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THE SUPREMUM OF A PARTICULAR GAUSSIAN FIELD

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We find exact upper and lower bounds for the distribution of the supremum of a homogeneous Gaussian random field with pyramidal covariance function. The upper bound comes from a reflection principle type argument. The lower bound is found by exploiting a relationship between this random field and a particular Banach space valued process in one-dimensional time.

1. Introduction. A real valued, two parameter Gaussian random field $X(\mathbf{t})$, $\mathbf{t} = (t_1, t_2)$, is called a τ-pyramidal covariance (τ-PC) field if it has zero mean and covariance function given by

$$\begin{aligned} R_\tau(\mathbf{s}, \mathbf{t}) &= E\{X(\mathbf{s})X(\mathbf{t})\} \\ &= (1 - |t_1 - s_1|/|\tau_1|)^+ \times (1 - |t_2 - s_2|/|\tau_2|)^+. \end{aligned} \tag{1.1}$$

Such fields are homogeneous, and represent a generalization of one of the few stationary one-parameter Gaussian processes for which the exact distribution of the maximum is known. (Slepian, 1961; Shepp, 1966, 1971.) Our aim will be to obtain information on the distribution of the maximum of a τ-PC field $X(\mathbf{t})$ as its parameter varies over a rectangle. Our results, unlike most others currently in the literature and dealing with the maxima of Gaussian fields, will provide exact upper and lower bounds on this distribution which are valid throughout the full range of the variable and not just at asymptotically high levels (e.g. Hasofer, 1978; Bickel and Rosenblatt, 1973; Piterbarg, 1972).

An earlier result, similar in spirit to the upper bound (2.4) we shall obtain below, is due to Cabaña and Wschebor (1981), who showed that for a τ-PC field $X(\mathbf{t})$

$$P\{M(X, \mathbf{T}) > u\} \le 16\Phi(-u(\tau_1\tau_2/(\tau_1 + T_1)(\tau_2 + T_2))^{1/2}), \tag{1.2}$$

where $M(X, \mathbf{T}) = \max\{X(\mathbf{t}): 0 \le t_i \le T_i\}$ and Φ is the standard normal distribution function. This result suffers from two drawbacks. Firstly, because of the large constant on the right hand side of (1.2), the upper bound is greater than one throughout much of the range of u (cf. Table 1, column 4). Secondly, it is easy to check that for large u, when the bound is less than one, it is the wrong order of magnitude. For example, when $\tau_i = T_i = 1$, the bound is $O(\exp\{-u^2/8\})$, whereas a general result on Gaussian processes (Landau and Shepp, 1971; Fernique, 1975; Marcus and Shepp, 1975) indicates that bounds of the form $O(\exp\{-u^2/v\})$ exist for every $v > 2$.

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The upper bound that we shall present in Section 2 to replace (1.2) suffers from the same defects as that of Cabaña and Wschebor, so that the improvement that it offers is in degree rather than in kind. The improvement in degree is, however, substantial.

In Section 4 we shall develop a lower bound for the excursion probability (1.2), which we believe to be close to the true probability. The technique we shall use there relies on a relationship between random fields and Banach space valued processes noted by Kuelbs (1973), and exploited by Goodman (1976) to obtain lower bounds on the excursion probabilities of the tied and untied Brownian sheets.

We note the obvious fact that the bounds obtained below are of interest not merely because of the information they provide about triangular covariance processes, but because by virtue of Slepian's inequality (Slepian, 1962), they lead to bounds for excursion probabilities of random fields whose covariance functions are either dominated by, or dominate, (1.1).

In the one-dimensional case, the distribution of $M(X, T)$ for X a standard PC process has been given by Slepian (1961) and Shepp (1966), for $T \le 1$, and Shepp (1971) for $T \ge 1$. The results we shall give in the following sections will depend upon these known probabilities, and, in particular, on the probability

$$g(u) := P\{M(X, 1) \ge u\} = 1 - \Phi^2(u) + \phi(u)\{u\Phi(u) + \phi(u)\}, \tag{1.3}$$

which is immediately derivable from the result of Slepian and Shepp. Here $\phi := \Phi'$. Note that straightforward analysis of the above shows that for large u

$$g(u) \sim (u + 2/u)\phi(u), \tag{1.4}$$

a result we shall exploit later.

