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ON THE ALMOST SURE CONVERGENCE OF RANDOMLY WEIGHTED SUMS OF RANDOM ELEMENTS

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Let $\{X_n\}$ be random elements in a separable Banach space which is p-smoothable and let $\{a_k\}$ and $\{A_k\}$ denote positive random variables such that almost surely A_k is monotonically increasing to ∞ and that $A_k/a_k \to \infty$. Convergence almost surely is obtained for the weighted sum $A_n^{-1} \sum_{k=1}^{n} a_k X_k$ and is related to a moment condition on the random elements and a growth condition on the random weights.

1. Introduction. Let E denote a real separable Banach space. Random elements are functions from a probability space (Ω, F, P) to E which are F-measurable with respect to the Borel subsets of E. The Bochner integral EX will be used as an expected value for the random element X. The random elements $\{X_n\}$ are said to be stochastically bounded by a positive random variable X, $\{X_n\} \leq X$, if $\sup_n P[\|X_n\| > t] \leq P[X > t]$ for each $t > 0$. Recall that uniformly bounded moments for $\{\|X_n\|\}$ can imply stochastic boundedness and that $\{X_n\} \leq \|X_1\|$ for identically distributed random elements $\{X_n\}$. In considering convergence results for randomly weighted sums, it is easy to show that the relationship between the random weights and the random elements cannot be ignored in general. Anderson and Taylor (1976) considered least squares estimates in linear regression problems where the weight a_{nk} could possibly depend on the observations $X_1, \cdots, X_{k-1}$ but not on X_k. This would be a natural restriction for estimates in sequential analysis. Some of these conditions are reflected in the hypotheses of the main result.

The following lemma can be proved by conditioning arguments and will be needed in the next section.

LEMMA. *Let $\{X_k\}$ be a sequence of random elements in E such that $\{X_k\} \leq X$. If A is a nonnegative random variable which is independent of $\{X_k\}$ and X, then $\{AX_k\} \leq AX$.*

2. Almost sure convergence. The format will be similar to the non-randomly weighted sums results in Howell, Taylor, and Woyczynski (1981). Let $\{a_k\}$ and $\{A_k\}$ denote positive random variables and assume that A_k is monotonically increasing to ∞ almost surely and that $A_k/a_k \to \infty$ a.s. For each $\omega \in \Omega$ and each $x \in R$, define the integer-valued random variable $N(x)$ by $N(x, \omega) = \text{card}\{k : A_k(\omega)/a_k(\omega) \leq x\}$. In the following theorem martingale techniques will be used to handle the possible dependence between the random weights and the random elements, and hence the space is assumed to be p-smoothable.

THEOREM. *Let E be a p-smoothable space, $1 < p \leq 2$. Let $\{X_n\}$ be a sequence of random elements in E such that $\{X_n\} \leq X$ with $EX < \infty$ and $EX_n = 0$ for each n. Assume that X_{n+1} is independent of $G_n = \sigma\{a_1/A_1, \cdots, a_{n+1}/A_{n+1}, X_1, \cdots, X_n\}$ for each n. If $EN(X) < \infty$ and*

$$(2.1) \qquad \int_0^\infty t^{p-1} P[X > t] \int_t^\infty \frac{EN(y)}{y^{p+1}}\, dy\, dt < \infty,$$

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then

$$\frac{1}{A_n}\sum_{k=1}^n a_k(X_k - V_k) \to 0 \quad \text{a.s.}$$

where $\{V_k\}$ *are random elements which are* G_{k-1}*-measurable and converge* a.s. *to* 0.

PROOF OF THEOREM 2.1. It can be assumed that X is independent of the σ-field $\sigma(\cup G_n)$. Define $Y_k = X_k I_{[\|X_k\| \le A_k/a_k]}$ and $V_k = E(Y_k \mid G_{k-1})$. Since $EX_n = 0$,

$$\begin{aligned}\|V_k\| &= \|E(X_k I_{[\|X_k\|>A_k/a_k]} \mid G_{k-1})\| \\ &\le E\left(\sum_{m=0}^\infty \|X_k\| I_{[\|X_k\|>m]} I_{[m \le A_k/a_k < m+1]} \mid G_{k-1}\right) \\ &\le \sum_{m=0}^\infty E(X I_{[X>m]}) I_{[m\le A_k/a_k<m+1]} \\ &\le E(X I_{[X>A_k/a_k-1]} \mid G_{k-1}) \quad \text{a.s.}\end{aligned} \tag{2.2}$$

