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GAUSSIAN MEASURE OF NORMAL SUBGROUPS

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Let $(\mu_t)_{t>0}$ be a Gaussian semigroup on a metric, separable, complete group G. If H is a Borel measurable normal subgroup of G such that $\mu_t(H) > 0$ for all t, then $\mu_t(H) = 1$ for every t. If, in addition, μ_t are symmetric, then $\mu_t(H) > 0$ for a single t implies $\mu_t(H) = 1$ for all t.

1. Let G be a separable complete metric group and let $(\mu_t)_{t>0}$ be a semigroup of probability measures on G. We say cf. e.g. [5], for locally compact groups, that $(\mu_t)_{t>0}$ is Gaussian if

$$\lim_{t\to 0}(1/t)\mu_t(U^c) = 0, \tag{1}$$

for every open neighbourhood of the identity e of G.

It is known cf. [2], [1], [8] that if G is Abelian and H is a Borel subgroup of G then for all $t > 0$ either $\mu_t(H) = 0$ or $\mu_t(H) = 1$. If moreover $(\mu_t)_{t>0}$ is symmetric then either $\mu_t(H) = 0$ for all $t > 0$ or $\mu_t(H) = 1$ for all $t > 0$.

The aim of this note is to show that this last statement holds also for non-Abelian G provided H is normal.

Because the measure induced by a symmetric Gaussian process with values in a locally compact group, on the product group, is embeddable into a Gaussian semigroup, as defined by (1), such a theorem might be of interest for G being the group of trajectories of a Gaussian process. Of course, having this application in mind, the assumption that H is normal is pretty restrictive. Unfortunately, the authors are unable to prove the theorem without it.

In [8] Tortrat introduced a notion of a p-stable measure on an arbitrary group G. For such a measure ν he has proved that for a Borel normal subgroup H either $\nu(H) = 0$ or $\nu(H) = 1$.

We show that for most non-commutative Lie groups G there exists a semigroup of symmetric Gaussian measures $(\mu_t)_{t>0}$ none of which is p-stable in the sense of Tortrat, whichever p. As a matter of fact, such a semigroup exists on the Heisenberg group and since this group is contained in very many non-commutative non-compact Lie groups as a Lie subgroup, the example is fairly general. The authors do not know of any non-commutative Lie group G and a symmetric Gaussian measure μ on G such that supp μ generates a dense subgroup of G which is p-stable in the sense of Tortrat.

2. Throughout the whole paper, G stands for a separable complete metric group. By a probability measure μ on G we mean a σ-additive Borel measure such that $\mu(G) = 1$. A sequence μ_n of probability measures converges weakly to μ if

$$\lim_n \int f\,d\mu_n = \int f\,d\mu,$$

for every continuous bounded function f on G. By $C_u = C_u(G)$ we denote the subspace consisting of all left uniformly continuous bounded functions on G.

The main tool used in this note is that of probability operators. For any probability measure μ on G we define the operator T_μ on C_u by the formula:

$$T_\mu f(x) = \int f(xy)\mu(dy), \quad f \in C_u.$$

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It is easy to see that $T_\mu f \in C_u$ and $T_{\mu_n} f \to T_\mu f$ uniformly, for every $f \in C_u$ if and only if $\mu_n \to \mu$ weakly. It is clear that

$$T_{\mu * \nu} = T_\mu T_\nu.$$

Now, suppose that $(\mu_t)_{t>0}$ is a convolution semigroup of probability measures on G, that is

$$\mu_t * \mu_s = \mu_{t+s}, \qquad \text{for all} \quad t, s > 0.$$

$(\mu_t)_{t>0}$ is called continuous if $\lim_{t\to 0} \mu_t = \delta_e$. From what has been said before it follows immediately that if $(\mu_t)_{t>0}$ is continuous, then the corresponding family $(T_{\mu_t})_{t>0}$ of probability operators forms a strongly continuous semigroup of contractions acting on C_u considered as a Banach space under the supremum norm. This semigroup is uniquely determined by its infinitesimal generator N defined on its domain $\mathscr{D}(N)$ which is dense in C_u. It is evident that N commutes with left translations: $L_x N f = N L_x f$ for $f \in \mathscr{D}(N)$. Therefore, it is enough to consider the generating functional A,

$$Af = (Nf)(e), \quad f \in \mathscr{D}(N).$$

If $f \in \mathscr{D}(N)$ then $Nf = AL_x f$.

