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CONDITIONED LIMIT THEOREMS AND HEAVY TRAFFIC

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In this note we prove a generalisation of a conditioned functional central limit theorem of Bolthausen (cf.[3]). This generalisation explains the nature of the discontinuity between such conditioned limit theorems for random walks (or in queueing for waiting times) with or without drift.

1. Introduction. For each $n \geq 1$, let $Y_{n1}, Y_{n2}, \cdots$ be i.i.d. random variables with mean μ_n and variance σ_n^2. We assume that as $n \to \infty$, $\sigma_n^2 \to \sigma^2$, $\sigma^2 > 0$, and $\mu_n\sqrt{n} \to -\lambda\sigma$, $0 \leq \lambda < \infty$. By S_{nk} we denote the partial sums: $S_{n0} = 0$, $S_{nk} = Y_{n1} + \cdots + Y_{nk}$, $k \geq 1$. Let $C = C[0, 1]$ be the set of continuous functions on $[0, 1]$, with the uniform topology, and denote by $\mathscr{C}$ its Borel σ-field. We define Y_n as the random element of C that is linear on each interval $[(k-1)/n, k/n]$, $1 \leq k \leq n$, and has values: $Y_n(k/n) = S_{nk}/\sigma\sqrt{n}$, $0 \leq k \leq n$. Furthermore let $T_n = \inf\{k: S_{nk} < 0\}$, $(\inf \phi = \infty)$.

We shall now introduce the limiting random function that will occur in the theorem. This random function $Y^{(\lambda)}$ is expressed in terms of Brownian excursion in the following way. If W denotes standard Brownian motion with zero drift, starting at the origin, $\tau^- = \sup\{t \leq 1: W(t) = 0\}$, $\tau^+ = \inf\{t \geq 1: W(t) = 0\}$, then the meander W^+ and the excursion W_0^+ are defined by

$$W^+(t) = (1 - \tau^-)^{-1/2} \,|\, W(\tau^- + (1 - \tau^-)t) \,|,$$
$$W_0^+(t) = (\tau^+ - \tau^-)^{-1/2} \,|\, W(\tau^- + (\tau^+ - \tau^-)t) \,|, \quad 0 \leq t \leq 1.$$

The finite dimensional distributions of $Y^{(\lambda)} \in C$, which completely determine this random function, are given by: for $0 \leq t_1 \leq t_2 \cdots \leq t_k \leq 1$ and $y_1, y_2, \cdots, y_k \geq 0$,

$$\Pr\{Y^{(\lambda)}(t_1) \leq y_1, \cdots, Y^{(\lambda)}(t_k) \leq y_k\}$$
$$(1.1) \qquad = (\psi(\lambda))^{-1} \int_0^1 \exp(\tfrac{1}{2}\lambda^2(1 - u^{-2}))\Pr\{W_0^+(u^2 t_1) \leq u y_1, \cdots, W_0^+(u^2 t_k) \leq u y_k\}\, du,$$

where $\psi(\lambda) = \{1 - \lambda e^{\lambda^2/2} \int_\lambda^\infty e^{-v^2/2}\, dv\}$.

We shall prove the following theorem.

THEOREM. *As $n \to \infty$,*

$$(1.2) \qquad (Y_n \,|\, T_n > n) \to_d Y^{(\lambda)} \quad on \quad (C, \mathscr{C}).$$

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To explain why the theorem formulated above is interesting, consider a random walk S_k, $k \geq 0$, generated by one sequence of i.i.d. random variables $X_1, X_2, \cdots$. If $T = \inf\{k: S_k < 0\}$ then for $EX_1 = 0$ and $EX_1^2 = \sigma^2$, Bolthausen's theorem tells us that the random sequence $(Z_n(\cdot) \mid T > n)$, where $Z_n \in C$ is linear on $[(k-1)/n, k/n]$ and $Z_n(k/n) = S_k/\sigma\sqrt{n}$, weakly converges to Brownian meander W^+. However for $EX_1 < 0$, and when certain conditions are imposed on the distribution of X_1, the limit is no longer W^+, but some renormalisation of Brownian excursion W_0^+. (To obtain W_0^+, S_k must be divided by some other multiple of $\sqrt{n}$.) The reader may consult [1] or [5] for these results. In the above theorem the mean values EY_{n1} depend on n in such a way that an intermediate result is obtained. For $\lambda = 0$ the random function $Y^{(\lambda)} = W^+$, the meander, while it is seen from (1.1) and by partial integration that for $\lambda \to \infty$ the finite dimensional distributions of $Y^{(\lambda)}$ approach those of the excursion W_0^+. In the following example we give an application of the theorem.

