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ON THE PROBABILITY OF LARGE DEVIATIONS IN BANACH SPACES

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Probabilities of large deviations for sums of i.i.d. Banach space valued random variables are investigated when the laws of the random variables converge weakly and a uniform exponential integrability condition is satisfied. Furthermore, a discussion of possible improvements of the estimates is given, when the probability is estimated that the sum lies in a convex set.

1. Introduction. Let B be a real separable Banach space, equipped with the Borel-σ-field $\mathscr{B}$ and let $\mathbb{P}$ be the set of probability measures on $(B, \mathscr{B})$. B^* denotes the (topological) dual of B.

If $\mu \in \mathbb{P}$, $\varphi \in B^*$, let $M(\varphi \mid \mu) = \int \exp(\varphi(x))\mu(dx)$ and if $a \in B$, let $h(a \mid \mu) = \sup\{\varphi(a) - \log M(\varphi \mid \mu)\colon \varphi \in B^*\}$. The following result is due to Donsker and Varadhan [6] and Bahadur and Zabell [3]:

THEOREM 1. *If* $\int \exp(t \| x \|)\, \mu(dx) < \infty$ *for all* $t > 0$, *then*

(1.1) *if* $A \subset B$ *is closed*, $\limsup_{n\to\infty}(1/n)\log \mu^{*n}(nA) \le -h(A \mid \mu)$.

(1.2) *If* $A \subset B$ *is open*, $\liminf_{n\to\infty}(1/n)\log \mu^{*n}(nA) \ge -h(A \mid \mu)$,

where μ^{*n} *is the n-fold convolution of* μ *and* $h(A \mid \mu) = \inf\{h(a \mid \mu)\colon a \in A\}$.

We shall prove here the following extension:

THEOREM 2. *Let* $\mu_n, \mu \in \mathbb{P}$, $n \in \mathbb{N}$, *such that* $\{\mu_n\}$ *converges weakly to* μ *and*

$$\sup_n \int \exp(t \| x \|)\mu_n(dx) < \infty \quad \textit{holds for all} \quad t > 0. \tag{1.3}$$

Then

(1.4) *if* $A \subset B$ *is closed*, $\limsup_{n\to\infty}(1/n)\log \mu_n^{*n}(nA) \le -h(A \mid \mu)$,

(1.5) *if* $A \subset B$ *is open*, $\liminf_{n\to\infty}(1/n)\log \mu_n^{*n}(nA) \ge -h(A \mid \mu)$.

The special case, where the μ_n are Gaussian, has been treated by Ellis and Rosen [7] and S. Chevet [4]. In this case (1.3) is automatically satisfied. In fact, inspection of Fernique's proof of the existence of exponential moments for Gaussian measures shows that if μ_n, $n \in \mathbb{N}$, are Gaussian and the μ_n converge

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weakly, then there are constants a, b, $c > 0$, not depending on n, such that

$$\mu_n(\{x: \|x\| \geq u\}) \leq a \exp(-bu^2) \quad \text{as} \quad u \geq c.$$

From this (1.3) follows (see [8]).

The proof given here is a rather elementary modification of the Donsker-Varadhan proof. In contrast, the proofs of Ellis/Rosen and Chevet rely on non-trivial properties of Gaussian measures in Banach spaces.

If $B = \mathbb{R}$ and A is an interval, the results which have been obtained are much better than Theorem 1 or 2 (see e.g. Bahadur and Rao [2] or Höglund [9]). Partly, this possibility of improvements depends only on the convexity of A. Although I have only very incomplete results in this direction, it seemed worth pointing out how the convexity of A leads to improvements of (1.1) and (1.2). This is done in Section 4. This has also been investigated by P. Ney [11] in the case $B = \mathbb{R}^n$.

2. The upper estimate. If $\nu, \mu \in \mathbb{P}$ let $k(\nu \mid \mu)$ be the Kullback/Leibler information, i.e. $k(\nu \mid \mu) = \nu(\log(d\nu/d\mu))$ if $\nu << \mu$ and $\nu(|\log(d\nu/d\mu)|) < \infty$ and $k(\nu \mid \mu) = \infty$ else. We write $\mu(f)$ for the expectation of f with respect to μ. Then

$$h(a \mid \mu) = \inf\{k(\nu \mid \mu): \nu(\text{id}) \text{ exists and equals } a\}. \tag{2.1}$$

Here id is the identity mapping $B \to B$ (see [6], Theorem 5.2. (iv)). Although there is in general no $\varphi \in B^*$ with $h(a \mid \mu) = \varphi(a) - \log M(\varphi \mid \mu)$, there is always a $\nu \in \mathbb{P}$ satisfying $\nu(\text{id}) = a$ and $h(a \mid \mu) = k(\nu \mid \mu)$, at least if $h(a \mid \mu) < \infty$. Furthermore, ν is then unique (see Csiszar [5]).

LEMMA 1. *Let μ_n, μ satisfy the condition of the theorem and $a_n \in B$ converge weakly to $a \in B$. Then* $\liminf_{n\to\infty} h(a_n \mid \mu_n) \geq h(a \mid \mu)$.

PROOF. From (1.3) it follows that for any $\varphi \in B^*$

$$\lim_{n\to\infty} M(\varphi \mid \mu_n) = M(\varphi \mid \mu). \tag{2.2}$$

Given $\varepsilon > 0$, there is a $\varphi \in B^*$ with $\varphi(a) - \log M(\varphi \mid \mu) \geq h(a \mid \mu) - \varepsilon$. Therefore, if n is large enough, we have $h(a_n \mid \mu_n) \geq \varphi(a_n) - \log M(\varphi \mid \mu_n) \geq h(a \mid \mu) - 2\varepsilon$. This proves the lemma.

