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A STRONG LAW OF LARGE NUMBERS FOR PARTIAL-SUM PROCESSES INDEXED BY SETS

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Let $J = \{1, 2, \cdots\}^d$ and let $\{X_{\mathbf{j}}, \mathbf{j} \in J\}$ be iid with finite mean. Let $S(nA)$ be the sum of those $X_{\mathbf{j}}$'s for which $\mathbf{j}/n \in A$. It is proved in this paper that $S(\cdot)$ satisfies a strong law of large numbers that is uniform over $A \in \mathscr{A}$, where $\mathscr{A}$ is a family of subsets of $[0, 1]^d$ satisfying a mild condition.

1. Introduction. Let $J = \{1, 2, \cdots\}^d$ and let $\{X_{\mathbf{j}}; \mathbf{j} \in J\}$ be a family of iid random variables with $E|X_{\mathbf{j}}| < \infty$ and $EX_{\mathbf{j}} = \mu$. If $B \subseteq [0, \infty)^d$ is Borel measurable, let $|B|$ denote the Lebesgue measure of B and let $S(B) = \sum_{j \in B} X_{\mathbf{j}}$. A natural question is, if B_n is a sequence of sets (not necessarily nested) with $|B_n| \nearrow \infty$, will $S(B_n)/|B_n| \to \mu$, a.s.? And will this convergence be uniform over a large family of such sequences?

We provide answers to these questions by proving the following result. Given a set B, let $nB = \{nx: x \in B\}$ and let $B(\delta) = \{x: \rho(x, \partial B) < \delta\}$ be the δ-annulus of ∂B, where $\rho(\cdot, \cdot)$ is Euclidean distance and ∂B is the boundary of B.

THEOREM 1. *Suppose $\mathscr{A}$ is a collection of Borel measurable subsets of* $[0, 1]^d$ *such that*

$$r(\delta) \equiv \sup_{A \in \mathscr{A}} |A(\delta)| \to 0 \text{ as } \delta \to 0.$$

With $X_{\mathbf{j}}$ and $S(\cdot)$ as above,

$$\sup_{A \in \mathscr{A}} \left| \frac{S(nA)}{n^d} - \mu |A| \right| \to 0, \quad \text{a.s. as } n \to \infty.$$

Theorem 1 provides a strong law of large numbers that is uniform over $\mathscr{A}$. In Section 3 we show how this uniformity provides an answer to the first problem posed in the first paragraph. What may be a bit surprising is, that in strong contrast to most theorems involving processes indexed by sets, $\mathscr{A}$ need not satisfy any metric entropy condition. Thus, for example, if $\mathscr{A}$ were the collection of convex subsets of $[0, 1]^d$, it is easy to verify that $\mathscr{A}$ would satisfy the hypothesis of Theorem 1 for any d; however, only for $d = 1, 2$ are the convex subsets a small enough collection for most other purposes, including existence of Brownian processes and uniform convergence results for partial-sum and empirical processes. The particular case of Theorem 1 where $\mathscr{A}$ is the set of rectangles

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with one vertex at $\mathbf{0}$ was considered by Dunford (1951), Zygmund (1951), and Smythe (1973).

In addition to the strong law of large numbers of this paper, the partial-sum processes $S(\cdot)$ also satisfy a uniform central limit theorem (Pyke, 1983, and Bass and Pyke, 1984) and a functional law of the iterated logarithm (Bass and Pyke, 1984). However, for these latter results much stronger conditions are necessary; in particular metric entropy is crucial.

2. Proof of Theorem 1. First of all, if $\mathbf{x} = (x_1, \cdots, x_d)$ is fixed, $(\mathbf{0}, \mathbf{x}] = \{(y_1, \cdots, y_d)\colon 0 < y_i \le x_i,\ i = 1, \cdots, d\}$, and $\#$ denotes cardinality, then by Kolmogorov's strong law,

$$(1) \qquad n^{-d}S(n(\mathbf{0}, \mathbf{x}]) = \frac{\#(J \cap n(\mathbf{0}, \mathbf{x}])}{n^d} \frac{S(n(\mathbf{0}, \mathbf{x}])}{\#(J \cap n(\mathbf{0}, \mathbf{x}])} \to |(\mathbf{0}, \mathbf{x}]|\,\mu, \quad \text{a.s.}$$

Secondly, if A can be obtained by a finite number of unions and differences of rectangles of the form $(\mathbf{0}, \mathbf{x}]$, by linearity,

$$(2) \qquad n^{-d}S(nA) \to |A|\,\mu, \quad \text{a.s.}$$

Now let $\nu = E|X_{\mathbf{j}}|$ and let $T(A) = \sum_{\mathbf{j}\in A} |X_{\mathbf{j}}|$. If m is an integer, let $C_{\mathbf{j}} = m^{-1}(\mathbf{j} - \mathbf{1}, \mathbf{j}]$, and for any $A \in \mathscr{A}$, let $R_m^-(A) = \cup_{C_{\mathbf{j}} \subseteq A} C_{\mathbf{j}}$, $R_m^+(A) = \cup_{C_{\mathbf{j}} \cap A \ne 0} C_{\mathbf{j}}$. Thus $R_m^-(A)$ and $R_m^+(A)$ are inner and outer rectilinear fits of A by cubes of size $1/m$.

Since the furthest any point of $R_m^+(A)\backslash R_m^-(A)$ can be from the boundary of A is the diameter of a cube of size $1/m$, we have by assumption

$$\sup_{A\in\mathscr{A}} |R_m^+(A)\backslash R_m^-(A)| \le r(d^{1/2}/m).$$

Let $\mathscr{R}_m^- = \{R_m^-(A)\colon A \in \mathscr{A}\}$ and $\mathscr{R}_m^\Delta = \{R_m^+(A)\backslash R_m^-(A)\colon A \in \mathscr{A}\}$. Since each $A \in \mathscr{A}$ is contained in $[0, 1]^d$, it should be evident that $\#\mathscr{R}_m^-$ and $\#\mathscr{R}_m^\Delta$ are finite.

