly stable processes. We will be concerned with the case $h \neq 0$. Since looking at the process $-X_t$ corresponds to changing the sign of h, we will assume throughout that $h > 0$, i.e. $p > q$. Since we can think of p and q as the weighting factors for positive and negative jumps, we are thereby assuming that positive jumps are more likely than negative jumps. When $p = 1$, $h = 2/\pi$, and the process is called completely asymmetric. It has no negative jumps (but it is not an increasing process) and several of its properties can be established by exploiting this fact—see Millar (1972) and Mijnheer (1974). In some cases the results are different in this extreme case and will be stated separately.

It is easy to check either directly or by using a general criterion (see Kingman, 1964) that all asymmetric Cauchy processes are transient, i.e. $|X_t| \to \infty$ as $t \to \infty$. In the completely asymmetric case X_t is bounded below and so $X_t \to \infty$. However, if $h \in (0, 2/\pi)$ then more careful analysis shows that X_t assumes both large positive values and large negative values when t is large. For large times, the process is more likely to be positive than negative in this case and this means that the part of the range of the process in $(-\infty, -M]$ is much less dense than the part in $[M, \infty)$ for large M. Some further remarks about this appear at the end of Section 7.

We will need the asymptotic behavior of the density $p(1, x)$ of X_1. The density $p(1, x)$ can be expressed as an integral by the inversion formula and then the asymptotics can be carried out. This has been done by Skorohod (1954). We only need his leading term which is, in our notation,

$$p(1, x) = \frac{2p}{\pi}\left(\frac{1}{x^2} + 2h\frac{\log x}{x^3}\right) + O\left(\frac{1}{x^3}\right), \quad x \to +\infty, \tag{2.1}$$

$$p(1, x) = \frac{2q}{\pi}\left(\frac{1}{x^2} + 2h\frac{\log|x|}{x^3}\right) + O\left(\frac{1}{|x|^3}\right), \quad x \to -\infty. \tag{2.2}$$

We will also occasionally need the asymptotic behavior of the first two derivatives of the density. Since they have similar integral expressions which may be obtained by differentiation of the inversion formula, their asymptotic behavior can be determined by the same techniques. As expected, the leading terms result from differentiating the leading terms in (2.1) and (2.2).

We make the usual assumption that we are dealing with a version of the process which satisfies the strong Markov property and that the sample paths are right continuous with left limits everywhere. Since the process has positive probability of hitting a point (Theorem 1 of [11]), any compact set K has a unique capacitary distribution μ_K concentrated on K and satisfying

$$\Phi(x, K) \equiv \int g(y - x)\mu_K(dy) \leq 1 \quad \text{for all} \quad x \in \mathbb{R}^1, \tag{2.3}$$

$$\Phi(x, K) = 1 \quad \text{for all} \quad x \in K \tag{2.4}$$

where g is the potential kernel defined in (1.3) (Proposition 4.7 of Chapter 6 of [1]). Furthermore,

$$\Phi(x, K) = P^x\{X_t \in K \quad \text{for some} \quad t > 0\}. \tag{2.5}$$

The total mass, $\mu_K(K)$, is called the capacity of K.

We will need to use a generalized version of the Borel-Cantelli lemma such as the one in [7]. If $\{E_k\}$ is a sequence of events such that $\sum P(E_k)$ diverges and

$$\liminf_{N\to\infty} \frac{\sum_{j=1}^N \sum_{k=1}^N P(E_jE_k)}{\{\sum_{k=1}^N P(E_k)\}^2} < \infty, \tag{2.6}$$

then $P(E_k \text{ i.o.}) > 0$. Note that (2.6) is satisfied if for each fixed j the integers $k > j$ split into two sets I_j^1, I_j^2 such that

$$P(E_jE_k) \leq cP(E_j)P(E_k) \text{ for } k \in I_j^1, \quad \sum_{k \in I_j^2} P(E_jE_k) \leq cP(E_j).$$

In the end we will always be able to assert, under these conditions, that $P(E_k \text{ i.o.}) = 1$ by using a zero-one law. What we need here is a continuous time version of the Hewitt-Savage Zero One Law. This is well known but we cannot find a reference that gives a complete proof in the present context. The following will be sufficient for our purposes: if $A \in \bigcap_n \sigma\{X_t : t \geq n\}$, then $P(A)$ is zero or one. A proof (in a somewhat more general setting) is given in [10].

3. The asymptotic behavior of the kernel. In this section we will obtain asymptotic estimates for the potential kernel $g(x)$ both for $x \to +\infty$ and $x \to -\infty$. As mentioned above we will assume without loss of generality that $h > 0$. The leading term in Theorem 3.1 has been obtained before [11] but the sharp error term is new and it is essential for our later results. It will be evident from the proof that care is needed to obtain the actual error term of order $\log^{-2} x$ instead of an error of order $\operatorname{loglog} x/\log^2 x$ which is not good enough for our purposes.

THEOREM 3.1. *Define g by* (1.3). *Then*

$$g(x) = \frac{b}{\log x} + O\left(\frac{1}{\log^2 x}\right), \quad x \to +\infty,$$

$$g(x) = \frac{d}{\log |x|} + O\left(\frac{1}{\log^2 |x|}\right), \quad x \to -\infty,$$

where $b = 2p/\pi h^2$, $d = 2q/\pi h^2$.

REMARK. In the completely asymmetric case, $d = q = 0$ so that we have not really given the asymptotic behavior of g in this case as $x \to -\infty$. It is possible here to use the fact that the process has no negative jumps in a renewal argument to obtain $g(x) = \pi e^x/2$, $x \leq 0$, so that the negative tail of the kernel is much smaller in this case. We will not use this in the present paper.

PROOF. The techniques involved in the two proofs are different so we will prove both results. In the first part, we will use the function

$$F(u) = u + h \log u, \quad u > 0.$$

This increases and hence has an inverse function G defined on $\mathbb{R}^1$. From $v = G(v) + h \log G(v)$, we obtain first that for v large, $\log G(v) = \log v + o(1)$ and then

$$G(v) = v - h \log v + o(1), \quad v \to +\infty. \tag{3.1}$$

Also since $G(v) > 0$, we have

$$G(v) < \exp\{vh^{-1}\}, \quad v < 0. \tag{3.2}$$

Now we are prepared to estimate g. We have by the scaling property that

$$g(x) = \int_0^\infty p(t, x)dt = \int_0^\infty p(1, xt^{-1} - h \log t)t^{-1}\, dt. \tag{3.3}$$

Now for $x > 0$, we have

$$\begin{aligned} g(x) &= \int_0^\infty p(1, F(xt^{-1}) - h \log x)t^{-1}\, dt \\ &= \int_{-\infty}^\infty p(1, v - h \log x)(G(v) + h)^{-1}\, dv \end{aligned} \tag{3.4}$$

where we have made the substitution $v = F(xt^{-1})$ at the last step. Now we split the range of integration into five pieces; we will denote the corresponding integrals as I_1, I_2, I_3, I_4, I_5. On $(-\infty, 0]$, we replace $G(v) + h$ by h. The difference is of order $G(v)$ which is integrable by (3.2) while the density is $O(\log^{-2} x)$ on this range by (2.2). Thus we have

$$\begin{aligned} I_1 &= h^{-1}P\{X_1 \leq -h \log x\} + O(\log^{-2} x) \\ &= \frac{2q}{\pi}\frac{1}{h^2}\frac{1}{\log x} - \frac{2q}{\pi}\frac{1}{h^2}\frac{\operatorname{loglog} x}{\log^2 x} + O(\log^{-2} x) \end{aligned} \tag{3.5}$$

where we have used (2.2) to estimate the probability. For the interval $[0, 1]$ we have $G(v) + h \geq h$ and the density is of order $\log^{-2} x$ so that

$$I_2 = O(\log^{-2} x). \tag{3.6}$$

Next we consider the interval $[1, \frac{1}{2}h \log x]$. Here we replace $G(v) + h$ by v. The difference will give a term of order $v^{-2} \log v$ which is integrable and again we have that the density is $O(\log^{-2} x)$ on this range so the error is acceptable. Thus

$$
\begin{aligned}
I_3 &= \int_1^{(h/2)\log x} p(1, v - h \log x)v^{-1}dv + O(\log^{-2} x) \\
&= \int_1^{(h/2)\log x} \left[\frac{2q}{\pi}\frac{1}{(v - h\log x)^2} + O\left(\frac{\log\log x}{\log^3 x}\right)\right]\frac{dv}{v} + O(\log^{-2} x) \\
&= \frac{2q}{\pi}\frac{1}{h^2}\frac{1}{\log^2 x}\int_1^{(h/2)\log x}\left(1 + O\left(\frac{v}{\log x}\right)\right)\frac{dv}{v} + O(\log^{-2} x) \\
&= \frac{2q}{\pi}\frac{1}{h^2}\frac{\log\log x}{\log^2 x} + O(\log^{-2} x).
\end{aligned}
\tag{3.7}
$$

For the interval $[\frac{1}{2}h \log x, 2h \log x]$ we use the expansion

$$
\begin{aligned}
\frac{1}{G(v) + h} &= \frac{1}{v - h\log v + O(1)} \\
&= \frac{1}{h\log x} - \frac{1}{h^2\log^2 x}(v - h\log v + O(1) - h\log x) \\
&\quad + O\left(\frac{1}{\log^3 x}\right)(v - h\log v + O(1) - h\log x)^2.
\end{aligned}
$$

The $O(1)$ in the second term leads to a term of order $\log^{-2} x$ since the density is integrable. For the final error term, we use

$$(v - h\log v + O(1) - h\log x)^2 \le 2(v - h\log x)^2 + 2(h\log v + O(1))^2$$

and note that the first of these leads to

$$O\left(\frac{1}{\log^3 x}\right)\int_{-(h/2)\log x}^{h\log x} u^2 p(1, u)\, du = O\left(\frac{1}{\log^2 x}\right),$$

while for the second, $\log v$ is of order $\log\log x$ and the density is integrable so we have a term of order $(\log\log x)^2/\log^3 x$ which is even smaller. Finally, note that $\log v = \log\log x + O(1)$ on the range in question. Thus we have

$$
\begin{aligned}
I_4 &= \left(\frac{1}{h\log x} + \frac{\log\log x}{h\log^2 x}\right)P\left\{-\frac{h}{2}\log x \le X_1 \le h\log x\right\} \\
&\quad - \frac{1}{h^2\log^2 x}\int_{-(h/2)\log x}^{h\log x} up(1, u)du + O\left(\frac{1}{\log^2 x}\right) \\
&= \frac{1}{h\log x} + \frac{\log\log x}{h\log^2 x} - \frac{2p}{\pi}\frac{1}{h^2\log^2 x}\log\log x \\
&\quad + \frac{2q}{\pi}\frac{1}{h^2\log^2 x}\log\log x + O\left(\frac{1}{\log^2 x}\right) \\
&= \frac{1}{h\log x} + \frac{\log\log x}{h^2\log^2 x}\left(h - \frac{2p}{\pi} + \frac{2q}{\pi}\right) + O\left(\frac{1}{\log^2 x}\right) \\
&= \frac{1}{h\log x} + O\left(\frac{1}{\log^2 x}\right).
\end{aligned}
\tag{3.8}
$$