EXAMPLE. Consider a sequence of GI/G/1 queues: $Y_{ni} = v_{ni} - u_{ni}$, where the service times v_{ni} have a distribution with mean β_n and the interarrival times u_{ni} have a distribution with mean α_n and independent of v_{ni}. In this case $(S_{nk}/\sigma\sqrt{n} \mid T_n > n)$ denotes the normalized waiting time of the kth customer conditioned by the event that the number of customers served during the first busy period exceeds n. Furthermore $\mu_n = \alpha_n(a_n - 1)$ where $a_n = \beta_n/\alpha_n$ is the traffic intensity, and $\sigma_n^2 = \operatorname{var} u_{ni} + \operatorname{var} v_{ni}$. The conditions of the theorem require $\alpha_n(a_n - 1)\sqrt{n} \to -\lambda\sigma$ and $\sigma_n^2 \to \sigma^2$. This is the situation of heavy traffic where the traffic intensity is approaching 1. Hence, dependent on the value of the traffic intensity parameter and the variance of the queue we may choose the best approximation for the conditional distribution of the normalized waiting time.

For a generalization of Bolthausen's theorem in another direction, consult the paper by Shimura (cf. [7]).

2. Proof of the theorem. According to the Lindeberg form of Donsker's theorem (cf. [2], page 77) Y_n converges weakly to W_λ, standard Brownian motion with negative drift $-\lambda$. We denote by Q_λ the measure induced by W_λ on the Borel σ-field of $C[0, \infty)$. We now follow Bolthausen's paper [3]. For $f \in C[0, \infty)$ we define

$$\tau(f) = \inf\{t: f(s) \geq f(t),\ t \leq s \leq t+1\}, \quad \inf \phi = \infty;$$

then, as in [3], Lemma 2.2, $Q_\lambda(\tau < \infty) = 1$ and

$$(Y_n \mid T_n > n) \to_d Y^{(\lambda)}, \quad \text{on} \quad (C, \mathscr{C}),$$

where $Y^{(\lambda)}(t) = W_\lambda(\tau_\lambda + t) - W_\lambda(\tau_\lambda)$, $0 \leq t \leq 1$, with $\tau_\lambda = \tau(W_\lambda)$. To obtain this result, the only thing to check is whether Lemma 3.1 of [3] still holds. However this is clear, because the lemma only uses the independence and identical distribution of the sequence involved and *not* the mean value. To complete the proof we show (1.1). Introduce the function $\xi_\tau: C[0, \infty) \to [1, \infty)$ defined by

$$\xi_\tau(f) = \inf\{t - \tau: t > \tau + 1, f(t) = f(\tau)\}, \quad \inf \phi = \infty.$$

From $Q_\lambda(\tau < \infty) = 1$ we obtain $Q_\lambda(\xi_\tau < \infty) = 1$. Now take an arbitrary set $A \in \mathscr{S}$ and denote by B its pre-image induced by the identity $Y^{(\lambda)}(t) = W_\lambda(\tau_\lambda + t) - W_\lambda(\tau_\lambda)$, $0 \le t \le 1$.

Then according to the Cameron-Martin formula (cf. [4], Section 1.11),

$$\begin{aligned} \Pr\{Y^{(\lambda)} \in A\} &= Q_\lambda(B) \\ &= \int_B \exp\{-\lambda f(\tau + \xi_\tau) - \tfrac{1}{2} \lambda^2(\tau + \xi_\tau)\}\, dQ_0(f) \qquad (2.1) \\ &= \int_B \exp\{-\lambda f(\tau) - \tfrac{1}{2}\lambda^2\tau\}\exp(-\tfrac{1}{2}\lambda^2\xi_\tau(f))\, dQ_0(f). \end{aligned}$$

Notice that τ_0 is a splitting time for W, so τ_0 and $W(\tau_0)$ are independent of $\{W(\tau_0 + t) - W(\tau_0), t \ge 0\}$, cf. [6]. Furthermore, $\tau_0 + 1$ is a stopping time and so

$$\{Y^{(0)}(t), 0 \le t \le 1\} = \{W(\tau_0 + t) - W(\tau_0), 0 \le t \le 1\}$$

and

$$\{W(\tau_0 + t + 1) - W(\tau_0 + 1), t \le 0\}$$

are independent and (cf. [3], page 484) distributed as $\{W^+(t), 0 \le t \le 1\}$ and $\{W(t), t \ge 0\}$, respectively. Putting these facts together, up to a multiplicative constant the right hand side of (2.1) is equal to

$$\int_1^\infty \exp(-\tfrac{1}{2}\lambda^2 x)\Pr\{W^+ \in A, \xi \in dx\},$$

where $\xi = (\tau^+ - \tau^-)/(1 - \tau^-)$ and so is the first return time to 0 beyond $t = 1$ of ordinary zero drift Brownian motion starting from $Y = W^+(1)$. It is easy to derive from the first passage time density in Brownian motion and from the Raleigh-distribution of $W^+(1)$ that

$$\Pr\{\xi \in dx\} = \tfrac{1}{2}x^{-3/2}\, dx, \quad x \ge 1.$$

Hence from the definitions of W^+ and W_0^+,

$$\begin{aligned} \Pr\{W^+ \in A, \xi \in dx\} &= \Pr\{\xi \in dx\}\Pr\{W^+(\cdot) \in A \mid \xi = x\} \\ &= \tfrac{1}{2}x^{-3/2}\Pr\{W_0^+(\cdot/x) \in x^{-1/2} A\}\, dx. \end{aligned}$$

Relation (1.1) follows after setting $u = x^{-1/2}$.

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