The Annals of Probability
1984, Vol. 12, No. 1, 272–273

AN ALTERNATE PROOF OF A CORRELATION INEQUALITY OF HARRIS

BY J. THEODORE COX

Syracuse University

A theorem of Harris states that a monotone Markov process on a finite partially ordered set has positive correlations at time t (assuming positive correlations at time 0) if and only if each jump of the process is either up or down. A new proof of the sufficiency of the jump condition is presented.

The purpose of this note is to present an elementary proof of a correlation inequality (see [1]) of Harris. This proof may be of some value since it avoids an auxiliary process, the FKG inequality for product measure, and an approximation theorem of Trotter used in the original proof. We start by repeating some of the notation and definitions of [1].

Let E be a finite set with partial ordering $\leq$. For any pair of real functions μ and f defined on E we write $\mu(f) = \sum_{x\in E}\mu(x)f(x)$. A probability measure on E is just a nonnegative function μ on E such that $\mu(1) = 1$. A real function f on E is called *increasing* if $f(x) \leq f(y)$ whenever $x \leq y$. C_i will denote the set of increasing functions. A probability measure μ on E is said to have *positive correlations* if $\mu(fg) \geq \mu(f)\mu(g)$ whenever $f, g \in C_i$. M_p will denote the set of probability measures with positive correlations.

Let $\{X_t, t \geq 0\}$ be a Markov process with step-function paths in E and transition function $p(t, x, y)$. T_t and U_t are the usual semigroup operators,

$$T_tf(x) = \textstyle\sum_{y\in E}p(t, x, y)f(y) \quad \text{and} \quad U_t\mu(y) = \textstyle\sum_{x\in E}\mu(x)p(t, x, y).$$

The generator is A,

$$A(x, y) = \lim_{t\to 0}t^{-1}(p(t, x, y) - p(0, x, y)).$$

$\{X_t\}$ is called *monotone* if $T_tC_i \subset C_i$.

The Harris correlation inequality is

THEOREM. *Let $\{X_t\}$ be a monotone process in a finite partially ordered state space E. In order that $U_tM_p \subset M_p$ for each $t > 0$ it is necessary and sufficient that each jump of $\{X_t\}$ is up or down.*

We consider only the proof of the sufficiency of the jump condition; the proof of the necessity given in [1] is short and simple. As in [1] the problem is to show that for all $t > 0$, $f, g \in C_i$, $x \in E$,

$$T_t(fg)(x) \geq T_tf(x)T_tg(x). \tag{1}$$

The fact that $U_tM_p \subset M_p$ follows easily from (1).

Received January 1983.
AMS 1980 *subject classifications.* Primary 60B99; secondary 60K35.
Key words and phrases. Correlation inequality; partial order; Markov.

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For any real function h on E, $0 \le u \le 1$ and $x \in E$

$$T_u h(x) = h(x) + u \sum_{y \in E} A(x, y) h(y) + \theta_1 c_1 u^2 \| h \| \tag{2}$$

where c_1 is a finite constant independent of u, h, and x, $|\theta_1| \le 1$, and $\| h \| = \max_{y \in E} | h(y) |$. Using this expansion, and the fact that $\sum_{y \in E} A(x, y) = 0$ we can write

$$\begin{aligned} T_u(fg)(x) - T_u f(x) T_u g(x) = u \sum_{y \in E} A(x, y)[f(x) - f(y)][g(x) - g(y)] \\ + \theta_2 c_2 u^2 \|f\| \|g\| \end{aligned} \tag{3}$$

where c_2 is a finite constant independent of u, f, g, and x, and $|\theta_2| \le 1$. Assuming $f, g \in C_i$ and using the "up or down" jump property (3) implies

$$T_u(fg)(x) \ge T_u f(x) T_u g(x) - c_2 u^2 \|f\| \|g\|. \tag{4}$$

Using the standard properties of T_t we can iterate (4) to obtain

$$T_{u+v}(fg)(x) \ge T_{u+v} f(x) T_{u+v} g(x) - c_2 (u^2 + v^2) \|f\| \|g\|. \tag{5}$$

If $t > 0$ and n is a positive integer larger than t, (5) implies

$$T_t(fg)(x) = T_{n(t/n)}(fg)(x) \ge T_t f(x) T_t g(x) - c_2 n (t/n)^2 \|f\| \|g\|. \tag{6}$$

Let $n \to \infty$ in (6) to obtain (1).

NOTE. After completing this work the author learned that Professor Thomas Liggett had previously discovered a proof of (1) which is similar to the one presented above. This proof will appear in Professor Liggett's forthcoming book, *Interacting Particle Systems.*

REFERENCES

[1] HARRIS, T. E. (1977). A correlation inequality for Markov processes in partially ordered state spaces. *Ann. Probab.* **5** 451–454.

DEPARTMENT OF MATHEMATICS
SYRACUSE UNIVERSITY
200 CARNEGIE
SYRACUSE, NEW YORK 13210