On the last interval $[2h \log x, \infty)$ we have $G(v) \sim v \geq 2h \log x$ and the integral of the density is of order $\log^{-1} x$. Thus

$$I_5 = O(\log^{-2} x). \tag{3.9}$$

Adding the estimates for $I_1 - I_5$ in (3.5) – (3.9) gives the first part of the theorem. The second part is somewhat easier since the variable in the density in (3.3), $xt^{-1} - h \log t$, has a maximum value of $-h(1 + \log(-xh^{-1}))$ when $x < 0$ and this tends to $-\infty$ as $x \to -\infty$. Thus we may use the asymptotic behavior of the density for all values of t and work directly from (3.3). First note that for x sufficiently negative $xt^{-1} - h \log t < x/2t$ for all t and so by (2.2) we have

$$\int_0^{|x|/\log|x|} p(1, xt^{-1} - h \log t)t^{-1}\, dt = O\left(\int_0^{|x|/\log|x|} \frac{t^2}{x^2}\frac{1}{t}\, dt\right)$$
$$= O(\log^{-2} |x|).$$

For the rest of the integral, using (2.2) again, we have

$$\begin{aligned}&\int_{|x|/\log|x|}^{\infty} \frac{2q}{\pi}\frac{1}{(xt^{-1} - h \log t)^2} t^{-1}\, dt + \frac{4hq}{\pi}\int_{|x|/\log|x|}^{\infty} \frac{\log|xt^{-1} - h \log t|}{(xt^{-1} - h\log t)^3} t^{-1}\, dt \\ &\qquad + O\left(\int_{|x|/\log|x|}^{\infty} \frac{1}{|xt^{-1} - h \log t|^3} t^{-1}\, dt\right).\end{aligned} \tag{3.10}$$

The final term is of order $\log^{-2} |x|$ since we obtain a bound by integrating $t^{-1} \log^{-3} t$. For the first term, we expand the integrand as follows:

$$\frac{2q}{\pi}\frac{1}{h^2 \log^2 t}\frac{1}{t} + O\left(\frac{1}{\log^3 t}\frac{1}{t} \cdot \frac{|x|}{t}\right).$$

After integrating this we have

$$\frac{2q}{\pi}\frac{1}{h^2}\frac{1}{\log|x| - \log\log|x|} + O\left(\frac{1}{\log^2|x|}\right)$$
$$= \frac{2q}{\pi}\frac{1}{h^2}\left(\frac{1}{\log|x|} + \frac{\log\log|x|}{\log^2|x|}\right) + O\left(\frac{1}{\log^2|x|}\right).$$

Finally the main contribution from the middle term in (3.10) is

$$\frac{4hq}{\pi}\int_{|x|/\log|x|}^{\infty} \frac{\log\log t}{-h^3\log^3 t}\frac{1}{t}\, dt = -\frac{4hq}{\pi}\frac{1}{h^3}\frac{\log\log|x|}{2\log^2|x|} + O\left(\frac{1}{\log^2|x|}\right)$$

and it can be easily checked that the error is of order $\log\log|x|/\log^3|x|$. Adding these estimates we again see that the $\log\log|x|/\log^2|x|$ term disappears and we have proved the theorem.

The other result we will need about the kernel g will be the leading term for the difference $g(x) - g(y)$. One would guess from Theorem 3.1 that this should behave like $b \log(yx^{-1})/\log y \log x$ as $x, y \to +\infty$. Note, however, that even if $y = 2x$, for example, the error term in Theorem 3.1 does not allow us to conclude this. Nevertheless, further detailed analysis does show that this is the correct asymptotic behavior of $g(x) - g(y)$ provided only that y/x is bounded above and below.

THEOREM 3.2. *Suppose that there exists a $k > 0$ such that $k^{-1} \le yx^{-1} \le k$. Then*

$$g(x) - g(y) \sim b \frac{\log(yx^{-1})}{\log y \log x} \quad as \quad x, y \to +\infty,$$

$$g(x) - g(y) \sim d \frac{\log(yx^{-1})}{\log|y| \log|x|} \quad as \quad x, y \to -\infty,$$

where $b = 2p/\pi h^2$, $d = 2q/\pi h^2$.

PROOF. We first consider positive x and y. Since x and y may be interchanged, there is no loss in considering only $x \le y$. By (3.4)

$$(3.11) \quad g(x) - g(y) = \int_{-\infty}^{\infty} p(1, u) \frac{G(u + h \log y) - G(u + h \log x)}{\{G(u + h \log x) + h\}\{G(u + h \log y) + h\}} du.$$

By the mean value theorem there is an $\eta_u \in (u + h \log x, u + h \log y)$ such that

$$(3.12) \quad \begin{aligned} 0 \le G(u + h \log y) - G(u + h \log x) &= \frac{G(\eta_u)}{G(\eta_u) + h} h \log(yx^{-1}) \\ &\le \frac{G(u + h \log y)}{G(u + h \log y) + h} h \log(yx^{-1}). \end{aligned}$$

Fix $\varepsilon \in (0, h/2)$. We split the range of integration in (3.11) into five intervals. The main contributions will come from the intervals $[-(h + \varepsilon) \log x, -(h - \varepsilon) \log x]$ and $[-\varepsilon \log x, \varepsilon \log x]$. We first show that the integrals over the remaining three intervals are all of smaller order. On $(-\infty, -(h + \varepsilon) \log x]$, $p(1, u) = O(\log^{-2} x)$, the three factors in the denominator are all at least h, and the $\log(yx^{-1})$ term comes from (3.12). Then by (3.2)

$$\int_{-\infty}^{-(h+\varepsilon)\log x} G(u + h \log y)\, du < \int_{-\infty}^{-(h+\varepsilon)\log x} ye^{uh^{-1}}\, du = hyx^{-1-\varepsilon h^{-1}} = o(1).$$

Since $\log y = \log x + O(1)$ under the boundedness assumption on yx^{-1}, this is sufficient for this first interval. On the interval $[-(h - \varepsilon) \log x, -\varepsilon \log x]$, we work directly with (3.11). Since G is increasing, the integrand is positive and we may use $p(1, u) = O(\log^{-2} x)$ and then

$$\begin{aligned} &\int_{-(h-\varepsilon)\log x}^{-\varepsilon\log x} \left\{\frac{1}{G(u + h \log x) + h} - \frac{1}{G(u + h \log y) + h}\right\} du \\ &\quad = \int_{\varepsilon\log x}^{\varepsilon\log x + h\log(yx^{-1})} \frac{1}{G(v) + h} dv - \int_{(h-\varepsilon)\log x}^{(h-\varepsilon)\log x + h\log(yx^{-1})} \frac{1}{G(v) + h} dv \\ &\quad = o(\log(yx^{-1})) \end{aligned}$$

by (3.1). On $[\varepsilon \log x, \infty)$, we again use (3.12) in (3.11). The expression in (3.12) is $O(\log(yx^{-1}))$, the two terms in the denominator in (3.11) give $O(\log^{-1}x\ log^{-1}y)$ by (3.1) and then the integral of $p(1, u)$ over the interval is $o(1)$. Now we are ready for the two main intervals. In each case we need separate upper and lower bounds but they are essentially the same so we will only obtain the upper bounds. On the first interval we have by (2.2) that for x sufficiently large

$$p(1, u) \le \frac{2q}{\pi} (1 + \varepsilon) \frac{1}{(h - \varepsilon)^2 \log^2 x}$$

for all u in the interval. Then working directly from (3.11) we have this factor times

$$\int_{-(h+\varepsilon)\log x}^{-(h-\varepsilon)\log x} \left\{ \frac{1}{G(u+h\log x)+h} - \frac{1}{G(u+h\log y)+h} \right\} du$$

$$= \int_{-\varepsilon\log x}^{-\varepsilon\log x + h\log(yx^{-1})} \frac{1}{G(v)+h}\, dv - \int_{\varepsilon\log x}^{\varepsilon\log x + h\log(yx^{-1})} \frac{1}{G(v)+h}\, dv$$

$$= \log(yx^{-1})(1+o(1))$$

since the integrands are $\sim h^{-1}$ and $o(1)$ respectively by (3.2) and (3.1). Thus for this part of the integral we have an upper bound of

$$\frac{2q}{\pi}(1+\varepsilon)^2 \frac{1}{(h-\varepsilon)^2} \frac{\log(yx^{-1})}{\log x \log y}. \tag{3.13}$$

On the final interval we use (3.12) in (3.11). We then have as a bound for all the terms except $p(1, u)$

$$\frac{1}{G((h-\varepsilon)\log x)} \frac{1}{G((h-\varepsilon)\log y)} h \log(yx^{-1}) \le (1+\varepsilon) \frac{h}{(h-\varepsilon)^2} \frac{\log(yx^{-1})}{\log x \log y}.$$

This bound comes from (3.1) and is uniform for all $u \in [-\varepsilon \log x, \varepsilon \log x]$ for x sufficiently large. Of course the integral of $p(1, u)$ over this interval is then ~ 1. Thus we have as an upper bound for the part of the integral over this interval

$$(1+\varepsilon) \frac{h}{(h-\varepsilon)^2} \frac{\log(yx^{-1})}{\log x \log y}. \tag{3.14}$$