Thus, $\|V_k\| \to 0$ a.s. By the previous lemma

$$\sum_{k=1}^\infty P[X_k \ne Y_k] = \sum_{k=1}^\infty P\left[\|X_k\| > \frac{A_k}{a_k}\right] \le \sum_{k=1}^\infty P\left[\frac{a_k}{A_k} X > 1\right]$$

$$\le \sum_{k=1}^\infty E(I_{[X \ge A_k/a_k]}) = EN(X) < \infty.$$

Hence, it is enough to show that

$$\frac{1}{A_n}\sum_{k=1}^n a_k(Y_k - V_k) \to 0 \quad \text{a.s.},$$

or by Kronecker's lemma that

$$\sum_{k=1}^\infty \frac{a_k}{A_k}(Y_k - V_k) \quad \text{converges a.s.} \tag{2.3}$$

By construction, $\left\{\frac{a_k}{A_k}(Y_k - V_k), G_k\right\}$ is a martingale difference sequence. Thus, by Woyczynski (1976) it is sufficient to show that

$$\sum_{k=1}^\infty E\left\|\frac{a_k}{A_k}(Y_k - V_k)\right\|^p < \infty. \tag{2.4}$$

For each positive integer m, let $E_{m\ell}^k = \left[\ell/m < \frac{A_k}{a_k} \le (\ell+1)/m\right]$. Then, for each m

$$\begin{aligned}&\sum_{k=1}^\infty \mathrm{E}\left\|\frac{\mathrm{a}_k}{\mathrm{A}_k}(\mathrm{Y}_k - \mathrm{V}_k)\right\|^p \\ &\le 2^p \sum_{k=1}^\infty E\left\|\frac{a_k}{A_k} Y_k\right\|^p \\ &\le 2^p \sum_{k=1}^\infty E\left(\sum_{\ell=0}^\infty \left\|\frac{a_k}{A_k} X_k\right\|^p I_{[\|X_k\|\le A_k/a_k]} I_{E_{m\ell}^k}\right) \\ &\le 2^p \sum_{k=1}^\infty \left(\sum_{\ell=0}^\infty E\left(\left(\frac{a_k}{A_k}\right)^p I_{E_{m\ell}^k}\right) \int_0^{(\ell+1)/m} t^p \, dP[\|X_k\| \le t]\right) \\ &\le 2^p \sum_{k=1}^\infty \sum_{\ell=0}^\infty E\left(\left(\frac{a_k}{A_k}\right)^p I_{E_{m\ell}^k} p \int_0^{A_k/a_k+1/m} t^{p-1} P[X>t]\, dt\right) \\ &= 2^p \sum_{k=1}^\infty E\left(p\left(\frac{a_k}{A_k}\right)^p \int_0^{A_k/a_k+1/m} t^{p-1} P[X>t]\, dt\right)\end{aligned} \tag{2.5}$$

using integration by parts and the independence of X_k and A_k/a_k. Letting $m \to \infty$ in (2.5) and following steps similar to the proof of Theorem 2.1 of Howell, Taylor, and Woyczynski (1981),

$$\sum_{k=1}^{\infty} E\left\|\frac{a_k}{A_k}(Y_k - V_k)\right\|^p \le p^2 2^p E\left(\int_0^{\infty} t^{p-1}P[X > t]\int_0^{A_k/a_k}\frac{N(y)}{y^{p+1}}\,dy\,dt\right)$$

which is finite from Condition (2.1) and Fubini's Theorem. □

Since $V_n \to 0$ a.s., $(1/A_n)\sum_{k=1}^{n} a_k V_k \to 0$ a.s. if $A_n^{-1}\sum_{k=1}^{n} a_k \le M$ a.s. for all n. Also, from a modification of the inequalities in (2.2)

$$\sum_{k=1}^{\infty} E\left\|\frac{a_k}{A_k}E(Y_k \mid G_{k-1})\right\| \le \sum_{k=1}^{\infty} E\left|\frac{a_k}{A_k}XI_{[X \ge A_k/a_k]}\right| < \infty \tag{2.6}$$

is sufficient to imply that $(1/A_n)\sum_{k=1}^{n} a_k E(Y_k \mid G_{k-1}) \to 0$ a.s.

The sharpness of the theorem has been illustrated for the constant weights case and for real-valued random variables. It is also important to observe that Condition (2.1) is both a moment condition on the random elements and a growth condition on the random weights. When the weights are non-random, the condition is the same as considered by previous authors. For example, if $A_k/a_k \ge Mk^{\alpha}$ a.s. for some positive constants M and α, $\alpha > 1/p$, then $N(x) \le M^{-1/\alpha}x^{1/\alpha}$ a.s. and

$$\int_0^{\infty} t^{p-1}\,P[X > t]\int_t^{\infty}\frac{EN(y)}{y^{p+1}}\,dy\,dt \le \left(p - \frac{1}{\alpha}\right)^{-1} M^{-1/\alpha}\int_0^{\infty} t^{1/\alpha - 1}\,P[X > t]\,dt$$

which is finite if $EX^{1/\alpha} < \infty$. Thus, the usual moment condition is much more restrictive than Condition (2.1) even for constant weights.

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