The main tool in the proof of our theorem is the well-known:

Trotter Approximation Theorem. *Let $T_t^{(n)}$ be a sequence of strongly continuous semigroups of operators on a Banach space X, satisfying the condition*

$$\| T_t^{(n)} \| \leq e^{Kt},$$

where K is independent of n and t. Let N_n be the infinitesimal generator of $T_t^{(n)}$. Assume that lim *$N_n x$ exists in the strong sense on a dense linear subspace D. Define*

$$Nx = \lim_n N_n x, \qquad x \in D.$$

Suppose additionally that for some $\lambda > K$ the range of $\lambda I - N$ is dense in X. Then the closure of N is the infinitesimal generator of a strongly continuous semigroup T_t such that

$$T_t x = \lim_n T_t^{(n)} x \quad \textit{for} \quad x \in X.$$

The other crucial point is the use of $L^1(\mu)$ space for μ defined by

$$\mu = \int_0^\infty e^{-t} \mu_t \, dt.$$

It is easy to check that μ is a probability measure. By L^1 we will denote the space of all Borel measurable and μ-integrable functions on G.

3. We begin with a preliminary result needed in the sequel.

Proposition 1. *Assume that $(\mu_t)_{t>0}$ is a continuous semigroup of probability measures on G. $(\mu_t)_{t>0}$ then acts, as a strongly continuous semigroup, on L^1. If H is a Borel subgroup of G such that $\mu(H) > 0$ then $\mu_t(H) \to 1$, as $t \to 0$.*

Proof. Let f be a nonnegative Borel function on G. We then have

$$\int T_{\mu_s} f \, d\mu = \int_0^\infty \left\{ \int f(xy) \mu_s(dy) e^{-t} \mu_t(dx) \right\} dt = \int_0^\infty \int f(z)(\mu_t * \mu_s)(dz) e^{-t} \, dt$$

$$= \int_0^\infty \int f(z) \mu_{t+s}(dz) e^{-t} \, dt = \int_s^\infty \int f(z) \mu_t(dz) e^{-(t-s)} \, dt \leq e^s \int f \, d\mu.$$

Consequently,

$$\| T_{\mu_s} \|_{L^1, L^1} \le e^s.$$

By the continuity of $(\mu_t)_{t>0}$ we have that $\| T_{\mu_t} f - f \|_{C_u} \to 0$, as $t \to 0$, for all $f \in C_u$. Since C_u is dense in L^1 and the uniform convergence is stronger than L^1 convergence, $T_{\mu_t} f \to f$ in L^1, for all $f \in L^1$, as $t \to 0$.

Suppose now that $\mu(H) > 0$. Then we have

$$\mu_t(H)\mu(H) = \int_H \int 1_H(xy)\mu_t(dx)\mu(dy) = \int_H T_{\mu_t} 1_H \, d\mu \to \int_H 1_H \, d\mu$$

$$= \mu(H), \text{ as } t \to 0,$$

which gives the desired conclusion.

Now, suppose that $(\mu_t)_{t>0}$ and H are as in Proposition 1 and, additionally, that H is normal. Let π be the canonical homomorphism of G onto G/H. Endow G/H with the measurable structure induced from G by π. Let $\lambda_t = \pi(\mu_t)$. We have the following:

COROLLARY. *Assume that $(\mu_t)_{t>0}$ and H are as above. Then*

$$\lambda_t = \pi(\mu_t) = \exp tc(\gamma - \delta_H), \quad c \ge 0,$$

for a certain probability measure γ on G/H. Hence

$$\lim_{s \to 0}(1/s)(1 - \mu_s(H)) \text{ exists.}$$

PROOF. λ_t is a semigroup of probability measures on G/H. Since $\mu(H) > 0$, $\mu_t(H) \to 1$, as $t \to 0$. Therefore, $(\lambda_t)_{t>0}$ acts on the space of all Borel measurable and bounded functions on G/H as a uniform semigroup:

$$\| T_{\mu_t} f - f \|_{C_u(G/H)} = \sup_{x \in G/H} \left| \int f(xy)(\lambda_t - \delta_H)(dy) \right| \le \| f \|_{C_u(G/H)} \| \lambda_t - \delta_H \|$$

$$= \| f \|_{C_u(G/H)} (\lambda_t|_{H^c} + (1 - \lambda_t|_H))$$

$$= 2(1 - \mu_t(H)) \| f \|_{C_u(G/H)}.$$

This concludes the proof.