Adding the bounds in (3.13) and (3.14) gives the upper bound for the first result in Theorem 3.2. As indicated above, the lower bounds are essentially the same. For negative x and y, it is easiest to develop the analogue of (3.4) which we did not need in the proof of Theorem 3.1. Here we use

$$F(u) = -u + h \log u, \quad u > 0$$

which increases on $(0, h]$ and then decreases on $[h, \infty)$. We let $\gamma = -h + h \log h$, the maximum value of F, and define on $(-\infty, \gamma]$ the two inverse functions of F, G_1 and G_2, with G_1 taking values in $(0, h]$ and G_2 in $[h, \infty)$. Then, for $x < 0$

$$g(x) = \int_0^\infty p(1, -|x|t^{-1} - h\log t)t^{-1}dt = \int_0^\infty p(1, F(|x|t^{-1}) - h\log|x|)t^{-1}\,dt$$

$$= \int_{-\infty}^{\gamma} p(1, v - h\log|x|)(G_2(v) - h)^{-1}\,dv$$

$$+ \int_{-\infty}^{\gamma} p(1, v - h\log|x|)(h - G_1(v))^{-1}\,dv$$

$$= \int_{-\infty}^{\gamma} p(1, v - h\log|x|)G_3(v)\,dv,$$

where

$$G_3(v) = (G_2(v) - G_1(v))(G_2(v) - h)^{-1}(h - G_1(v))^{-1}.$$

As $v \to -\infty$, $G_1(v) \to 0$ and $G_2(v) \to \infty$ so that $G_3(v) \to h^{-1}$. Also both $(h - G_1(v))^{-1}$ and $(G_2(v) - h)^{-1}$ and hence $G_3(v)$ are integrable near γ since, for example, making the substitution $v = F(u)$,

$$\int^{\gamma} (h - G_1(v))^{-1} dv = \int^{h} (h - u)^{-1}(-1 + hu^{-1})\, du = \int^{h} u^{-1}\, du.$$

Now we may complete the proof. We have, for $y \leq x$,

$$\begin{aligned} g(x) - g(y) &= \int_{-\infty}^{\gamma} \{p(1, v - h\log|x|) - p(1, v - h\log|y|)\} G_3(v)\, dv \\ &= \int_{-\infty}^{\gamma} p'(1, \eta_v)\, h \log(yx^{-1}) G_3(v)\, dv \end{aligned} \tag{3.15}$$

where $p'(1, u)$ is the derivative of $p(1, u)$ and $\eta_v \in (v - h\log|y|, v - h\log|x|)$. Since $p'(1, u)$ is increasing for u sufficiently negative, we may obtain upper and lower bounds in (3.15) by putting in the extreme values for η_v. Fix $\varepsilon > 0$ and choose M so that $G_3(v) < (1 + \varepsilon)h^{-1}$ for $v \leq -M$. Then

$$\begin{aligned} \int_{-\infty}^{-M} p'(1, \eta_v)\, h \log(yx^{-1}) G_3(v)\, dv &\leq (1 + \varepsilon) \log(yx^{-1}) \int_{-\infty}^{-M} p'(1, v - h\log|x|)\, dv \\ &= (1 + \varepsilon) \log(yx^{-1})\, p(1, -M - h\log|x|) \\ &\sim (1 + \varepsilon) \log(yx^{-1}) \frac{2q}{\pi} \frac{1}{h^2 \log^2|x|}. \end{aligned}$$

For the interval $[-M, \gamma]$ we have $p'(1, \eta_v) = O(\log^{-3}|x|)$ and so by the integrability of G_3, the integral over this interval is of order $\log(yx^{-1}) \log^{-3}|x|$. This is sufficient for the upper bound; the lower bound clearly follows in the same way.

We conclude this section with three observations about the properties of g. By (3.11) and the monotonicity of G, we see that

$$g \text{ decreases on } [0, \infty). \tag{3.16}$$

On the negative side, we cannot say as much. But since $p'(1, u) > 0$ for u sufficiently negative, it follows from (3.15) that there is an $M > 0$ such that

$$g \text{ increases on } (-\infty, -M]. \tag{3.17}$$

Finally, we note that g is continuous [11].

4. Capacitary distributions. In this section we will obtain the asymptotic estimates that we need for the capacitary distribution on a large interval and on the union of two such intervals. As above, we assume that $h > 0$. In this case, most of the mass of the capacitary distribution for a long interval is spread relatively close to the right end of the interval. Perhaps the most interesting result here is Theorem 4.5 which gives a very precise estimate for this part of the distribution.

We state two results initially since it is more convenient to prove them simultaneously. The constants in all these results are uniform in all the variables so long as they are in the given range.

THEOREM 4.1. *Let $C(a)$ denote the capacity of an interval of length a. Then*

$$C(a) = b^{-1}\log a + O(1)$$

as $a \to \infty$, where $b = 2p/\pi h^2$.

LEMMA 4.2. *Let μ be the capacitary distribution on* $[0, a]$. *Then*

$$\mu([0, x]) = O(xa^{-1}), \quad 10 \le ax^{-1} \le \log a/(\log\log a)^3,$$

as $a \to \infty$.

PROOFS. Since g decreases for $x \ge 0$ by (3.16), we have by (2.4)

$$1 = \int_0^a g(x)\mu(dx) \ge g(a)C(a). \tag{4.1}$$

By Theorem 3.1, this suffices for the upper bound in Theorem 4.1.

Next we prove the lemma. Divide the interval $[a/2, a]$ into $\log^3 a$ intervals of length $a/2\log^3 a$. Since $\mu([0, a]) = O(\log a)$ by (4.1), at least one of these intervals must have measure $O(\log^{-2} a)$. Take such an interval, say $[u, v]$, and let $y = (u + v)/2$. By (2.4), we have

$$\int_0^a \{g(z) - g(z-y)\}\mu(dz) = 1 - 1 = 0,$$

so that

$$\int_0^u \{g(z) - g(z-y)\}\mu(dz) = \int_u^a \{g(z-y) - g(z)\}\mu(dz). \tag{4.2}$$

We estimate the right side of (4.2):

$$\int_u^v \{g(z-y) - g(z)\}\mu(dz) = O(\mu([u, v])) = O(\log^{-2} a),$$

$$\int_v^a \{g(z-y) - g(z)\}\mu(dz) \le \{g(a/4\log^3 a) - g(a)\}C(a) = O(\log\log a/\log a)$$

where we have used Theorem 3.1 and (4.1) in the last estimate. The left hand side of (4.2) is at least

$$\left(\frac{b-\varepsilon}{\log a} - \frac{d+\varepsilon}{\log a}\right)\mu([0, u]) \tag{4.3}$$

for large a. Since $h > 0$ implies $b > d$, these estimates yield

$$\mu([0, a/2]) \le \mu([0, u]) = O(\log\log a). \tag{4.4}$$

With this estimate at our disposal we can now prove the lemma by essentially repeating the argument. Divide $[x, 2x]$ into $\log^3 a$ intervals of length $x/\log^3 a$. One of these must have measure $O(\log^{-2} a)$. We abuse the notation slightly by letting $[u, v]$ denote this new interval and $y = (u + v)/2$. We again use (4.2). In estimating the right side, the integral over $[u, v]$ is $O(\log^{-2} a)$ as before. Next,

$$\int_v^{a/2} \{g(z-y) - g(z)\}\mu(dz) = O\left(\frac{\log\log a}{\log^2 a}\right)\mu([0, a/2]) = O\left(\frac{(\log\log a)^2}{\log^2 a}\right)$$

where we have used the upper bound for ax^{-1} and Theorem 3.1 for the first equality and then (4.4). Adding these first two estimates and using the upper bound for ax^{-1} again we see they are of order $(\log\log a \log a)^{-1}xa^{-1}$. Finally, by Theorem 3.2,

$$\int_{a/2}^a \{g(z-y) - g(z)\}\mu(dz) \le (b+\varepsilon)\log^{-2} a \int_{a/2}^a \log\left(\frac{z}{z-y}\right)\mu(dz)$$

$$\le (b+\varepsilon)\log^{-2} a \frac{2x}{\frac{1}{2}a - 2x} C(a) = O((\log^{-1} a)xa^{-1})$$

where we have used (4.1) and the lower bound for ax^{-1} at the last step. Note that this term dominates the earlier two integrals. Since the lower bound for the left side of (4.2) given in (4.3) is still valid and $[0, x] \subset [0, u]$, this completes the proof of the lemma.

To obtain the lower bound for $C(a)$, divide the interval $[0, a/20]$ into $\log^2 a$ intervals of length $a/20\log^2 a$. One must have measure $O(\log^{-1} a)$; again we denote it by $[u, v]$ with $y = (u + v)/2$. Then

$$1 = \int_0^a g(z - y)\mu(dz)$$
$$= \int_0^{a/10} g(z - y)\mu(dz) + (b \log^{-1}a + O(\log^{-2}a))\mu([a/10, a]).$$

Now the integral over $[0, a/10]$ is of order $\log^{-1} a$ since the measure of $[u, v]$ is of this order and the integrand is of this order on the complement of $[u, v]$ while the measure of $[0, a/10]$ is $O(1)$ by Lemma 4.2. Thus we have

$$C(a) \geq \mu([a/10, a]) = b^{-1}\log a(1 + O(\log^{-1}a))$$

which completes the proof of the theorem.

Thus far we have seen that the capacity of an interval of length a grows like $\log a$ but that the capacitary measure of an interval of length $a/10$ at the left end is bounded. Actually most of the measure is concentrated relatively near the right end of the interval and we will obtain a very precise estimate for $\mu([a - x, a])$. As a first step in this direction we need the next lemma.

LEMMA 4.3. *Let μ be the capacitary distribution on $[0, a]$ and let $\gamma \in (1, bd^{-1})$ be a fixed constant. Then if $x \leq y \leq a \wedge x^\gamma$,*

$$\mu([a - y, a - x]) \leq \left(\frac{b(\delta - 1)}{b - d\delta} + O\left(\frac{\log \log x}{\log x}\right)\right)\mu([a - x, a])$$

as $x \to \infty$, where δ is defined by $y = x^\delta$.