Finally, we note that since the τ-PC field with $\tau = (1, 1)$ will occur often, we shall simply refer to it as a standard PC field. For a review of what is known (primarily in the asymptotic case) about excursion probabilities of general homogeneous Gaussian fields, the interested reader is referred to Section 6.9 of Adler (1981).

2. The upper bound.

To obtain an upper bound for the exceedance probability of a τ-pyramidal covariance field $X(\mathbf{t})$, we shall exploit a close relationship between X and the Brownian sheet $W(\mathbf{t})$; i.e., the zero mean Gaussian random field with covariance function

$$E\{W(\mathbf{s})W(\mathbf{t})\} = (s_1 \wedge t_1) \cdot (s_2 \wedge t_2), \quad s_i, t_i > 0.$$

The Brownian sheet can also be thought of as a function indexed by rectangles, by defining

$$W((\mathbf{s}, \mathbf{s} + \mathbf{t}]) = W(s_1 + t_1, s_2 + t_2) - W(s_1 + t_1, s_2) - W(s_1, s_2 + t_2) + W(s_1, s_2)$$

as the increment of W over the rectangle $(\mathbf{s}, \mathbf{s} + \mathbf{t}] = (s_1, s_1 + t_1] \times (s_2, s_2 + t_2]$. Increments of W over disjoint rectangles are independent, with variances equal to the areas of the rectangles. The relationship between X and W is simple: on the positive quadrant X is a version of the process Y defined by

$$Y(\mathbf{t}) = (\tau_1\tau_2)^{-1/2}W((\mathbf{t}, \mathbf{t} + \tau]), \quad t_1, t_2 > 0,$$

as is easily checked by evaluating the covariance function of Y. Thus, writing $\mathbf{t} < \mathbf{T}$ if $\mathbf{t} \in (\mathbf{0}, \mathbf{T}]$, we have

$$P\{\sup_{\mathbf{t}<\mathbf{T}} X(\mathbf{t}) \geq u\} = P\{\sup_{\mathbf{t}<\mathbf{T}} W((\mathbf{t}, \mathbf{t} + \tau]) \geq u(\tau_1\tau_2)^{1/2}\}. \tag{2.1}$$

We shall bound the right hand side of (2.1) via a "reflection principle" type of argument.

Fix $n \geq 1$ and let $\mathbf{t}_k$, $k = 1, \cdots, 2^{2n}$ be the point in $[\mathbf{0}, \mathbf{T}]$ with coordinates $t_{ki} = T_i(k_i + 1)2^{-n}$, $i = 1, 2$, where $k = 1 + k_1 + k_2 2^n$ and k_1 and k_2 run from 0 to $2^n - 1$. For any $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$ write

$$\mathbf{t}^* = (T_1 + \tau_1, t_2), \quad \mathbf{t}' = (t_1, T_2 + \tau_2)$$

and define

$$E_k := \{\sup_{m\leq k} W((\mathbf{t}_m, \mathbf{t}_m + \tau]) \geq u\ (\tau_1\tau_2)^{1/2}\} = \{\sup\ Y(\mathbf{t}_m) \geq u\}$$

$$F_k := E_k E_{k-1}^c \cdots E_1^c.$$

Suppose we are given F_k. Then since $(0, T_1 + \tau_1] \times (t_{k2} + \tau_2, T_2 + \tau_2]$ and $(\mathbf{0}, \mathbf{t}_1 + \tau]$ are disjoint (draw a picture), it follows that $W(((t_{k1}, t_{k2} + \tau_2), (t_{k1} + \tau_1, T_2 + \tau_2)])$ is independent of $W((\mathbf{t}_k, \mathbf{t}_k + \tau])$ and so has probability ½ of being positive and thus making $W((\mathbf{t}_k, (\mathbf{t}_k + \tau)']) > u(\tau_1\tau_2)^{1/2}$. Hence

$$P\{F_k\} \leq 2P\{F_k;\ \sup_{\mathbf{s}<\mathbf{T}} W((s, (s + \tau)']) \geq u(\tau_1\tau_2)^{1/2}\}.$$