We then have, for m fixed,

$$(3) \qquad \begin{aligned} &\limsup_{n\to\infty, A\in\mathscr{A}} |n^{-d}S(nA) - |A|\,\mu| \\ &\quad\le \limsup_{n\to\infty, A\in\mathscr{A}} n^{-d}|S(nA) - S(nR_m^-(A))| \\ &\qquad + \limsup_{n\to\infty, A\in\mathscr{A}} |n^{-d}S(nR_m^-(A)) - \mu\,|R_m^-(A)|| \\ &\qquad + \limsup_{n\to\infty, A\in\mathscr{A}} |\mu|\,|A\backslash R_m^-(A)| = I_1 + I_2 + I_3. \end{aligned}$$

Clearly, $I_3 \le |\mu|\, r(d^{1/2}/m)$.

$$\begin{aligned} I_2 &\le \limsup_{n\to\infty, B\in\mathscr{R}_m^-} |n^{-d}S(nB) - \mu\,|B|| \\ &\le \limsup_{n\to\infty} \max_{B\in\mathscr{R}_m^-} |n^{-d}S(nB) - \mu\,|B|| = 0, \quad \text{a.s,} \end{aligned}$$

since $\#\mathscr{R}_m^- < \infty$ and every set $B \in \mathscr{R}_m^-$ can be obtained by a finite number of unions and differences of rectangles of the form $(\mathbf{0}, \mathbf{x}]$, recall (2).

Finally,

$$\begin{aligned} I_1 &\le \limsup_{n\to\infty, A\in\mathscr{A}} n^{-d} T(nR_m^+(A)\setminus nR_m^-(A)) \\ &\le \limsup_{n\to\infty} \max_{B\in\mathscr{R}_m^\Delta} | n^{-d}T(nB) | \\ &\le \nu \max_{B\in\mathscr{R}_m^\Delta} | B | \le \nu r(d^{1/2}/m), \quad \text{a.s.} \end{aligned}$$

where we used the fact that $\#\mathscr{R}_m^\Delta$ was finite and the analogue of (2) for the partial-sum process T.

Summing, we have from (3)

$$\limsup_{n\to\infty, A\in\mathscr{A}} | n^{-d}S(nA) - |A|\mu | \le (|\mu| + \nu)r(d^{1/2}/m), \quad \text{a.s.}$$

Letting $m \to \infty$ concludes the proof. □

3. Remarks.

1. Suppose we are given a sequence of sets B_n such that $|B_n| \to \infty$, as in the first paragraph of the introduction. Let $A_n = n^{-1}B_n$, and let $\mathscr{A} = \{A_n\}$. If $\mathscr{A}$ satisfies the hypothesis of Theorem 1 and $|A_n|$ is bounded away from 0, then

$$\limsup_{n\to\infty} \left| \frac{S(B_n)}{|B_n|} - \mu \right| \le \limsup_n |A_n|^{-1} \limsup_{A\in\mathscr{A}} \left| \frac{S(nA)}{n^d} - \mu |A| \right| = 0.$$

2. Without further conditions on $\mathscr{A}$, one cannot say much about the necessity of $E|X_{\mathbf{j}}| < \infty$, as the following trivial example shows. Let $d = 1$, let $\mathscr{A}$ consist of the single set $A = \{x_0\}$, where x_0 is irrational. Then $S(nA) \equiv 0$ for all n, no matter what the distribution of $X_{\mathbf{j}}$ is.

3. By requiring $E|X_{\mathbf{j}}| \log^+ |X_{\mathbf{j}}|^{d-1} < \infty$, Theorem 1 can be extended to allow $n \to \infty$ in more than one way. That is, one considers $\limsup_{A\in\mathscr{A}} |S(\mathbf{n}\cdot A)/\|\mathbf{n}\| - \mu|A|\|$, where $\mathbf{n} = (n_1, \cdots, n_d)$, $\|\mathbf{n}\| = n_1 \cdot n_2 \cdot \cdots \cdot n_d$ and $\mathbf{n}\cdot A = \{(n_1y_1, \cdots, n_dy_d): (y_1, \cdots, y_d) \in A\}$, and the limits over $\mathbf{n}$ are as in Smythe (1973). To prove the extension, replace the use of Kolmogorov's strong law in the proof of (1) by the use of Smythe's strong law.

4. In Pyke (1983) and Bass and Pyke (1984), it was necessary to consider a smoothed version of the partial sum process. In both cases, $S(nA)$ was replaced by

$$\hat{S}_n(A) = \textstyle\sum_{\mathbf{j}} |(\mathbf{j} - \mathbf{1}, \mathbf{j}] \cap nA| X_{\mathbf{j}}.$$

Only minor modifications are needed to the proof of Theorem 1 to make it applicable to this case as well.

5. Theorem 1 and the above remark suggest that one could formulate a more general uniform strong law. That is, let $X_1, X_2 \cdots$ be an infinite sequence of iid

random variables. For each n, let A_n be a subset of $l_1 \equiv \{(a_1, \cdots): \sum_{i=1}^{\infty} |a_i| < \infty\}$, and let

$$D_n = \sup_{(a_1, \cdots) \in A_n} \left| \sum_{i=1}^{\infty} a_i X_i - \mu \sum_{i=1}^{\infty} a_i \right|.$$

It may be verified that Theorem 1 and its extension given in Remark 4 are special cases of this general formulation. It would be of interest to find the most general conditions on the A_n's so that $D_n \to 0$, a.s. as $n \to \infty$.

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