PROOF. Divide $[a - x, a]$ into $\log^2 x$ intervals of length $x/\log^2 x$. One will have measure no larger than $\log^{-2} x\mu([a - x, a])$; denote it by $[u, v]$ and let $w = (u + v)/2$. Since both w and $a - y$ are in $[0, a]$, we have

$$1 = \int_0^a g(z - w)\mu(dz) = \int_0^a g(z - a + y)\mu(dz)$$

and so

$$\int_0^u \{g(z - a + y) - g(z - w)\}\mu(dz) = \int_u^a \{g(z - w) - g(z - a + y)\}\mu(dz). \tag{4.5}$$

We start by obtaining a lower bound for the left side of (4.5). First the integrand is nonnegative for $z \in [0, a - y]$ since g is increasing for sufficiently negative arguments by (3.17) and since $z - w$ is very large negative on this range. For $z \in [a - y, u]$,

$$g(z - a + y) - g(z - w) \geq \frac{b}{\log y} - \frac{d}{\log(x/2 \log^2 x)} + O(\log^{-2} x)$$
$$= (b\delta^{-1} - d)\log^{-1} x + O(\log \log x \log^{-2} x).$$

Thus the left side of (4.5) is at least

$$(b\delta^{-1} - d)\log^{-1} x(1 + O(\log \log x \log^{-1} x))\mu([a - y, a - x]). \tag{4.6}$$

Now we need an upper bound for the right side of (4.5). For the integral over $[u, v]$ we simply use the measure which is $O(\log^{-2}x\mu([a-x, a]))$. Then

$$\int_v^a \{g(z-w) - g(z-a+y)\}\mu(dz) \le \left(\frac{b}{\log(x/2\log^2 x)} - \frac{b}{\log y} + O\left(\frac{1}{\log^2 x}\right)\right)\mu([v, a])$$

$$\le \left(\frac{b(\delta-1)}{\delta \log x} + O\left(\frac{\log\log x}{\log^2 x}\right)\right)\mu([a-x, a]),$$

and using these bounds together with the bound in (4.6) completes the proof.

Now we can prove the key lemma and the theorem which give the estimates for $\mu([a-x, a])$. We state and prove them together since the first half of the theorem is needed in the proof of the second half of the lemma.

LEMMA 4.4. *Let μ be the capacitary distribution on $[0, a]$ and let $\lambda = b/(b-d)$, $\eta = (\log\log a)^{-2}$. Then if $x \ge \exp\{(\log\log a)^5\}$ and $y = x^{1+\eta}$, $y \le a$, it follows that*

$$\mu([a-y, a]) = (1 + \lambda\eta + O(\eta^2))\mu([a-x, a])$$

as $a \to \infty$.

THEOREM 4.5. *Let μ be the capacitary distribution on $[0, a]$ and let $\lambda = b/(b-d)$. Then*

$$\mu([a-x, a]) \sim \left(\frac{\log x}{\log a}\right)^\lambda \frac{\log a}{b}$$

as $a \to \infty$, uniformly for $x \in [\exp\{(\log\log a)^5\}, a]$.

PROOFS. The upper bound in Lemma 4.4 follows immediately from Lemma 4.3 since

$$\frac{b(\delta-1)}{b-d\delta} = \frac{b\eta}{b-d-d\eta} = \lambda\eta + O(\eta^2)$$

and

$$\frac{\log\log x}{\log x} \le \frac{5\log\log\log a}{(\log\log a)^5} \le \eta^2 \tag{4.7}$$

for large a.

Next we prove the lower bound of Theorem 4.5. Define

$$x_i = a^{(1+\eta)^{-i}}$$

and let $j = \min\{i : x_i < x\}$. Since $x_{j-1} \ge x$, we have

$$(1+\eta)^{j-1} \le \log a \log^{-1} x \le \log a \tag{4.8}$$

which implies that $j\eta = O(\log\log a)$. Thus

$$j = O((\log\log a)^3) \quad \text{and} \quad j\eta^2 = o(1). \tag{4.9}$$

Now we use the half of Lemma 4.4 that has been proved and Lemma 4.3 to obtain

$$\begin{aligned} \mu([a-x, a]) &= \mu([0, a]) \prod_{i=1}^{j-1} \frac{\mu([a-x_i, a])}{\mu([a-x_{i-1}, a])} \cdot \frac{\mu([a-x, a])}{\mu([a-x_{j-1}, a])} \\ &\ge \mu([0, a])(1+\lambda\eta + O(\eta^2))^{-j+1}(1+O(\eta)) \\ &\ge \mu([0, a])\exp\{-(\lambda\eta + O(\eta^2))j\}(1+o(1)) \\ &= \mu([0, a])e^{-j\lambda\eta}(1+o(1)) \end{aligned} \tag{4.10}$$

where (4.9) has been used at the last step. Now by (4.8),

$$\exp\{(\eta - \eta^2)(j-1)\} \le \log a \log^{-1} x$$

which leads to $e^{j\eta} \le \log a \log^{-1}x(1 + o(1))$ by (4.9). Using this in (4.10) and recalling Theorem 4.1 completes the proof of the lower bound of Theorem 4.5.

Now we complete the proof of Lemma 4.4. Divide the interval $[a - y, a - y/2]$ into $\log^2 y$ intervals of length $y/2\log^2 y$. One of these must have measure no larger than $\log^{-2}y\,\mu([a - y, a])$; denote this interval by $[u, v]$ and its midpoint by w. Then as in (4.5) we have

$$\int_0^{a-x} \{g(z - w) - g(z - a + x)\}\mu(dz) = \int_{a-x}^{a} \{g(z - a + x) - g(z - w)\}\mu(dz). \tag{4.11}$$

This time we need an upper bound for the left side and this makes things a bit more complicated than in the proof of Lemma 4.3. We will split this integral into four parts. First

$$\begin{aligned}\int_v^{a-x} &\{g(z - w) - g(z - a + x)\}\mu(dz) \\ &\le \left(\frac{b}{\log(y/4\log^2 y)} - \frac{d}{\log y} + O\left(\frac{1}{\log^2 y}\right)\right)\mu([a - y, a - x]) \\ &= ((b - d)\log^{-1}y + O(\log\log y \log^{-2}y))\mu([a - y, a - x]).\end{aligned} \tag{4.12}$$

Then

$$\int_u^{v} \{g(z - w) - g(z - a + x)\}\mu(dz) = O(\log^{-2}y)\mu([a - y, a]) \tag{4.13}$$

by the choice of the interval $[u, v]$. Next choose $\delta \in (1, bd^{-1})$, let $s = y^\delta \wedge a$, and consider the integral over $[a - s, u]$. The integrand is $O(\log\log y \log^{-2}y)$ on $[a - 2y, u]$ by Theorem 3.1 and $O(\log^{-2}y)$ on $[a - s, a - 2y]$ by Theorem 3.2. Since $\mu([a - s, a]) = O(\mu([a - x, a]))$ by Lemma 4.3 we have

$$\int_{a-s}^{u} \{g(z - w) - g(z - a + x)\}\mu(dz) = O(\log\log x \log^{-2}x)\mu([a - x, a]). \tag{4.14}$$

Note that since $\mu([a - y, a]) \le \mu([a - s, a])$, the error terms in (4.12) and (4.13) are also of this order. Finally, on $[0, a - s]$ we have by Theorem 3.2 that the integrand is of order

$$\log^{-2}y \log\left(\frac{a - x - z}{w - z}\right) \le (\log^{-2}y)\frac{a - x - w}{w - z} \le (\log^{-2}y)\frac{y}{y^\delta - y} \sim y^{1-\delta}\log^{-2}y.$$

Now since $y \ge x \ge \exp\{(\log\log a)^5\}$, it follows that

$$\begin{aligned}y^{1-\delta} &\le \exp\{(1 - \delta)(\log\log a)^5\} \le \exp\{-\lambda \log\log a\} \le (\log y \log^{-1}a)^\lambda \\ &= O(\log^{-1}a)\mu([a - y, a])\end{aligned}$$

where we have used the half of Theorem 4.5 that has been proved at the last step. Thus, since the total mass of μ is of order $\log a$, we have

$$\int_0^{a-s} \{g(z - w) - g(z - a + x)\}\mu(dz) = O(\log^{-2}y)\mu([a - y, a]). \tag{4.15}$$

Combining (4.12) – (4.15) and recalling that $\mu([a - y, a]) = O(\mu([a - x, a]))$, we see that the left side of (4.11) is at most

$$(b - d)\log^{-1}y\,\mu([a - y, a - x]) + O(\log\log x \log^{-2}x)\mu([a - x, a]).$$

Now we need a lower bound for the right side of (4.11). If we take a large enough so that $y \ge 3x$, then by Theorem 3.1 we have this integral is at least

$$\left(\frac{b}{\log x} - \frac{b}{\log y} + O\left(\frac{1}{\log^2 x}\right)\right)\mu([a - x, a]) = \left(\frac{b\eta}{\log y} + O\left(\frac{1}{\log^2 x}\right)\right)\mu([a - x, a]).$$

Putting these two bounds together and recalling (4.7) completes the proof of Lemma 4.4.

Now we can finish the proof of Theorem 4.5. With x_i as before we have as in (4.10)

$$\mu([a-x,a]) \le \mu([a-x_{j-1},a]) \le \mu([0,a])(1+\lambda\eta+O(\eta^2))^{-j+1}$$
$$\le \mu([0,a])\exp\{(\lambda\eta+O(\eta^2))(-j+1)\} = \mu([0,a])e^{-j\lambda\eta}(1+o(1))$$

by (4.9). Finally, $x_j < x$ implies

$$\log a \log^{-1} x \le (1+\eta)^j \le e^{j\eta}$$

and this completes the proof.

We also need bounds similar to those in Lemma 4.2 and Theorem 4.5 for the capacitary distribution on the union of two intervals. Since the proofs are very similar to the ones given above (and somewhat more tedious), we will simply state the results.

LEMMA 4.6. *Let μ be the capacitary distribution on $[0,\alpha]\cup[\beta,a]$ where $2<\alpha<\beta$. Then*

$$\mu([0,x]) = O(xa^{-1}), \quad \text{if } x\ge\beta,\ 10\le ax^{-1}\le \log a/(\log\log a)^3,$$
$$\mu([0,\alpha]) = O(\beta\log\alpha/a\log a), \quad \text{if } 10\le a\beta^{-1}\le \log a/(\log\log a)^3,$$

as $a\to\infty$. (In these results α may tend to ∞ as long as $\alpha<\beta$.) If $10\le a\beta^{-1}\le \log a/(\log\log a)^3$ and $10\le\beta\alpha^{-1}$, then

$$\mu([0,x]) = O(xa^{-1}), \quad \text{if } \beta\le\alpha\log\alpha,\ 10\le\alpha x^{-1}\le\log\alpha/(\log\log\alpha)^3,$$
$$\mu([0,x]) = O(x\beta/\alpha a\log a), \quad \text{if } \beta\ge\alpha\log\alpha,\ 10\le\alpha x^{-1}\le\log\alpha/(\log\log\alpha)^3,$$

as $\alpha\to\infty$.