Summing over k, letting $n \to \infty$, and using separability we obtain the bound

$$\begin{aligned} &P\{\sup_{\mathbf{t}<\mathbf{T}} W((\mathbf{t}, \mathbf{t} + \tau]) \geq u(\tau_1\tau_2)^{1/2}\} \\ &\qquad\qquad \leq 2P\{\sup_{\mathbf{t}<\mathbf{T}} W((\mathbf{t}, (\mathbf{t} + \tau)']) \geq u(\tau_1\tau_2)^{1/2}\}. \end{aligned} \tag{2.2}$$

We need now to apply a reflection argument once more to obtain the final bound. To do this, we first note that

$$\sup_{\mathbf{t}<\mathbf{T}}\{W((\mathbf{t}, (\mathbf{t} + \tau)'])\} =_{\mathscr{D}} \sup_{\mathbf{t}<\mathbf{T}}\{W(((t_1, 0), \mathbf{t} + \tau])\}$$

where $=_{\mathscr{D}}$ denotes equivalence in distribution. Defining, with the above conventions, E_k as $\{\sup_{m\leq k} W(((t_{m1}, 0), (\mathbf{t}_m + \tau)]) \geq u(\tau_1\tau_2)^{1/2}\}$, following the above argument again easily yields that the right hand side of (2.2) is bounded by

$$4P\{\sup_{t_1<T_1} W(((t_1, 0), (t_1 + \tau_1, T_2 + \tau_2)]) \geq u(\tau_1\tau_2)^{1/2}\}. \tag{2.3}$$

(Draw another picture). The supremum within this probability is, however, easily seen to be distributionally equivalent to that of the PC process with covariance $\tau_1(T_2 + \tau_2)(1 - |t|/\tau_1)^+$ on the interval $0 \leq t \leq T_1$. This, in turn, can be expressed in terms of the standard PC process to give

THEOREM 2.1. *Let X be a τ-PC field on the plane and Z a standard PC process on the real line. Then if $T_1, T_2 > 0$*

$$(2.4)\qquad \begin{aligned} &P\{\sup_{0<\mathbf{t}<\mathbf{T}} X(\mathbf{t}) \ge u\} \\ &\quad \le 4 \min\left\{P\left\{\sup_{0<t<T_1/\tau_1} Z(t) \ge u\left(\frac{\tau_2}{T_2+\tau_2}\right)^{1/2}\right\},\right. \\ &\qquad\qquad \left. P\left\{\sup_{0<t<T_2/\tau_2} Z(t) > u\left(\frac{\tau_1}{T_1+\tau_1}\right)^{1/2}\right\}\right\}. \end{aligned}$$

PROOF. All that remains to be proven is that the above bound extends from the first term in the min to the minimum. But this is a trivial consequence of the fact that the argument leading to (2.3) could have also been applied with the roles of $\mathbf{t}^*$ and $\mathbf{t}'$, and k_1 and k_2, interchanged.

Theorem 2.1 allows us to use one-dimensional results to bound excursion probabilities for τ-PC fields. In particular, if $\tau_i = T_i = 1$, (2.4), (1.3) and (1.4) yield

$$(2.5)\qquad P\{M(X, (1, 1)) \ge u\} \le 4g(u/\sqrt{2}) \sim O(\exp\{-u^2/4\}), \quad u \text{ large.}$$

Comparison with the general bound mentioned in the previous section indicates that although the above represents a substantial improvement on the Cabaña and Wschebor upper bound (cf. Table 1) it is still, asymptotically, of the wrong order of magnitude. Note, however, that despite this problem, this bound still yields quite reasonable order of magnitude bounds to the percentiles of $M(X, \mathbf{T})$.

