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ON MARCINKIEWICZ SLLN IN BANACH SPACES

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Let $S_n = X_1 + \cdots + X_n$ where (X_n) is a sequence of 0-mean i.i.d. random vectors in a B-space such that $P(\| X_n \| > t) \le CP(|X_0| > t)$ for some random variable $X_0 \in L_p$. We show that $S_n/n^{1/p} \to 0$ in L_p iff B is p-stable $(1 \le p < 2)$.

It is well known that for the real valued 0-mean i.i.d. with $E|X_1|^p < \infty$ $(1 \le p < 2)$ $S_n/n^{1/p} \to 0$ a.s. (Marcinkiewicz) and in L_p (Pyke and Root [6]). Our aim in this note is to characterize those B-spaces for which the above carries over. For the definitions of the p-smoothness and p-stability, p-Rademacher type or finite representability of l_p we refer the reader to [3] and [4] respectively.

We start with a vector version of a result due to Dharmadhikari and Sreehari [2].

THEOREM 1. *Let $1 \le p < q \le 2$ and let B be q-smooth. Then for a martingale increments sequence (X_n) in B $S_n/n^{1/p} \to 0$ in L_p provided* $\sup E\| X_n \|^p < \infty$ *and* $E\| X_n \| \chi_{\{\| X_n \| > a_n\}} \to 0$ *for some sequence (a_n) satisfying $\sum_1^n a_k^{q-p} = o(n^{q/p})$.*

Let us remark that $p < q$ is essential because for $B = l_p$ and $q = p$ taking a sequence $X_n = \varepsilon_n e_n$, where e_n is a standard basis and ε_n are independent with $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = ½$, one gets $E\| S_n \|^p = n$.

Since the proof is a minor modification of that for the real line we omit it. An analogous theorem with a slightly stronger assumption i.e., $q = 2$ was proved by Rao [5]. Furthermore, Theorem 1 carries over for 0-mean independent random vectors with values in q-Rademacher type space.

Consider a sequence (X_n) of 0-mean independent random vectors in B satisfying the following "tails" condition

$$\mathrm{T_p} \qquad P(\| X_n \| > t) \le CP(|X_0| > t), \quad t \ge 0, \quad n \in N$$

where X_0 is a random variable in L_p (cf. [7]).

Notice that for $(X_n) \in T_p$ one can apply Theorem 1 because $a_n = (n/\log n)^{1/p}$ meets the required conditions.

THEOREM 2. *Let $1 \le p < 2$, then the following assertions are equivalent*:

i) *l_p is not finitely representable in B*
ii) *B is p-stable*
iii) *$S_n/n^{1/p} \to 0$ in L_p for $(X_n) \in T_p$*
iv) *$S_n/n^{1/p} \to 0$ in probability for $(X_n) \in T_p$*
v) *$S_n/n^{1/p} \to 0$ almost surely for $(X_n) \in T_p$.*

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PROOF. (i) $\Leftrightarrow$ (ii) belongs to Maurey and Pisier [4]. (ii) $\Rightarrow$ (iii) is a consequence of Theorem 1 because p-stability implies q-Rademacher type for some $p < q$. (iii) $\Rightarrow$ (iv) is obvious. (iv) $\Rightarrow$ (v) is a slight modification of a Theorem 3.1 due to A. de Acosta [1] i.e., a sequence (X_n) of 0-mean i.i.d. is replaced by $(X_n) \in T_p$. (v) $\Rightarrow$ (i) follows from Corollary 2.4 of [4].

The above theorem besides new conditions (iii) and (iv) unifies the results of Woyczynski [7] who proved (i) $\Leftrightarrow$ (v) for $1 < p < 2$ whereas for $p = 1$ under the additional assumption that $X_0 \in L\log^+ L$.

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