THEOREM 4.7. *Let μ be the capacitary distribution on $[0,a-\beta]\cup[a-\alpha,a]$ where $\beta>\alpha$, $\beta\ge\exp\{(\log\log a)^5\}$. Then*

$$\mu([a-x,a]) \sim (\log x\log^{-1}a)^\lambda\mu([0,a])$$

as $a\to\infty$, uniformly for $x\in[2\beta,a]$. Also

$$\mu([a-\alpha,a]) \sim \frac{(b-d)\log\alpha}{b\log\beta-d\log\alpha}\mu([a-2\beta,a])$$

and if $\beta\ge\alpha^2$

$$\mu([a-x,a]) \sim (\log x\log^{-1}\alpha)^\lambda\mu([a-\alpha,a])$$

as $\alpha\to\infty$, uniformly for $x\in[\exp\{(\log\log\alpha)^5\},\alpha]$.

5. Hitting probabilities. We can now obtain the estimates we require for the probability of hitting a long interval or the union of two such intervals. Recall that $\Phi(x,K)$ is the probability of hitting K, starting from x, and by (2.3) is the potential of the capacitary distribution on K. With the information that we have obtained about the kernel and the capacitary distribution, these probabilities are now fairly easy to estimate. Note that the probability of hitting a long interval is very close to one so it is more natural to give the estimates for the "missing probabilities" $1-\Phi(x,K)$.

The first two theorems give estimates for the probability of missing a large interval starting from relatively nearby points on the left and right respectively. Recall that we are assuming that $h>0$.

THEOREM 5.1. *Let $K=[0,a]$. Then*

$$\frac{1}{2}\frac{x}{a\log a} \le 1-\Phi(-x,K) \le 4\frac{x}{a\log a} \quad \text{if } 10\le ax^{-1}\le\log a/(\log\log a)^4,$$

and a is sufficiently large.

REMARK. The additional assumption $ax^{-1} \to \infty$ leads almost as easily to $1 - \Phi(-x, K) \sim x/a \log a$.

PROOF. Divide the interval $[0, x/\log\log x]$ into $\log x$ intervals of length $x/\log x \log\log x$. By Lemma 4.2 one of these intervals must have capacitary measure $O(x/a \log x \log\log x)$. Denote it by $[u, v]$ with $y = (u + v)/2$. Then

$$1 - \Phi(-x, K) = \int_0^a \{g(z - y) - g(z + x)\}\mu(dz)$$

where μ is the capacitary distribution for K. We split the range of integration into four intervals. By Theorem 3.1, Lemma 4.2, and (4.4),

$$\int_0^u \{g(z - y) - g(z + x)\}\mu(dz) = O(\log^{-1} x)O(x/a \log\log x),$$

$$\int_u^v \{g(z - y) - g(z + x)\}\mu(dz) = O(x/a \log x \log\log x),$$

$$\int_v^{a/2} \{g(z - y) - g(z + x)\}\mu(dz) = O(\log\log a/\log^2 a)O(\log\log a)$$

$$= O(x/a \log a(\log\log a)^2),$$

where we have used the upper bound for ax^{-1} in the last bound. Since $\log x \sim \log a$, these terms are all of smaller order than the main term and so can be ignored. Finally, by Theorem 3.2 and Theorem 4.1

$$\int_{a/2}^a \{g(z - y) - g(z + x)\}\mu(dz)$$

$$\leq (b + \varepsilon)\log^{-2} a \log\left(\frac{x + a/2}{-y + a/2}\right)b^{-1}\log a \tag{5.1}$$

$$\leq (b + \varepsilon)b^{-1}\log^{-1}a\frac{x + y}{-y + a/2} \sim (b + \varepsilon)b^{-1}2x/a \log a$$

which gives the upper bound. Recalling (4.4), we can estimate the lower bound for (5.1) similarly.

THEOREM 5.2. *Let $K = [0, a]$. Then*

$$1 - \Phi(a + x, K) \sim \lambda^{-1}(\log x \log^{-1}a)^{\lambda-1} \quad \textit{if } \exp\{(\log\log a)^5\} \leq x \leq a,$$

as $a \to \infty$, uniformly for $x \in [\exp\{(\log\log a)^5\}, a]$, where $\lambda = b/(b - d)$.

PROOF. For the lower bound,

$$1 - \Phi(a + x, K) = \int_0^a \{g(z - a + x) - g(z - a - x)\}\mu(dz)$$

$$\geq \int_{a-x}^a \{g(z - a + x) - g(z - a - x)\}\mu(dz)$$

since the integrand is nonnegative for $z \leq a - x$ as in the proof of Lemma 4.3. Then

$$1 - \Phi(a + x, K) \geq \left\{\frac{b}{\log x} - \frac{d}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right\}\mu([a - x, a])$$
$$\sim (b - d)b^{-1}(\log x \log^{-1} a)^{\lambda - 1}$$

by Theorems 3.1 and 4.5. For the upper bound, divide $[a - x, a - x/2]$ into $\log^2 x$ intervals of length $x/2\log^2 x$. One must have measure no larger than $\log^{-2} x\mu([a - x, a])$. Denote it by $[u, v]$ and its midpoint by y. Then

$$1 - \Phi(a + x, K) = \int_0^a \{g(z - y) - g(z - a - x)\}\mu(dz)$$

and

$$\int_v^a \{g(z - y) - g(z - a - x)\}\mu(dz) \leq \left\{\frac{b}{\log(x/4\log^2 x)} - \frac{d}{\log x} + O\left(\frac{1}{\log^2 x}\right)\right\}\mu([v, a])$$
$$\leq \left\{\frac{b - d}{\log x} + O\left(\frac{\log\log x}{\log^2 x}\right)\right\}\mu([a - x, a])$$

which has the right asymptotic behavior. Thus it remains to show the integral over $[0, v]$ is smaller. The interval $[u, v]$ is all right by the way it was chosen. Let $s = x^2 \wedge a$; then

$$\int_{a-s}^u \{g(z - y) - g(z - a - x)\}\mu(dz) = O(\log\log x \log^{-2} x)\mu([a - s, a])$$

and this is small enough since $\mu([a - s, a]) \doteq O(\mu([a - x, a]))$ by Theorem 4.5. Finally, using Theorems 3.2 and 4.5, we have

$$\int_0^{a-s} \{g(z - y) - g(z - a - x)\}\mu(dz) = O(x^{-1}\log^{-2} x)\mu([0, a])$$
$$= O(x^{-1}\log^{-2} x(\log a \log^{-1} x)^{\lambda})\mu([a - x, a])$$
$$= O(e^{\lambda\log\log a - (\log\log a)^5}\log^{-2} x)\mu([a - x, a])$$
$$= O(\log^{-2} x)\mu([a - x, a])$$

for large a, where we have used the lower bound for x. This completes the proof of the theorem.

REMARK. It is a consequence of Theorems 5.1 and 5.2 that the probability of hitting an interval of length a, a large, starting from a point whose distance from the interval is of order a is close to one if the starting point is to the left of the interval and close to d/b if the starting point is to the right. This is another indication of the asymmetry. Theorem 5.2 is not very informative in the completely asymmetric case ($\lambda = 1$), but we will not need it then.

We need similar bounds for the probability of missing the union of two intervals. Since the proofs are similar we simply state the results.

THEOREM 5.3. *Let* $K = [0, \alpha] \cup [\beta, a]$ *where* $10 \leq a\beta^{-1} \leq \log a/(\log\log a)^3$ *and* $10 \leq \alpha x^{-1} \leq \log \alpha/(\log\log \alpha)^4$. *Then*

$$1 - \Phi(-x, K) = O(x/a\log a) \quad \text{if } \beta \leq \alpha\log\alpha,$$
$$= O(x\beta/\alpha a\log\alpha\log a) \quad \text{if } \beta \geq \alpha\log\alpha,$$

as $x \to \infty$.

THEOREM 5.4. *Let* $K = [0, a - \beta] \cup [a - \alpha, a]$ *where* $\beta \geq \alpha^2$, $\beta \geq \exp\{(\log\log a)^5\}$. *Then*

$$1 - \Phi(a + x, K) = O((\log x \log^{-1}\alpha)^{\lambda-1}(\log \beta \log^{-1} a)^{\lambda-1}),$$

if $\exp\{(\log\log \alpha)^5\} \leq x \leq \alpha$ *and* $\alpha \to \infty$, *where* $\lambda = b/(b-d)$.

For the rate of escape problem we will need estimates for delayed hitting probabilities. These are given in the next three lemmas. As above we assume that $h > 0$.

LEMMA 5.5. *Suppose* $\varepsilon > 0$. *Then for* a, T *sufficiently large*

$$P^x\{|X_t| \leq a \text{ for some } t \geq T\} \leq (1+\varepsilon)\log a \log^{-1} T \quad \text{for all } x.$$

PROOF. First we write

$$P^x\{|X_t| \leq a \quad \text{for some } t \geq T\} = \int p(T, y - x)\Phi(y, [-a, a])\, dy.$$

Now let μ be the capacitary distribution for $[-a, a]$, use (2.3) and (1.3) and interchange order of integration to obtain

$$P^x\{|X_t| \leq a \quad \text{for some } t \geq T\} = \int_{-a}^{a}\int_0^\infty p(t + T, z - x)\, dt\, \mu(dz)$$

$$\leq \sup_y \int_T^\infty p(s, y)\, ds\, \mathrm{Cap}([-a, a]).$$

First consider $|y| \leq T$. Then for $s \geq T$ we have $|ys^{-1}| \leq 1$ so that

$$\int_T^\infty p(s, y)\, ds = \int_T^\infty p(1, ys^{-1} - h\log s)s^{-1}\, ds \sim \int_T^\infty \frac{2q}{\pi}\frac{1}{h^2\log^2 s}\frac{ds}{s} = \frac{d}{\log T}.$$

For $|y| > T$, we simply bound the integral by $g(y)$ and use Theorem 3.1. Thus we obtain

$$\sup_y \int_T^\infty p(s, y)\, ds \leq (1+\varepsilon)b\log^{-1} T$$

for all sufficiently large T. Using this with the capacity estimate of Theorem 4.1 yields the result.

