The Annals of Probability
1984, Vol. 12, No. 1, 262–267

CHARACTERISTIC FUNCTIONS OF MEANS OF DISTRIBUTIONS CHOSEN FROM A DIRICHLET PROCESS[1]

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Let P be a Dirichlet process with parameter α on (R, B), where R is the real line, B is the σ-field of Borel subsets of R and α is a non-null finite measure on (R, B). By the use of characteristic functions we show that if $Q(\cdot) = \alpha(\cdot)/\alpha(R)$ is a Cauchy distribution then the mean $\int_R x\, dP(x)$ has the same Cauchy distribution and that if Q is normal then the distribution of the mean can be roughly approximated by a normal distribution. If the 1st moment of Q exists, then the distribution of the mean is different from Q except for a degenerate case. Similar results hold also in the multivariate case.

1. Introduction and summary. Ferguson (1973) introduces the Dirichlet process for use in Bayesian nonparametric inference and gives its properties and applications. With respect to the distribution of the quantity associated with the Dirichlet process, the distribution function of the quantile is given in 5(d) of Ferguson (1973).

Let P be a Dirichlet process with parameter α on (R, B), where R is the real line, B is the σ-field of Borel subsets of R and α is a non-null finite measure on (R, B). We denote $\alpha(\cdot)/\alpha(R)$ by $Q(\cdot)$ and a random functional $\int_R h(x)\, dP(x)$ by $\int h\, dP$ for a real-valued measurable function h.

The author (1980) shows that the mean $\mu(P) = \int x\, dP$ is distributed symmetrically about ξ if the measure α is symmetric about a constant ξ and $\int_R |x|\, d\alpha(x)$ is finite, and gives the moment of the mean when there exists the moment of the distribution Q. Generally, the mean $\mu(P)$ is distributed symmetrically about ξ provided $\mu(P)$ makes sense and α is symmetric about ξ (see Hannum and others (1981), page 669).

For the random functional $\int h\, dP$, Hannum and others (1981) obtain the equality

$$\Pr\left\{\int h\, dP \le x\right\} = \Pr\{T^x \le 0\},$$

where $-\infty < x < \infty$ and T^x is a random variable with characteristic function $\exp\{-\int_R \log[1 - it\{h(t) - x\}]\, d\alpha(t)\}$. By the use of this result they show that when h is odd and α is symmetric about 0, then the distribution of $\int h\, dP$ is symmetric about 0 and that if $P, P_n, n = 1, 2, \cdots$ are random probability measures chosen by Dirichlet processes on (R, B) with parameters $\alpha, \alpha_n, n = 1, 2, \cdots$ respectively, and if α_n converges weakly to α as $n \to \infty$, then under mild regularity conditions $\int h\, dP_n$ converges in distribution to $\int h\, dP$.

Received January 1982; revised February 1983.

[1]This work was supported in part by Kagoshima prefectural scholarship foundation, Japan.

AMS 1980 *subject classifications.* Primary 60K99; secondary 60E05.

Key words and phrases. Dirichlet process, mean, characteristic function.

In Section 2, we give several properties about the Dirichlet process for use in the following sections.

In Section 3, by the use of characteristic functions it is shown that if Q is a Cauchy distribution then the mean $\mu(P)$ has the same Cauchy distribution. If Q is the normal distribution $N(\xi, \sigma^2)$ then the characteristic function of the mean is given by (9) and the distribution of the mean can be roughly approximated by the normal distribution $N(\xi, \sigma^2/(\alpha(R)+1))$. If there exists the 1st moment of Q, then the distribution of the mean $\mu(P)$ is different from Q except for the case in which Q degenerates.

Similar results hold also in the multivariate case, which are shown in Section 4.

2. Preliminaries. Let $V_1, V_2, \cdots$ be a sequence of independent and identically distributed random variables with values in R and the probability measure Q. Let $P_1, P_2, \cdots$ be a sequence of real-valued random variables with the particular distribution, given in Ferguson (1973), depending on α only through $\alpha(R)$ and satisfying $\sum_{j=1}^{\infty} P_j = 1$ and $P_j \geq 0$ $(j = 1, 2, \cdots)$. Let $V_1, V_2, \cdots$ be independent of $P_1, P_2, \cdots$. Then the Dirichlet process P with parameter α on (R, B) has the form

$$P(A) = \sum_{j=1}^{\infty} P_j \delta_{V_j}(A) \quad \text{for} \quad A \in B, \tag{1}$$

where δ_v denotes the unit measure on (R, B) concentrated at the point v. In addition,

$$E(\textstyle\sum_{j=1}^{\infty} P_j^2) = (\alpha(R) + 1)^{-1}. \tag{2}$$

Hereafter we shall denote $\alpha(R)$ by M. A generalization of (2) is the following Lemma 1, which is immediately obtained from Lemma 4 of Yamato (1977).