TABLE 1.
Exceedence probabilities for the pyramidal covariance field on the unit square

u	Lower bounds (6.1)	Lower bounds (5.6)	Upper bounds (2.4)	Upper bounds (1.2)
0	.9917	.9990	3.637	8.000
0.4	.9610	.9936	3.358	6.732
0.8	.8743	.9714	2.971	5.513
1.2	.7093	.9084	2.495	4.388
1.6	.4911	.7813	1.971	3.390
2.0	.2833	.5942	1.456	2.538
2.4	.1354	.3880	.9993	1.841
2.8	.0538	.2136	.6356	1.292
3.2	.0179	.0983	.3737	.8768
3.6	.0050	.0377	.2027	.5749
4.0	.0012	.0120	.1014	.3640
4.4	.0002	.0032	.0467	.2224
4.8	.0001	.0007	.0198	.1312

3. A Banach space valued version of PC fields. We shall now introduce new, Banach space valued, processes that are essentially equivalent to PC random fields, and which we shall use, in the following section, to derive lower bounds for excursion probabilities. In order to introduce these processes we shall use the standard formalism detailed, for example, in Carmona (1977).

Thus, let $B = C[0, 1]$ be the space of continuous functions on $[0, 1]$, equipped with the supremum norm $\|\cdot\|$. Let B^* be its dual, $\mathscr{B}$ its Borel σ-field, and μ the measure induced on $\mathscr{B}$ by the zero mean Gaussian process with covariance function $(1 - |t - s|)^+$. It is well known that B^* can be embedded in $L^2(B, \mathscr{B}, \mu)$ by a one-one, continuous and weakly continuous mapping. Let H denote the closure of the range of this embedding. Then H is a Hilbert space with inner product $(\cdot, \cdot)_H$. If δ_x is the delta function in B^* defined by $\langle \delta_x, f \rangle = f(x)$, $f \in B$, then it is a simple calculation to check that

$$(3.1) \qquad (\delta_x, \delta_y)_H = \int \langle \delta_x, f \rangle \langle \delta_y, f \rangle \mu(df) = (1 - |x - y|)^+,$$

a result we shall use later.

We would now like to define and study the B-valued, zero mean, Gaussian process $\{X_t, t \in R\}$ satisfying, for each $s, t \in R$, and each $f, g \in B^*$,

$$(3.2) \qquad E\{\langle f, X_s \rangle \langle g, X_t \rangle\} = (1 - |t - s|)^+ \cdot (f, g)_H.$$

This process does not seem to have been studied before, so that we must prove

THEOREM 3.1. *There exists a continuous sample path, zero mean, Gaussian* $C[0, 1]$*-valued process on* R *satisfying* (3.2).

Note that such a process is necessarily stationary, and that X_s and X_t are independent whenever $|s - t| > 1$. The importance of this process, for our purposes, lies in

THEOREM 3.2. *A separable, continuous version of the standard* PC *random field is given by the* $C[0, 1]$*-valued Gaussian process satisfying* (3.2) *under the correspondence* $X(s, t) = X_s(t)$.

PROOF OF THEOREM 3.1. Comparing this result with Carmona's (1977) proof of the existence of the generalised Ornstein-Uhlenbeck process, it is easy to see that a Gaussian process satisfying (3.2) exists, and that its almost sure continuity will follow from the existence of a continuous and increasing p on R_+ with $p(0) = 0$ and

$$\int^{\infty} p(e^{-u^2})\, du < \infty$$

for which

$$(3.3) \qquad E\{\|X_t - X_s\|^2\} \le p(t - s).$$

To show that such a p exists, fix t and s and define the real valued process

$$Y(r) = X_t(r) - X_s(r), \quad r \in [0, 1].$$

Clearly $\| X_t - X_s \| = \sup\{ Y(r), r \in [0, 1]\}$. Furthermore, it follows from (3.1) and (3.2) that

$$\begin{aligned} E\{Y(u)Y(v)\} &= E\{[\langle \delta_u, X_t\rangle - \langle \delta_u, X_s\rangle] \cdot [\langle \delta_v, X_t\rangle - \langle \delta_v, X_s\rangle]\} \\ &= 2\,|t - s|\,(1 - |u - v|)^+, \end{aligned}$$

so that Y is a version of $\sqrt{2\,|t - s|}\,Z$, where Z is a standard PC process on $[0, 1]$. From (1.3) and (1.4) it is obvious that $\| Z \|$ has all moments, so if we write M_α for $2^{\alpha/2}E\{ \| Z \|^\alpha\}$ then

$$E\{ \| X_t - X_s \|^\alpha\} = E\{ \| Y \|^\alpha\} M_\alpha\, |t - s|^{\alpha/2}, \quad \alpha > 0.$$

Setting $\alpha = 2$ now yields (3.3) with $p(u) = M_2\,|u|$ and so the proof is complete.