LEMMA 5.6. *If* $c < d/2b$, *then*

$$P^x\{|X_t| \leq a \quad \text{for some } t \leq T\} \geq c\log a \log^{-1}|x|,$$

provided that $T \geq |x|^{2b/d}$, $|x| \geq 2a$, *and* a *is sufficiently large.*

PROOF. We write

$$P^x\{|X_t| \leq a \quad \text{for some } t \leq T\} \geq \Phi(x, [-a, a]) - P^x\{|X_t| \leq a \quad \text{for some } t \geq T\}.$$

By (2.3) and Theorems 3.1 and 4.1, the first term on the right is at least

$$(1-\varepsilon)\, d\log^{-1}|x|b^{-1}\log a,$$

while the previous lemma implies that the second term is at most

$$(1+\varepsilon)\log a \log^{-1} T \leq (1+\varepsilon)(d/2b)\log a \log^{-1}|x|.$$

Subtracting the two estimates gives the result.

LEMMA 5.7. *Suppose that* $h < 2/\pi$. *Then there is a* $c > 0$ *such that*

$$P\{|X_t| \le a \quad \text{for some } t \in [T_1, T_2]\} \ge c \log a \log^{-1} T_1,$$

provided that $T_2 \ge T_1^{5b/d}$, $a \le hT_1 \log T_1/2$, *and a is sufficiently large.*

PROOF. First write

$$\begin{aligned} P\{|X_t| \le a \quad &\text{for some } t \in [T_1, T_2]\} \\ &= \int p(T_1, x) P^x\{|X_t| \le a \quad \text{for some } t \le T_2 - T_1\}\, dx \\ &\ge c \log a \log^{-1} T_1^2 P\{hT_1 \log T_1 \le |X_{T_1}| \le T_1^2\}, \end{aligned}$$

where at the last step we have restricted the range of integration as indicated and used Lemma 5.6, the assumptions being valid for large enough a. To estimate the probability, we have

$$\begin{aligned} P\{hT_1 \log T_1 \le |X_{T_1}| \le T_1^2\} &= \int_F p(1, xT_1^{-1} - h \log T_1) T_1^{-1}\, dx \\ &\ge \int_G p(1, u)\, du, \end{aligned}$$

where

$$F = \{hT_1 \log T_1 \le |x| \le T_1^2\}, \quad G = \{0 \le x \le T_1 - h \log T_1\}.$$

Since the last integral is asymptotically $P\{X_1 \ge 0\} > 0$, this is sufficient to complete the proof.

Finally, for the result on the Lebesgue measure of the range, we need some results on the probability of hitting one and two point sets before moving too far toward $+\infty$. Again we assume that $h > 0$.

LEMMA 5.8. *If* $a \log^{-\alpha} a < x < y < a - a \log^{-\alpha} a$ *and* $y - x > a \log^{-\alpha} a$ *for some* $\alpha > 0$, *then as* $a \to \infty$

(i) $P\{X_t \textit{ hits } \{x\} \textit{ before } [a, \infty)\} = \dfrac{1}{\log a}\left(1 + O\left(\dfrac{\log \log a}{\log a}\right)\right)$,

(ii) $P^x\{X_t \textit{ hits } \{y\} \textit{ before } [a, \infty)\} = \dfrac{1}{\log a}\left(1 + O\left(\dfrac{\log \log a}{\log a}\right)\right)$,

(iii) $P^y\{X_t \textit{ hits } \{x\} \textit{ before } [a, \infty)\} = O\left(\dfrac{\log \log a}{\log^2 a}\right)$.

PROOF. Since the proofs of (i) and (ii) are essentially the same, we will only prove (ii). Let $A = [a, \infty)$, $B = [a, a + 2a/\log^{2\alpha} a]$, $C = A \backslash B$, $D = [a, a + a/\log^{2\alpha} a]$. For any set E, T_E will denote the first hitting time of E, i.e.

$$T_E = \inf\{t > 0 : X_t \in E\},$$

with the usual provision that $T_E = \infty$ if $X_t \notin E$ for all t. If $E = \{z\}$ we write T_z for T_E. Now

$$P^x\{T_y < T_B\} - P^x\{T_C < T_y < T_D\} \le P^x\{T_y < T_A\} \le P^x\{T_y < T_B\}$$

and we estimate the error term by

$$\begin{aligned} P^x\{T_C < T_y < T_D\} &\le E^x\{1\{T_C < T_D\} P^{X_{T_C}}\{T_y < \infty\}\} \\ &= O(\log^{-1} a) P^x\{T_C < T_D\} = O(\log \log a \log^{-2} a) \end{aligned}$$

where the last estimate follows Lemma 1 of [12] and the estimate for $P^z\{T_y < \infty\}$ follows from Theorem 3.1 since the capacity of a one point set must be $1/g(0)$. To estimate $P^x\{T_y < T_B\}$, we use

$$
\begin{aligned}
P^x\{T_y < \infty\} &= P^x\{T_y < T_B\} + P^x\{T_B < T_y < \infty\} \\
&= P^x\{T_y < T_B\} + P^x\{T_B < T_y\}\left(\frac{d}{\log a} + O\left(\frac{\log\log a}{\log^2 a}\right)\right)\frac{1}{g(0)},
\end{aligned}
\tag{5.2}
$$

where we have started the process over when it hits B, and used Theorem 3.1 to estimate the probability of hitting $\{y\}$ starting from some point in B. Similarly, we have

$$
\begin{aligned}
P^x\{T_B < \infty\} &= P^x\{T_y < T_B\}\left(\frac{b}{\log a} + O\left(\frac{\log\log a}{\log^2 a}\right)\right)\left(\frac{\log a}{b} + O(\log\log a)\right) \\
&\quad + P^x\{T_B < T_y\}.
\end{aligned}
\tag{5.3}
$$

Solving these two equations yields

$$
\begin{aligned}
&P^x\{T_y < T_B\}\left(1 - \frac{d}{g(0)}\frac{1}{\log a}\right) \\
&\quad = P^x\{T_y < \infty\} - P^x\{T_B < \infty\}\frac{d}{g(0)}\frac{1}{\log a} + O\left(\frac{\log\log a}{\log^2 a}\right) \\
&\quad = \frac{b-d}{g(0)}\frac{1}{\log a} + O\left(\frac{\log\log a}{\log^2 a}\right),
\end{aligned}
\tag{5.4}
$$

where we have used Theorems 3.1 and 4.1 at the last step. Now $g(0)$ can be evaluated from (3.3) and is equal to $h^{-1} = b - d$. Thus we have proved (ii). It is easier to prove (iii). We use

$$P^y\{T_x < T_A\} \le P^y\{T_x < T_B\}$$

and the latter probability can be estimated as in (5.2)–(5.4). The fact that the roles of x and y have been interchanged only affects the term $P^y\{T_x < \infty\}$ which is $d(g(0)\log a)^{-1} + O(\log\log a \log^{-2} a)$ and this means that the principal term on the right side of (5.4) disappears.

LEMMA 5.9. *Under the conditions of Lemma* 5.8, *as* $a \to \infty$

$$P\{X_t \textit{ hits both } \{x\} \textit{ and } \{y\} \textit{ before } [a, \infty)\} = \frac{1}{\log^2 a}\left(1 + O\left(\frac{\log\log a}{\log a}\right)\right).$$

PROOF. First note that with $A = [a, \infty)$

$$P\{T_y < T_x < T_A\} \le P\{T_y < \infty\}P^y\{T_x < T_A\} = O\left(\frac{\log\log a}{\log^3 a}\right)$$

by (iii) of Lemma 5.8 and the strong Markov property. The main contribution comes from hitting x first and so we have to use more care there:

$$P\{T_x < T_y < T_A\} = P\{T_x < T_y,\, T_x < T_A\}P^x\{T_y < T_A\}.$$

The estimate for the second term on the right comes from (ii) of Lemma 5.8. The estimate for the first term comes from (i) of Lemma 5.8 since with $E = [y, \infty)$,

$$P\{T_x < T_E\} \le P\{T_x < T_y,\, T_x < T_A\} \le P\{T_x < T_A\}.$$

6. Rate of escape. Since the asymmetric Cauchy processes are transient we know that $|X_t| \to \infty$ as $t \to \infty$. The most natural question to ask about the rate at which it approaches infinity is: which monotone functions will eventually lie below $|X_t|$? In the completely asymmetric case, $h = 2/\pi$, we have $|X_t| = X_t$ for all sufficiently large t and Millar (1972) has shown that in this case there is an exact lower function:

$$\liminf_{t\to\infty} X_t/t \log t = 2/\pi \quad \text{a.s.}$$

(Even more is known in this case; an integral test is given in Mijnheer (1974).) This is not the case for the other asymmetric Cauchy processes however. But there is a simple integral test that answers the question.