LEMMA 1. *For any positive integers $m, r(1), \cdots, r(m)$,*

$$E \textstyle\sum_{j(1)\neq\cdots\neq j(m)} P_{j(1)}^{r(1)} \cdots P_{j(m)}^{r(m)} = (r(1)-1)! \cdots (r(m)-1)! M^m/M^{(k)},$$

where $k = \sum_{j=1}^{m} r(j)$, and for any real a, $a^{(0)}$ denotes 1 and $a^{(j)}$ denotes $a(a+1) \cdots (a+j-1)$ for a positive integer j.

From the above lemma, we have

$$E(\textstyle\sum_{j=1}^{\infty} P_j^2)^2 = M(M+6)/M^{(4)}, \quad E(\sum_{j=1}^{\infty} P_j^2)^3 = M(M^2+18M+120)/M^{(6)}$$

and therefore

$$E(\textstyle\sum_{j=1}^{\infty} P_j^2 - (M+1)^{-1})^2 = 2M^2/(M+1)M^{(4)} \tag{3}$$

$$E(\textstyle\sum_{j=1}^{\infty} P_j^2 - (M+1)^{-1})^3 = 16M^2(2M-1)/(M+1)^2 M^{(6)}. \tag{4}$$

More generally by applying Lemma 1 to

$$(\textstyle\sum_{j=1}^{\infty} P_j^2)^t = \sum_{u=1}^{t} \sum^* (t!/s(1)! \cdots s(u)! \prod_{l=1}^{u} K_l(s(1), \cdots, s(u))!) \times \sum_{j(1)\neq\cdots\neq j(u)} P_{j(1)}^{2s(1)} \cdots P_{j(u)}^{2s(u)}$$

with a positive integer t, we have

$$E(\textstyle\sum_{j=1}^{\infty} P_j^2)^t = \sum_{u=1}^{t} \sum^* c(t; s(1), \cdots, s(u))M^u/M^{(2t)},$$

where $\sum^*$ represents the summation over all sequences of integers $1 \le s(1) \le \cdots \le s(u)$ such that $\sum_{j=1}^{u} s(j) = t$, $K_l(s(1), \cdots, s(u))$ is the number of j such that $s(j) = l$ $(j = 1, 2, \cdots, u)$ for $l = 1, 2, \cdots, t$ and

$$c(t; s(1), \cdots, s(u))$$
$$= (2s(1) - 1)! \cdots (2s(u) - 1)!t!/s(1)! \cdots s(u)! \textstyle\prod_{l=1}^{t} K_l(s(1), \cdots, s(u))!.$$

Therefore we have for a positive integer r

$$E(\textstyle\sum_{j=1}^{\infty} P_j^2 - (M + 1)^{-1})^r = (-(M + 1))^{-r}$$

$$\times \left[1 + \sum_{t=1}^{r} \binom{r}{t}(-(M + 1))^t \sum_{u=1}^{t} \sum^* c(t; s(1), \cdots, s(u))M^u \Big/ M^{(2t)}\right]. \tag{5}$$

3. Univariate case. We shall consider the characteristic function of the mean of P, $\mu(P)$, for special distributions Q, where P is a Dirichlet process with parameter α on (R, B). We denote the characteristic functions of Q and $\mu(P)$ by ϕ and ψ, respectively. At first we show the following

LEMMA 2. *If the mean $\mu(P)$ exists almost surely* (a.s.) *and $\prod_{j=1}^{\infty} \phi(P_j z)$ exists* a.s. *for $-\infty < z < \infty$, then*

$$\psi(z) = E \textstyle\prod_{j=1}^{\infty} \phi(P_j z) \quad \text{for} \quad -\infty < z < \infty. \tag{6}$$

PROOF. We can express by (1)

$$\mu(P) = \int x \, dP = \textstyle\sum_{j=1}^{\infty} V_j P_j,$$

whose characteristic function is

$$\psi(z) = \lim_{n\to\infty} E(\exp[iz \textstyle\sum_{j=1}^{n} V_j P_j]). \tag{7}$$

The independence of $V_1, V_2, \cdots$ yields

$$\begin{aligned} E(\exp[iz \textstyle\sum_{j=1}^{n} V_j P_j]) &= E \textstyle\prod_{j=1}^{n} E(\exp[iz V_j P_j] \mid P_1, \cdots, P_n) \\ &= E \textstyle\prod_{j=1}^{n} \phi(P_j z). \end{aligned} \tag{8}$$

Since $| \prod_{j=1}^{n} \phi(P_j z) | \le 1$, under the assumption we obtain (6) from (7) and (8).