The next proposition clarifies somehow the role of the assumption $\mu(H) > 0$.

PROPOSITION 2. *Let $(\mu_t)_{t>0}$, μ and H be as in Proposition 1. Assume additionally that for every t, μ_t is symmetric. Then $\mu_{t_0}(H) > 0$ for a certain t_0 implies that $\mu_t(H) > 0$ for all $t > 0$. Conversely, if $\mu(H) > 0$, then $\mu_t(H) > 0$ for all $t > 0$.*

PROOF. Assume first that μ_t are symmetric and $\mu_{t_0}(H) > 0$. Then for all s such that $0 < s/2 < t_0$ we have

$$0 < \mu_{t_0}(H) = \int \mu_{s/2}(x^{-1}H)\mu_{t_0 - s/2}(d\dot{x}).$$

Therefore $\mu_{s/2}(x_1^{-1}H) > 0$, for an $x_1 \in G$. By symmetry of $\mu_{s/2}$ we also have $\mu_{s/2}(Hx_1) > 0$, hence

$$\mu_s(H) = \mu_{s/2} * \mu_{s/2}(H) = \mu_{s/2} \times \mu_{s/2}(\{(x, y); xy \in H\})$$

$$\ge \mu_{s/2} \times \mu_{s/2}(\{(x, y); x \in Hx_1, y \in x_1^{-1}H\}) = \mu_{s/2}(x_1^{-1}H)^2 > 0.$$

We have thus shown that

$$\mu_{t_0}(H) > 0 \quad \text{implies} \quad \mu_t(H) > 0, \qquad \text{for all} \quad t > 0.$$

This, of course, implies that $\mu(H) > 0$.

On the other hand, if $\mu(H) > 0$ then, by Proposition 1, $\mu_t(H) \to 1$, as $t \to 0$, so it is positive for $t \in (0, \varepsilon]$, $\varepsilon > 0$. However, it is easily seen that the set of all $t > 0$ such that $\mu_t(H) > 0$ is an additive semigroup. Since it contains $(0, \varepsilon]$, it must coincide with R^+.

Now, we are able to formulate our main result.

THEOREM. *Assume that $(\mu_t)_{t>0}$ is a Gaussian semigroup on G. If H is a Borel measurable normal subgroup of G such that $\mu_t(H) > 0$, for all $t > 0$, then $\mu_t(H) = 1$, for every $t > 0$. If μ_t are symmetric, then for a normal Borel subgroup H, $\mu_t(H) > 0$ for a single $t > 0$ implies $\mu_t(H) = 1$ for all $t > 0$.*

PROOF. Let μ_s^H be the conditional probability of μ_s with respect to H. Since

$$\mu_s = \mu_s|_{H^c} + \mu_s(H)\mu_s^H$$

and $\mu_s(H) \to 1$, as $s \to 0$, μ_s^H converges weakly to δ_e, as $s \to 0$. Next, if we write

$$(1/s)[\mu_s - \delta_e] = (1/s)[\mu_s - \mu_s^H] + (1/s)[\mu_s^H - \delta_e] \tag{2}$$

then, because of equality

$$(1/s)[\mu_s - \mu_s^H] = (1/s)\mu_s|_{H^c} - (1/s)(1 - \mu_s(H))\mu_s^H \tag{3}$$

and the corollary, the first part on the right side of (2) is norm bounded, as $s \to 0$. If f is continuous, nonnegative, bounded and $f(e) = 0$ then

$$(1/s)[\mu_s - \delta_e]f \geq (1/s)[\mu_s - \mu_s^H]f.$$

If additionally $f|_U = 0$, where U is a certain neighbourhood of e then

$$0 = \lim_{s\to 0}(1/s)[\mu_s - \delta_e]f \geq \lim_{s\to 0}(1/s)[\mu_s - \mu_s^H]f,$$

because $(\mu_s)_{s>0}$ is Gaussian. Because of the equality (3) and the fact that $\mu_s^H \to \delta_e$ weakly, as $s \to 0$, we obtain that

$$\lim_{s\to 0}(1/s)\mu_s|_{H^c}f = 0, \tag{4}$$

for all continuous, bounded f with the property that f vanishes on a neighbourhood U of e. Since such functions approximate uniformly functions vanishing at e, (4) implies that for all continuous bounded functions f

$$\lim_{s\to 0}(1/s)\mu_s|_{H^c}f = cf(e), \tag{5}$$

where $c = \lim_{s\to 0}(1/s)(1 - \mu_s(H))$. Now, (5) implies that for $f \in C_u$

$$\lim_{s\to 0}(1/s)\mu_s|_{H^c}({}_yf) = cf(y), \text{ uniformly in } y \in G.$$