PROOF OF THEOREM 3.2. Following the proof of Theorem 2 of Kuelbs (1973), which establishes a similar result relating the Brownian sheet to $C[0, 1]$-valued Brownian motion, we see that it suffices to show that for every pair of points (s_1, t_1), (s_2, t_2) we have

$$E\{X_{s_1}(t_1)X_{s_2}(t_2)\} = E\{X(s_1, t_1)X(s_2, t_2)\}.$$

However, in view of (3.1), (3.2), and (1.1), this is an easy calculation, which we leave to the reader.

4. The lower bound. In this section we shall use X to denote both a standard PC field and the $B = C[0, 1]$-valued process of Theorem 3.1. Exactly which interpretation is intended will always be clear from the context. If $u \in R$ is fixed, let us write $\mathscr{U}$ for the set $\{x \in B: \sup_{t\in[0,1]}x(t) \le u\}$. For Borel sets E, F of B we can define the *transition probability*

$$P(E, F) := P\{X_0 \in E, X_1 \in F\} = P\{X_0 \in E\}P\{X_1 \in F\}.$$

Since each $x \in B$ is a.s. continuous, and the sup norm is measurable, $\mathscr{U}$ is a measurable event and so we can also define the *transition probability for X absorbed by $\mathscr{U}$ as*

$$Q(E, F: \mathscr{U}) = P\{X_0 \in E, X_1 \in F, X_t \in \mathscr{U}\ \forall t \in [0, 1]\}.$$

Clearly $Q(E, F: \mathscr{U}) \le Q(E, F:B) = P(E, F)$, and so the *transition density*

$$q^u(x, y) \equiv \frac{Q(dx, dy: \mathscr{U})}{P(dx, dy)}$$

is well defined as a Radon-Nikodym derivative. Furthermore, $q^u(x, y) \le 1$ a.s. relative to $P(dx, dy)$.

Now let us note that the probability we are interested in is simply

$$(4.1)\qquad \begin{aligned} 1 - \int_{\mathscr{U}}\int_{\mathscr{U}} q^u(x, y)\, d\mu(x)\, d\mu(y) &= 1 - P\{\sup_s(\sup_t X_s(t)) \le u\} \\ &= 1 - P\{\sup_t \sup_s X(s, t) \le u\}, \end{aligned}$$

since $X_s(t)$ is a version of $X(s, t)$ by Theorem 3.1. This last probability is precisely what we are seeking, and we shall bound it by obtaining bounds for the transition densities $q^u(x, y)$. This is an approach originally developed by Goodman (1976) for studying the corresponding problem for the Brownian sheet. We adopt an approach similar to Cabaña and Wschebor's (1982) simpler formulation of Goodman's technique.

Let us first note that for each $r \in [0, 1]$ the three processes

$$(4.2)\qquad Y^r(s) := X(s, r) - sX(1, r) - (1 - s)X(0, r), \quad s \in [0, 1],$$

$X(1, t)$, $t \in [0, 1]$ and $X(0, t)$, $t \in [0, 1]$ are all independent, as is easily verified via a direct computation of covariance. Hence for fixed $r \in [0, 1]$ the conditional distribution of

$$X(s, r) = Y^r(s) + sX(1, r) + (1 - s) \cdot X(0, r)$$

given $X(0, t)$ and $X(1, t)$, for all $t \in [0, 1]$, is the same as the one given $X(0, r)$ and $X(1, r)$ only. If we now convert these statements to statements on the B-valued process, and let $x, y \in \mathscr{U}$, then we have

$$(4.3)\qquad \begin{aligned} &P\{X_s(r) < u \text{ for every } r \in [0, 1] \mid X_0 = x, X_1 = y\} \\ &\quad = P\{X_s(r) < u \text{ for every } r \in [0, 1] \mid X_0(r) = x(r), X_1(r) = y(r)\}. \end{aligned}$$