PROPOSITION 1. *If Q is a Cauchy distribution then the mean $\mu(P)$ has the same Cauchy distribution.*

PROOF. If Q is a Cauchy distribution, the mean of the random distribution

P exists a.s. (see Doss and Sellke, 1982). We can denote the characteristic function of Q, $\phi(z)$, in the form of $e^{i\xi z - c|z|}$, where ξ, c are constants with $-\infty < \xi < \infty$ and $c > 0$ and we have

$$\prod_{j=1}^{\infty} \phi(P_j z) = \exp[i\xi z \textstyle\sum_{j=1}^{\infty} P_j - c\,|\,z\,|\, \sum_{j=1}^{\infty} P_j] = e^{i\xi z - c|z|}.$$

By Lemma 2 we obtain $\psi(z) = e^{i\xi z - c|z|}$.

PROPOSITION 2. *If Q is the normal distribution $N(\xi, \sigma^2)$ then for a positive integer r the characteristic function of $\mu(P)$ is given by*

$$\psi(z) = \exp\left[i\xi z - \frac{\sigma^2 z^2}{2(M+1)}\right]$$

$$\times \left\{1 + \frac{M^2(\sigma^2 z^2)^2}{4(M+1)M^{(4)}} - \frac{M^2(2M-1)(\sigma^2 z^2)^3}{3(M+1)^2 M^{(6)}} + \cdots + \frac{(\sigma^2 z^2)^r}{r!2^r(M+1)^r}\right.$$

$$\times \left[1 + \textstyle\sum_{t=1}^{r} \binom{r}{t}(-(M+1))^t \sum_{u=1}^{t} \sum^{*} c(t; s(1), \cdots s(u))M^u/M^{(2t)}\right] \tag{9}$$

$$\left. + \frac{\theta(\sigma^2 z^2)^{r+1}}{(r+1)!}\right\}, \quad (-\infty < z < \infty, \ |\,\theta\,| \le 1).$$

REMARK. We can easily show $0 \le M^2/4(M+1)M^{(4)} \le 0.0064$ and

$$\left|\frac{M^2(2M-1)}{3(M+1)^2 M^{(6)}}\right| = \frac{M^2}{4(M+1)M^{(4)}} \times \left|\frac{4(2M-1)}{3(M+1)(M+4)(M+5)}\right| \le 0.00043.$$

If Q is the normal distribution $N(\xi, \sigma^2)$, then the distribution of $\mu(P)$ can be roughly approximated by the normal distribution $N(\xi, \sigma^2/(M+1))$.

PROOF. By Lemma 2 we have for $-\infty < z < \infty$

$$\psi(z) = e^{i\xi z} E(\exp[-\sigma^2 z^2 \textstyle\sum_{j=1}^{\infty} P_j^2/2]). \tag{10}$$

We note that for a positive integer r

$$\exp[-z^2(\textstyle\sum_{j=1}^{\infty} P_j^2 - (M+1)^{-1})/2]$$

$$= 1 - z^2(\textstyle\sum_{j=1}^{\infty} P_j^2 - (M+1)^{-1})/2 + [-z^2(\sum_{j=1}^{\infty} P_j^2 - (M+1)^{-1})/2]^2/2!$$

$$+ \cdots + [-z^2(\textstyle\sum_{j=1}^{\infty} P_j^2 - (M+1)^{-1})/2]^r/r! + \theta z^{2(r+1)}/(r+1)!,$$

where θ is a random variable such that $|\,\theta\,| \le 1$. Taking the expectation of the above equation and using (2), (3), (4) and (5) we have

$$E\left(\exp\left[\frac{-z^2(\sum_{j=1}^{\infty} P_j^2 - (M+1)^{-1})}{2}\right]\right)$$

$$= 1 + \frac{M^2 z^4}{4(M+1)M^{(4)}} - \frac{M^2(2M-1)z^6}{3(M+1)^2 M^{(6)}} + \cdots + \frac{z^{2r}}{r!2^r(M+1)^r} \tag{11}$$

$$\times\left[1 + \sum_{t=1}^{r}\binom{r}{t}(-(M+1))^t \sum_{u=1}^{t} \sum{}^{*} c(t; s(1), \cdots, s(u))M^u/M^{(2t)}\right]$$

$$+ \frac{\theta z^{2(r+1)}}{(r+1)!},$$

where θ is a constant such that $|\theta| \leq 1$. Application of (11) to (10) yields the desired result.