Since the same is true for $(1/s)(1 - \mu_s(H))\mu_s^H$, we finally obtain that for all $f \in C_u$

$$\lim_{s\to 0}(1/s)[\mu_s - \mu_s^H]({}_yf) = 0, \qquad \text{uniformly in} \quad y \in G. \tag{6}$$

Now, let N be the infinitesimal generator of $(\mu_t)_{t>0}$ and let N_s^H be the infinitesimal generator of the semigroup $\exp((t/s)[\mu_s^H - \delta_e])$. We have just proved that for all $f \in \mathscr{D}(N)$

$$\lim_{s\to 0}N_s^H f = Nf \text{ strongly on } C_u. \tag{7}$$

We now prove that the above fact implies that $(\mu_t)_{t>0}$ is concentrated on H. To show this, we use once more the space L^1. In the proof of Proposition 1 we obtained that $(\mu_t)_{t>0}$ acts as a strongly continuous semigroup on L^1 and $\| T_{\mu_t}\|_{L^1,L^1} \leq e^t$. Similarly we can easily verify that also the family μ_s^H acts on L^1 and

$$\| T_{\mu_s^H}\|_{L^1,L^1} \leq \mu_s(H)^{-1}e^s.$$

Using these facts we have the following estimate:

$$\| \exp((t/s)[T_{\mu_s^H} - I])\|_{L^1,L^1} \leq \exp(-t/s)\exp((t/s)\mu_s(H)^{-1}e^s)$$
$$= \exp((t/s)(\mu_s(H)^{-1}e^s - 1)).$$

Since $\lim_{s\to 0}(1/s)(\mu_s(H)^{-1}e^s - 1) = 1 + c < \infty$, the family of semigroups

$$T_t^{(s)} = \exp((t/s)[T_{\mu_s^H} - I]),\ s \in (0, 1]$$

has the property:

$$\| T_t^{(s)} \|_{L^1L^1} \leq e^{Kt},$$

for a $K > 0$, independent from s.

Let now $\mathcal{N}$ and $\mathcal{N}_s^H$ be infinitesimal generators of $(\mu_t)_{t>0}$ and $(\exp(t/s[\mu_s^H - \delta_e]))_{t>0}$, respectively, considered on L^1. Let $\bar{N}$ be the closure of N in L^1. By a standard trick $\bar{N} = \mathcal{N}$. Indeed, $\bar{N} \subset \mathcal{N}$ and since for a $\lambda > 0$ both $\lambda - \bar{N}$ and $\lambda - \mathcal{N}$ are invertible and map $\mathcal{D}(N)$ onto L^1, $\bar{N} = \mathcal{N}$. Moreover, by (7)

$$\lim_{s\to 0}\| \mathcal{N}_s^H f - Nf\|_{L^1} = 0 \quad \text{for} \quad f \in \mathcal{D}(N). \tag{8}$$

Since also for a $\lambda > 0$, $(\lambda - N)(\mathcal{D}(N))$ is dense in L^1, (8), by the Trotter Approximation Theorem, gives

$$\lim_{s\to 0}\| T_{\mu_t} f - \exp((t/s)[T_{\mu_s^H} - I])f\|_{L^1} = 0. \tag{9}$$

Putting $f = 1_H$, since $\exp((t/s)[\mu_s^H - \delta_e])$ are all concentrated on H, by (9) we get

$$\mu_t(H)\mu(H) = \int_H \int 1_H(yx)\mu_t(dx)\mu(dy) = \int_H 1_H\, d\mu = \mu(H).$$

Hence

$$\mu_t(H) = 1.$$

4. A symmetric measure μ on a group G is called stable with the exponent p (p-stable) in the sense of Tortrat [8], if for the mapping

$$\sigma_n : G \ni x \longrightarrow x^n \in G$$

we have

$$\mu(\sigma_{n'}^{-1}M) = \mu^{*n''}(M) \quad \text{for all Borel} \quad M \quad \text{in} \quad G,$$

with $n' = n^\ell$, $n'' = n^m$ and $p = m/\ell$.