THEOREM 6.1. *Assume that $h < 2/\pi$ and let $\varphi(t)\uparrow$. Then*

$$P\{|X_t| \le \varphi(t) \text{ i.o. } \text{ as } t \to \infty\} = {0 \atop 1} \quad \text{iff} \quad \int^{\infty} \frac{\log \varphi(t)}{t \log^2 t}\, dt \ {< \infty \atop = \infty}.$$

PROOF. First note that if φ is bounded then the integral converges and $|X|$ will eventually lie above φ. Thus we may assume that $\varphi(t) \uparrow \infty$. Suppose the integral converges and let $t_k = \exp\{e^k\}$. Then

$$\int_{t_{k+1}}^{t_{k+2}} \frac{\log \varphi(t)}{t \log^2 t}\, dt \ge \log \varphi(t_{k+1}) \int_{t_{k+1}}^{t_{k+2}} \frac{dt}{t \log^2 t} = e^{-1}(1 - e^{-1}) \log \varphi(t_{k+1}) \log^{-1} t_k$$

$$\ge e^{-1}(1 - e^{-1})(1 + \varepsilon)^{-1} P\{|X_t| \le \varphi(t_{k+1}) \quad \text{for some } t \ge t_k\}$$

by Lemma 5.5. Thus for k sufficiently large and $t_k \le t \le t_{k+1}$, we have by Borel-Cantelli that $|X_t| > \varphi(t_{k+1}) \ge \varphi(t)$. If the integral diverges, we first show that we may assume with no loss of generality that $\varphi(t) \le t$ for all large t. If not, then there is a sequence $t_k \uparrow \infty$ with $\varphi(t_k) > t_k$. Letting $\varphi_1(t) = \varphi(t) \wedge t$, we have

$$\int_{t_k}^{\infty} \frac{\log \varphi_1(t)}{t \log^2 t}\, dt \ge \log \varphi_1(t_k) \int_{t_k}^{\infty} \frac{dt}{t \log^2 t} = \frac{\log t_k}{\log t_k} = 1$$

so the integral diverges for φ_1. Once we have proved the theorem for φ such that $\varphi(t) \le t$, we will know it for φ_1 and so we will have $|X_t| \le \varphi_1(t) \le \varphi(t)$ i.o. as $t \to \infty$. Thus we will assume that $\varphi(t) \le t$. Choose $\gamma > 5b/d$ and let $t_k = \exp\{\gamma^k\}$,

$$E_k = \{|X_t| \le \varphi(t_k) \quad \text{for some} \quad t \in [t_k, t_{k+1}/2]\}.$$

Since $\varphi(t_k) \le t_k \le h t_k \log t_k/2$ for large k, Lemma 5.7 applies and

$$P(E_k) \ge c \log \varphi(t_k) \log^{-1} t_k \ge c(\gamma - 1)^{-1} \int_{t_{k-1}}^{t_k} \frac{\log \varphi(t)}{t \log^2 t}\, dt.$$

Thus we know that $\sum P(E_k)$ diverges. If $j < k$, then $t_{j+1}/2 < t_k$ and it is easy to use the Markov property and Lemmas 5.5 and 5.7 to obtain

$$P(E_j E_k) \le P(E_j)(1 + \varepsilon) \log \varphi(t_k) \log^{-1} t_k = O(P(E_j) P(E_k)). \tag{6.1}$$

Note that the fact that the upper bound in Lemma 5.5 is uniform in the starting point is used here when we start the process over at time $t_{j+1}/2$. Now (6.1) means that we may use the generalized Borel-Cantelli lemma and the zero-one law to see that infinitely many E_k occur with probability one and this is enough.

REMARKS. This proof does not apply in the completely asymmetric case since Lemma 5.6 does not provide a reasonable lower bound. An examination of the proofs of Lemmas 5.6 and 5.7 shows that they are also valid if $|X_t| \le a$ is replaced by $X_t \in [0, a]$ or

$X_t \in [-a, 0]$. This means that the integral test of Theorem 6.1 is also valid for the one sided rate of escape problems, i.e. for

$$0 \le X_t \le \varphi(t) \text{ i.o.} \quad \text{and} \quad -\varphi(t) \le X_t \le 0 \text{ i.o.}$$

We now compare the rate of escape with some similar known results. It follows from Theorem 6.1 that the process will lie below $\exp\{\log t/\log\log t\}$ infinitely often and this function grows more slowly than any power. However, if one considers $M_t = \sup_{0\le s\le t}|X_s|$, then one has $\liminf_{t\to\infty} M_t/t\log t = c$ a.s. (Theorem 4 of [4]). This gives some idea of the magnitude of the fluctuations. If we look for the large values of $|X_t|$, these are not very much larger than $t\log t$ but the best way to investigate this problem is to center the process at the median. It follows readily from the scaling property that

$$\text{med } X_t = ht\log t + t \text{ med } X_1.$$

The linear term is not important so we will center at $ht \log t$. Then one may prove in the usual fashion (see, for example, Theorem 5 on page 294 of [3]) that for any increasing φ

$$\limsup_{t\to\infty} \frac{|X_t - ht\log t|}{\varphi(t)} = \begin{matrix} 0 \\ \infty \end{matrix} \quad \text{iff} \quad \int \frac{dt}{\varphi(t)} \begin{matrix} <\infty \\ =\infty \end{matrix}.$$

Note that it is a consequence of this that

$$\limsup_{t\to\infty} \frac{|X_t|}{t\log t(\log\log t)^{1+\varepsilon}} = 0 \quad \text{a.s.}$$

for every $\varepsilon > 0$. But this also shows that the behavior of the large values of $|X_t|$ is quite different for these processes than for the strictly stable processes. In particular, it is possible to find an exact normalizing function for the lim sup problem for $|X_t|$. If one takes an increasing sequence of times t_k such that $\sum 1/\log t_k$ converges and then defines $\varphi(t) = \sum t_k \log t_k 1(t_{k-1}, t_k]$, it is easy to deduce from the above result that if $h > 0$, then

$$\limsup_{t\to\infty} \frac{X_t}{\varphi(t)} = h, \quad \liminf_{t\to\infty} \frac{X_t}{\varphi(t)} = 0, \quad \text{a.s.}$$

7. Holes in the range of the process. In this section we obtain integral tests for the existence of holes in the range of the process. The main tool is the missing probability estimates of Theorems 5.1–5.4. Theorem 7.1 gives the result for the positive axis and Theorem 7.2 for the negative axis. In the latter result, the completely asymmetric case ($h = 2/\pi$) is excluded but there are trivially holes that are arbitrarily large in this case since the process is bounded below.

Erickson [2] has recently obtained results of this type for random walks that drift to infinity. He also considers a local problem concerning small holes near zero for a class of Lévy processes that includes the subordinators.

THEOREM 7.1. *Suppose that $\varphi(x)\uparrow$ and let R denote the range of the process, i.e. $R = \{x : x = X_t$ for some $t \ge 0\}$. Then*

$$P\{R \cap [n, n\varphi(n)] = \varnothing \text{ i.o.}\} = \begin{matrix} 0 \\ 1 \end{matrix} \quad \text{iff} \quad \int^\infty \frac{dx}{x\log x\varphi(x)} \begin{matrix} <\infty \\ =\infty \end{matrix}.$$

PROOF. First note that we may assume that $\varphi(x) \uparrow \infty$ since otherwise the integral diverges and $\varphi(x) \le \log\log x$ for large x and the result for φ will follow from the one for $\log\log x$. Next we may assume that

$$\varphi(x) \le (\log\log x)^2. \tag{7.1}$$

To see this, let $\varphi_1(x) = \varphi(x) \wedge (\log\log x)^2$ and suppose that the theorem has been proved for φ_1. It is easy to check that the integral converges for φ iff it does for φ_1 and clearly

$R \cap [n, n\varphi_1(n)] \neq \varnothing$ implies the same for φ. Finally, in the divergent case, there must be a sequence $n_k = n_k(\omega)$ such that

$$R \cap [n_k, n_k\varphi_1(n_k)] = \varnothing, \quad R \cap [n_k, n_k(\log\log n_k)^2] \neq \varnothing.$$

Thus $\varphi_1(n_k) < (\log\log n_k)^2$ so that $\varphi_1(n_k) = \varphi(n_k)$ and this implies that $[n_k, n_k\varphi(n_k)]$ is also missed.

Now we prove the theorem with φ subject to (7.1). This condition will make it possible to verify the restrictions in Theorems 5.1–5.4. First suppose the integral converges and let

$$E_k = \{R \cap [2^{k+1}, 2^k\varphi(2^k)] = \varnothing\}.$$

By Theorem 5.1, we have for large k

$$P(E_k) \leq 5\frac{2^{k+1}}{2^k\varphi(2^k)k\log 2} \leq \frac{20}{k\varphi(2^k)} = \frac{20}{k\log 2\,\varphi(2^k)}\int_{2^{k-1}}^{2^k}\frac{dx}{x}$$

$$\leq 20\int_{2^{k-1}}^{2^k}\frac{dx}{x\log x\,\varphi(x)}.$$

By Borel-Cantelli this implies that

$$R \cap [2^{k+1}, 2^k\varphi(2^k)] \neq \varnothing$$

for all k sufficiently large. This is sufficient since

$$[2^{k+1}, 2^k\varphi(2^k)] \subset [n, n\varphi(n)], \quad 2^k < n \leq 2^{k+1}.$$

Now suppose that the integral diverges and let

$$F_k = \{R \cap [2^k, 2^k\varphi(2^k)] = \varnothing\}.$$

Then $\sum P(F_k)$ diverges by using Theorem 5.1 and comparing with the integral as above. It only remains to estimate $P(F_jF_k)$ so that we may apply the generalized Borel-Cantelli lemma. We assume that $j \leq k$. Fix j and let

$$k_1 = \max\{k : 2^k \leq 2^j\varphi(2^j)\log(2^j\varphi(2^j))\}.$$

For $k \leq k_1$, we have by Theorem 5.1 if $2^k \leq 2^j\varphi(2^j)$ or Theorem 5.3 otherwise that

$$P(F_jF_k) = O\left(\frac{2^j}{2^k\varphi(2^k)k}\right) = O(2^{j-k}P(F_k)) = O(2^{j-k}P(F_j))$$

so that $\sum_{k=j}^{k_1} P(F_jF_k) = O(P(F_j))$. If $k > k_1$, then the second part of Theorem 5.3 applies and we have $P(F_jF_k) = O(P(F_j)P(F_k))$. Now we may apply the generalized Borel-Cantelli lemma and the zero-one law to complete the proof.

THEOREM 7.2. *Suppose that $h < 2/\pi$, $\varphi(x) \uparrow$ and R denotes the range of the process. Then*

$$P\{R \cap [-\varphi(n), -n] = \varnothing \text{ i.o.}\} = \begin{matrix}0\\1\end{matrix} \quad \text{iff} \quad \int^{\infty}\left(\frac{\log x}{\log\varphi(x)}\right)^{\lambda-1}\frac{dx}{x\log x} \quad \begin{matrix}<\infty\\=\infty\end{matrix},$$

where $\lambda = b/(b-d)$.