From the previous Propositions 1, 2 we know that if Q is a Cauchy distribution then the distribution of the mean $\mu(P)$ is Q and that if Q is a normal distribution then the distribution of the mean $\mu(P)$ is different from Q. Generally, we have the following proposition.

PROPOSITION 3. *If there exists the* 1st *moment of Q, then the distribution of the mean $\mu(P)$ is different from Q except for the case in which Q degenerates.*

PROOF. To prove the proposition, we shall show that $E|\mu(P)| \leq E|V|$ with equality iff Q degenerates. Assume wlog that $EV = 0$. We have $E|\mu(P)| = E|\int x\, dP| \leq E \int |x|\, dP = \int |x|\, dQ = E|V|$, with equality iff $|\int x\, dP| = \int |x|\, dP$ a.s. This occurs iff $P((0, \infty)) = 0$ a.s. or $P((-\infty, 0)) = 0$ a.s., which is equivalent to $Q((0, \infty)) = 0$ or $Q((-\infty, 0)) = 0$ by Proposition 1 of Ferguson (1973). This implies that Q degenerates.

4. Multivariate case. In this section we shall consider a Dirichlet process on (R^d, B^d), where R^d is the d-dimensional Euclidean space and B^d is the σ-field of Borel subsets of R^d. We shall use the same notation as in the univariate case. P denotes a Dirichlet process with parameter α on (R^d, B^d), where α is a non-null finite measure on (R^d, B^d). We use M to denote $\alpha(R^d)$. $\boldsymbol{\mu}(P)$ denotes the mean vector of the distribution P. ϕ and ψ denote the characteristic function of $Q(\cdot) = \alpha(\cdot)/\alpha(R^d)$ and $\boldsymbol{\mu}(P)$, respectively. By a similar method to the proof of Lemma 2, we have the following

LEMMA 3. *If the mean vector $\boldsymbol{\mu}(P)$ exists* a.s. *and $\prod_{j=1}^{\infty} \phi(P_j\mathbf{z})$ exists* a.s. *for $\mathbf{z} \in R^d$, then we have*

$$\psi(\mathbf{z}) = E \prod_{j=1}^{\infty} \phi(P_j\mathbf{z}) \quad \textit{for} \quad \mathbf{z} \in R^d.$$

If Q is a multivariate Cauchy distribution, then its characteristic function $\phi(\mathbf{z})$ has the form $\phi(\mathbf{z}) = \exp[-g(\mathbf{z}) + i\gamma(\mathbf{z})]$, where $g(\mathbf{z}) \geq 0$ and $\gamma(\mathbf{z})$ are real functions satisfying the equations $g(a\mathbf{z}) = |a|g(\mathbf{z})$, $\gamma(a\mathbf{z}) = a\gamma(\mathbf{z})$ for every real number a

(see Ferguson, 1962). Therefore by a similar method to the proof of Proposition 1 using Lemma 3, we have the following.

PROPOSITION 4. *If Q is a multivariate Cauchy distribution then the mean vector $\mu(P)$ has the same multivariate Cauchy distribution.*

If Q is a multivariate normal distribution, then by a similar method to the proof of Proposition 2 using Lemma 3 we have the following.

PROPOSITION 5. *If Q is the multivariate normal distribution $N(\xi, \Sigma)$ with $\xi \in R^d$ and a $d \times d$ symmetric positive definite matrix Σ, then the characteristic function of the mean vector $\mu(P)$, $\psi(\mathbf{z})$, is given by* (9) *with $\xi'\mathbf{z}$ and $\mathbf{z}'\Sigma\mathbf{z}$ instead of ξz and $\sigma^2 z^2$, respectively.*

Thus, if Q is the multivarite normal distribution $N(\xi, \Sigma)$ then the distribution of $\mu(P)$can be roughly approximated by the multivarite normal distribution $N(\xi, \Sigma/(M + 1))$. Corresponding to Proposition 3, we have the following.

PROPOSITION 6. *If there exists the mean vector of a distribution Q, then the distribution of the mean vector $\mu(P)$ is different from Q, except for the case in which Q degenerates.*

The proof is accomplished by applying Proposition 3 to each component of the mean vector $\mu(P)$.

Acknowledgments. The author is grateful to Professor Kjell A. Doksum for his kindliness and helpful comments. The author also wishės to express his gratitude to Professor Thomas S. Ferguson and a referee for their many valuable comments and suggestions.

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