But this last probability is simple to evaluate, since it is only the transition density of a real-valued PC process on [0, 1] conditioned at its start and end points. To evaluate this, let $Z(t)$, $t \in [0, 1]$ be a standard PC process, and $W(t)$, $t \in [0, 1]$, a Brownian motion, with covariance $s \wedge t$ and arbitrary initial value. Then a check of covariances shows that $Z(t)$, conditioned on $\{Z(0) = x, Z(1) = y\}$ is a version of $\sqrt{2}W(t)$, conditioned on the event $\{W(0) = x/\sqrt{2}, W(1) = y/\sqrt{2}\}$. Thus

$$(4.4)\qquad \begin{aligned} &P\{Z(t) < u\ \forall t \in [0, 1] \mid Z(0) = x, Z(1) = y\} \\ &\quad = P\{W(t) < u/\sqrt{2}\ \forall t \in [0, 1] \mid W(0) = x/\sqrt{2}, W(1) = y/\sqrt{2}\} \\ &\quad = 1 - \exp\{-(x - u)(y - u)\}, \end{aligned}$$

where the last line follows from the known transition density for W (e.g. Feller, 1971, page 341).

Now let us return to the transition density q^u. Since $X_t \in \mathscr{U}$ implies $X(t, r) < u$ for each r, it immediately follows from the definition of q^u, (4.3) and (4.4), that for $x, y \in \mathscr{U}$,

$$q^u(x, y) \le 1 - \exp\{-[u - x(r)] \cdot [u - y(r)]\}, \quad \forall r \in [0, 1].$$

But this implies the seemingly stronger result

$$q^u(x, y) < 1 - \exp\{-[u - \sup_r x(r)] \cdot [u - \sup_r y(r)]\}. \tag{4.5}$$

However x and y represent values of X_0 and X_1, which are independent, as must therefore be the distributions of their suprema. These distributions are, of course, already known, with the tail probabilities given by $g(u)$, as at (1.3).

Thus, combining (4.1) and (4.5), we finally obtain

THEOREM 4.1. *Let $X(s, t)$ be a standard* PC *field on* $[0, 1]^2$. *Then*

$$P\{\sup_{[0,1]^2} X(s, t) \geq u\} \geq 1 - \int_{-\infty}^{u} \int_{-\infty}^{u} (1 - e^{-(u-x)(u-y)}) f(x) f(y)\, dx\, dy \tag{4.6}$$

where

$$f(x) = dg(x)/dx = \phi(x)\{\phi(x) + x\phi(x) + x^2\Phi(x)\}.$$

REMARK. The fact that in the above result we have considered the distribution of the supremum of $X(s, t)$ only over the unit square is, unfortunately, of crucial importance to the arguments used in obtaining the result. If a smaller rectangle were to be considered, the same argument would carry through until (4.5). However the two suprema appearing there would no longer be independent, and since their joint distribution is unknown (the marginal distributions are, of course, the same as before) the simplification of (4.5) to (4.6) is not possible.

If a rectangle larger than $[0, 1]^2$ is to be considered, the arguments relating the univariate processes $X(\cdot, t)$ to a standard Brownian motion break down, just as they do for the one-dimensional problem of Shepp (1966) and Slepian (1961), and no result is easily forthcoming.

5. Numerical results. Table 1 contains an evaluation of the bounds given in this paper for $P\{\sup_{[0,1]^2} X(s, t) \geq u\}$, when X is a standard PC field. Column 2 gives the lower bound of Theorem 4.1, Column 3 the upper bound of Theorem 2.1, and Column 4 the upper bound (1.2) of Cabaña and Wschebor. Of course, the upper bounds should in fact be replaced by unity when they exceed this value, but we thought it interesting to merely include them as they arise from the appropriate formulae. In Column 1 we have included a simple lower bound obtained from

$$\begin{aligned} P\{\sup_{[0,1]^2} X(s, t) \geq u\} &\geq P\{(\sup_{[0,1]} X(0, t) \geq u) \cup (sup_{[0,1]} X(1, t) \geq u)\} \\ &= 2g(u) - g^2(u), \end{aligned} \tag{5.1}$$

the last line following from (1.3) and the independence of $X(0, t)$ and $X(1, t)$.

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