The Heisenberg group $\mathbf{H}$ is defined as $\mathbf{C} \times \mathbf{R}$ with the multiplication given by

$$(z, s)(z', s') = (z + z', s + s' + 2Im z\overline{z'}). \tag{10}$$

It follows from (10) that

$$(z, s)^n = (nz, ns) \quad \text{for} \quad n \in \mathbf{Z}. \tag{11}$$

Let X, Y be the elements of the Lie algebra of $\mathbf{H}$ which correspond to the one-parameter subgroups

$$\mathbf{R} \ni x \longrightarrow (x + i0, 0) \in \mathbf{H}$$
$$\mathbf{R} \ni y \longrightarrow (0 + iy, 0) \in \mathbf{H},$$

respectively.

Next let

$$L = (1/2)(X^2 + Y^2). \tag{12}$$

In virtue of G. Hunt theory [7], L is the infinitesimal generator of a semigroup of symmetric Gaussian measures

$$(\mu_t)_{t>0} \tag{13}$$

on **H**. Moreover, by e.g. [3]

$$\mu_t(dz, ds) = p_t(z, s)\, dz\, ds, \tag{14}$$

where p_t is a C^∞ (in fact real analytic cf. [6]) function on **H**, and dz is the differential of the Lebesgue measure on **C**. Let

$$\alpha : \mathbf{H} \longrightarrow \mathbf{C} \times \mathbf{R}/\mathbf{R} = \mathbf{C}$$

be the homomorphism of **H** onto the additive group $\mathbf{C} = \mathbf{R}^2$. Then

$$\partial\alpha(X) = \frac{\partial}{\partial x}, \quad \partial\alpha(Y) = \frac{\partial}{\partial y},$$

whence

$$\partial L = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

and, consequently,

$$\int p_t(z, s)\, ds = (1/(2\pi t))\exp(-|z|^2/2t). \tag{15}$$

On the other hand, it is known cf. [4], [6] that

$$\int p_1(z, s)\, dz = p_1(\hat{0}, s) = (\cosh 2s)^{-1}. \tag{16}$$

For $r > 0$ let

$$\delta_r : \mathbf{H} \ni (z, s) \longrightarrow (rz, r^2 s) \in \mathbf{H}.$$

It is clear that δ_r is an automorphism of **H**. Moreover, it is easy to verify, cf. e.g. [3], that

$$p_t(z, s) = t^{-2} p_1(t^{-1/2} z, t^{-1} s). \tag{17}$$

PROPOSITION 3. *None of the Gaussian measures μ_t, $t > 0$, as defined by* (13) *on the Heisenberg group* **H** *is p-stable in the sense of Tortrat, whichever p.*

PROOF. In view of (11) and (14), it suffices to show that for every $r, t', t'' > 0$ identity

$$r^3 p_{t'}(rz, rs) = p_{t''}(z, s) \quad \text{for all} \quad (z, s) \quad \text{in} \quad \mathbf{H} \tag{18}$$

implies $r = 1$.

In virtue of (17) we rewrite (18) as

$$r^3 t'^{-2} p_1(t'^{-1/2} rz, t'^{-1} rs) = t''^{-2} p_1(t''^{-1/2} z, t''^{-1} s). \tag{19}$$

In view of (15), integrating both sides with respect to s we get

$$\frac{r^3}{2\pi t' r}\exp\left[-\frac{r^2|z|^2}{2t'}\right] = \frac{1}{2\pi t''}\exp\left[-\frac{|z|^2}{2t''}\right],$$

which implies

$$r^2 t'' = t'. \tag{20}$$

On the other hand, by (16), integrating both sides of (19) with respect to z we get

$$rt'^{-1}\left(\cosh\frac{2rs}{t'}\right)^{-1} = t''^{-1}\left(\cosh\frac{2s}{t''}\right)^{-1},$$

whence

$$rt'' = t',$$

which by (20) implies $r = 1$, $t' = t''$.

Added in proof. In Arnold Janssen, Zero-one Laws for Infinitely Divisible Probability Measures on Groups, *Z. Wahrsch. verw. Gebiete* **60** 119–138 (1982), the theorem of our paper has been proved under the assumption that the group G is locally compact.

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