PROOF. We will assume that $\varphi(x) \leq \exp\{\log^2 x\}$. One may show that this involves no loss of generality as in the proof of Theorem 7.1. For the convergent case, we let $x_k = \exp\{e^k\}$ and

$$E_k = \{R \cap [-\varphi(x_k), -x_{k+1}] = \varnothing\}.$$

Then

$$(7.2)\qquad \int_{x_k}^{x_{k+1}} \left(\frac{\log x}{\log \varphi(x)}\right)^{\lambda-1} \frac{dx}{x\log x} \geq \left(\frac{\log x_k}{\log \varphi(x_{k+1})}\right)^{\lambda-1} \int_{x_k}^{x_{k+1}} \frac{dx}{x\log x} = \left(\frac{e^k}{\log \varphi(x_{k+1})}\right)^{\lambda-1}.$$

Since for $x \in [x_{k+1}, x_{k+2}]$, we have

$$(7.3)\qquad \frac{\log x}{\log \varphi(x)} \leq \frac{\log x_{k+2}}{\log \varphi(x_{k+1})} \leq e^2 \frac{e^k}{\log \varphi(x_{k+1})},$$

we see by (7.2) and (7.3) that the convergence of the integral implies that $\log x = o(\log \varphi(x))$. Thus for x sufficiently large,

$$3 \log x \leq \log \varphi(x) \implies x^3 \leq \varphi(x)$$

so that $x^e/\varphi(x) \to 0$. Thus

$$\varphi(x_{k+1}) - x_{k+2} = \varphi(x_{k+1}) - x_{k+1}^e \sim \varphi(x_{k+1})$$

and then by Theorem 5.2, the last term in (7.2) is comparable to $P(E_{k+1})$. Since $n \in [x_k, x_{k+1})$ implies that

$$[-\varphi(n), -n] \supset [-\varphi(x_k), -x_{k+1}],$$

an application of Borel-Cantelli completes the proof of the convergent case. For the divergent case, we let

$$F_k = \{R \cap [-\varphi(x_k), -x_k] = \varnothing\}.$$

Essentially as in (7.2) we see that the integral over $[x_k, x_{k+1}]$ is bounded by a constant times $P(F_k)$. In order to make this estimate from Theorem 5.2, we need to know that $\varphi(x_k) \geq 2x_k$. There is no harm in assuming this since if $\varphi(n) < 2n$ i.o., then for such n, by Theorem 5.2 we have

$$P\{R \cap [-\varphi(n), -n] = \varnothing\} \geq P\{R \cap [-2n, -n] = \varnothing\} \sim \lambda^{-1} > 0$$

so that infinitely many of the intervals $[-\varphi(n), -n]$ will be empty for n in this subsequence. Thus we have $P(F_k)$ comparable to a piece of the integral so that $\sum P(F_k)$ diverges. It remains to check the supplementary condition on $P(F_jF_k)$. This seems to be a little harder in this case and we will have to sum over both j and k in order to obtain the needed bound. Define $I = \{(j, k) : 1 \leq j \leq k \leq N\}$ and

$$m(j) = \min\{k : x_k \geq \varphi(x_j)\}, \qquad G_1 = \{(j, k) \in I : k < m(j)\}$$

$$G_2 = \{(j, k) \in I : k = m(j)\}, \quad G_3 = \{(j, k) \in I : k > m(j)\}.$$

It is not hard to check that on G_3, Theorem 5.4 applies so that $P(F_jF_k) = O(P(F_j)P(F_k))$. Thus

$$\textstyle\sum_{G_3} P(F_jF_k) = O((\sum_{i=1}^N P(F_i))^2).$$

Also we clearly have

$$\textstyle\sum_{G_2} P(F_jF_k) \leq \sum_{j=1}^N P(F_j).$$

Finally on G_1 we note that $x_k < \varphi(x_j)$ so that F_jF_k is the event that the interval $[-\varphi(x_k), -x_j]$ is missed. By Theorem 5.2,

$$P(F_jF_k) = O\left(\left(\frac{\log x_j}{\log \varphi(x_k)}\right)^{\lambda-1}\right) = O(e^{(j-k)(\lambda-1)}P(F_k)).$$

Now summing first on j gives a bound of order $P(F_k)$ so we have

$$\sum_{G_1} P(F_j F_k) = O(\sum_{k=1}^{N} P(F_k)).$$

The generalized Borel-Cantelli lemma and the zero-one law now complete the proof.

Note that the size of the holes in the range is quite different in the two directions. Thus on the positive axis there will be infinitely many holes in the range of the type $[n, n \log\log n]$ but not of the type $[n, n(\log\log n)^{1+\epsilon}]$. On the negative axis there will be infinitely many holes of the type $[-n^k, -n]$ for any k. In fact there will even be infinitely many holes of the form

$$[-\exp\{\log n(\log\log n)^{1/(\lambda-1)}\}, -n].$$

A partial explanation for this phenomenon in that the process spends much more time on the positive axis than on the negative axis. Using the scaling property and the asymptotic behavior of the density, it is easy to see that

$$P(X_t \leq 0\} = P\{X_1 \leq -h \log t\} \sim \frac{2q}{\pi h} \frac{1}{\log t}$$

as $t \to \infty$ and then

$$E \int_0^t 1_{(-\infty,0]}(X_s)\, ds = \int_0^t P\{X_s \leq 0\}\, ds \sim \frac{2q}{\pi h} \frac{t}{\log t}.$$

Thus the expected proportion of time spent on the negative axis is going to zero like a constant times $\log^{-1} t$.

8. Measure of the range in a large interval. Since the process has positive probability of hitting any point, the range of the process will have positive Lebesgue measure. We have the following result on the growth of the Lebesgue measure of that part of the range that is included in a large interval.

THEOREM 8.1. *Let R denote the range of the process, $|\cdot|$ Lebesgue measure, and $\rho = q/p$. Then as $a \to \infty$,*

(i) $|R \cap [0, a]| \log a/a \to G$ *in distribution where G is a geometric distribution assigning measure $(1-\rho)\rho^{j-1}$ to j for $j = 1, 2, \cdots$.*
(ii) $|R \cap [-a, 0]| \log a/a \to H$ *in distribution where H is geometric but assigns measure $(1-\rho)\rho^j$ to j for $j = 0, 1, 2, \cdots$.*
(iii) $|R \cap [-a, a]| \log a/a \to K$ *in distribution where K assigns measure $(1-\rho)\rho^j$ to $2j+1$ for $j = 0, 1, 2, \cdots$.*

The Lebesgue measure of the part of the range in each of these three intervals is thus close to an integral multiple of $a/\log a$, when a is large, with high probability. We first discovered this surprising result by computing the moments of the distribution, using a combinatorial identity of Euler, and then applying the continuity theorem for Laplace transforms. In [12] we gave a more probabilistic proof which gives more intuition about the result. Here we give an alternative shorter version of the proof of Theorem 3 of [12]. This proof uses techniques that are closely related to those which we will use for the study of the local behavior of the sample paths in [13]. We state the result as:

THEOREM 8.2. *If $\tau = T_{[a,\infty)}$, the first hitting time of $[a, \infty)$, $R[0, \tau] = \{x : x = X_t$ for some $t \in [0, \tau]\}$, then as $a \to \infty$*

(i) $|R[0, \tau] \cap [0, a]| \log a/a \to 1$ *in probability;*
(ii) $|R[0, \tau] \cap [-a, 0]| \log a/a \to 0$ *in probability.*

REMARK. Part (ii) is a consequence of the fact that the process does not usually go very far in the negative direction before it hits $[a, \infty)$.

PROOF. The proof is a relatively simple consequence of Lemmas 5.8 and 5.9. Let

$$Z_1(a) = |R[0, \tau] \cap [0, a]| = \int_0^a 1\{X_t \text{ hits } \{x\} \text{ before } [a, \infty)\}\, dx.$$

We can estimate $EZ_1(a)$ by integrating the first estimate of Lemma 5.8. The contribution to the integral from those x in $[0, a/\log^2 a]$ is of order $a/\log^2 a$ which is small compared to the main term. Thus $EZ_1(a) \sim a/\log a$. Similarly, using Lemma 5.9 and ignoring the set where either x or y is in $[0, a/\log^3 a]$ or $[a(1 - 1/\log^3 a), a]$ or where $|x - y| \le a/\log^3 a$, we obtain

$$E\{Z_1(a)\}^2 \sim \frac{a^2}{\log^2 a} \sim \{EZ_1(a)\}^2.$$

Thus we have that $Z_1(a) \log a/a \to 1$ in mean square and therefore in probability. If we consider

$$Z_2(a) = |R[0, \tau] \cap [-a, 0]|,$$

then by the final part of Lemma 5.8 we have $EZ_2(a) = O(a \log \log a/\log^2 a)$ so that $Z_2(a)\log a/a \to 0$ in L_1 and hence in probability.

The argument that leads from Theorem 8.2 to the first statement of Theorem 8.1 is given in [12]. The idea is that $R \cap [0, a]$ is the union of k almost independent sets where k is the number of upcrossings from $[0, a(1 - 1/\log^2 a)]$ to $[a, \infty)$. The integer k is geometrically distributed and the sets do not overlap substantially so that, as $a \to \infty$, the measure of the union is close to the sum of the measures of the pieces of the range obtained on each pass from $[0, a(1 - 1/\log^2 a)]$ to $[a, \infty)$. If we consider $R \cap [-a, 0]$, the second part of Theorem 8.2 says that the first pass to $[a, \infty)$ will make no contribution. But now we consider successive returns to $[-a, a/\log^2 a]$; the geometric distribution of k remains the same and now this interval behaves as in the first case. The final case comes from picking up a set of measure approximately $a/\log a$ on the first pass to $[a, \infty)$ when $[-a, 0]$ does not contribute anything significant. But on any succeeding passes the process will treat the interval $[-a, a]$ the same as $[0, 2a]$ and so will pick up a set of measure approximately $2a/\log 2a \sim 2a/\log a$.

COROLLARY 8.3. *If X is completely asymmetric ($\rho = 0$), then*

$$|R \cap [0, a]| \log a/a \to 1 \textit{ in probability}$$

as $a \to \infty$.

REMARK. There is a similarity between this corollary and the prime number theorem. However, we emphasize that there is not almost sure convergence in the corollary. This is a consequence of Theorem 7.1 which guarantees large values of a for which $R \cap [a, a \log \log a] = \emptyset$. For such an a,

$$|R \cap [0, a]| = |R \cap [0, a \log \log a]|$$

and so both of these cannot be near one after normalization. Nevertheless, if we take our convergence to be convergence in probability, the random set R has many of the distribution properties of the discrete set of prime integers. It would be of interest to know which properties of the primes have appropriate continuous analogues for R.

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SCHOOL OF MATHEMATICS
127 VINCENT HALL
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA 55455

PURE MATHEMATICS DEPARTMENT
UNIVERSITY OF LIVERPOOL
BOX 147
LIVERPOOL, ENGLAND L693BX