LEMMA 2. *Let $A \subset B$ be closed, then*

$$h(A \mid \mu) \leq \liminf_{n\to\infty} h(A \mid \mu_n).$$

PROOF. We may assume that $\liminf_{n\to\infty} h(A \mid \mu_n) < \infty$. We select a subsequence $\{n_k\}$ with $\lim_{k\to\infty} h(A \mid \mu_{n_k}) = \liminf_{n\to\infty} h(A \mid \mu_n)$. Let $a_k \in A$ satisfy $h(a_k \mid \mu_{n_k}) \leq h(A \mid \mu_{n_k}) + 1/k$ and $\nu_k \in \mathbb{P}$ satisfy $k(\nu_k \mid \mu_{n_k}) = h(a_k \mid \mu_{n_k})$, $\nu_k(\text{id}) = a_k$. From Lemma 5.1 of [6] it follows that the sequence $[\nu_k\}$ is tight and furthermore

$$\limsup_{\rho\uparrow\infty} \sup_k \int_{\|x\|\geq\rho} \|x\| \, \nu_k(dx) = 0.$$

Therefore $\{a_k\}$ is relatively compact. Let $a \in A$ be a limit point of this sequence.

Then by Lemma 1

$$h(A \mid \mu) \le h(a \mid \mu) \le \lim_{k\to\infty} h(a_k \mid \mu_{n_k}) = \lim_{k\to\infty} h(A \mid \mu_{n_k}) = \liminf_{n\to\infty} h(A \mid \mu_n).$$

LEMMA 3. *If $A \subset B$ is open and convex, then*

$$\mu^{*n}(nA) \le \exp(-nh(A \mid \mu)).$$

PROOF. If A is open and convex, then $-h(A \mid \mu) = \lim_{n\to\infty}(1/n)\log \mu^{*n}(nA)$ (see [1], Theorem I 4.8). If A is convex, one has the following subadditivity: $\mu^{*n}(nA)\mu^{*m}(mA) \ge \mu^{*(n+m)}((n+m)A)$. From this, $h(A \mid \mu) = \inf_n(-(1/n)\log \mu^{*n}(nA))$. The lemma follows.

PROOF OF (1.4) IN THE CASE WHERE A IS COMPACT. Take $\varepsilon > 0$ and $A \subset \bigcup_{j=1}^m U_j$, where U_j are open balls with radius ε and center in A. Then

$$\begin{aligned}
\limsup_{n\to\infty} (1/n) \log \mu_n^{*n}(nA) &\le \limsup_{n\to\infty} (1/n)\log \mu_n^{*n}(\textstyle\bigcup_{j=1}^m nU_j)\\
&\le \limsup_{n\to\infty} (1/n)\log (\textstyle\sum_{j=1}^m \mu_n^{*n}(nU_j)).\\
&\le \max_{1\le j\le m} \limsup_{n\to\infty} (1/n)\log \mu_n^{*n}(nU_j)\\
&\le \max_{1\le j\le m} \limsup_{n\to\infty}(-h(U_j \mid \mu_n)) \quad \text{by Lemma 3}\\
&\le \max_{1\le j\le m}(-\liminf_{n\to\infty} h(\bar{U}_j \mid \mu_n))\\
&\le -\min_{1\le j\le m} h(\bar{U}_j \mid \mu) \quad \text{by Lemma 2}\\
&= -h(\textstyle\bigcup_{j=1}^m \bar{U}_j \mid \mu)\\
&\le -h(A^\varepsilon \mid \mu) \quad \text{where } A^\varepsilon \text{ is the closed } \varepsilon\text{-neighbourhood of } A.
\end{aligned}$$

If $\varepsilon \downarrow 0$, then $h(A^\varepsilon \mid \mu)$ increases to $h(A \mid \mu)$, as follows easily from the compactness of A and the fact that $h(a \mid \mu)$ is lower semicontinuous.

The general noncompact case can now be reduced to the compact case as is done in [6], by just showing that all arguments there work uniformly in n if (1.3) is satisfied.

Let μ_n^n be the n-fold product measure on B^n and θ_n: $B^n \to \mathbb{P}$ be defined by $\theta_n(x_1, \cdots, x_n) = (1/n) \sum_{j=1}^n \delta_{x_j}$ where δ_x is the one point measure in x.

LEMMA 4. *Given any $a > 0$, there is a compact set $C(a) \in \mathbb{P}$ (in the weak topology) with $\mu_n^n(\theta_n \notin C(a)) \le e^{-na}$ for all $n \in \mathbb{N}$.*

PROOF. This follows by a straightforward transcription of the corresponding result where the μ_n do not depend on n (see e.g. [1], Lemma I 7.4).

We construct now a sequence $0 = t_0 < t_1 < \cdots$, such that for $k \in \mathbb{N}$

$$\sup_n \int_{\|x\|\ge t_k} \exp(k \,\|x\|)\mu_n(dx) \le 2^{-k}.$$

Let f: $[0, \infty) \to [0, \infty)$ be such that $f(t)/t$ is continuous and increasing with

$\lim_{t\to\infty} f(t)/t = \infty$ and $f(t_k)/t_k \le k - 1$, $k \in \mathbb{N}$. Then it is easy to see that $\int \exp(f\|x\|)\mu_n(dx) \le 2$ for all n. If $a > 0$, let $G(a) = \{\nu \in \mathbb{P} : \int f(\|x\|)\nu(dx) \le a\}$. Then

$$\mu_n^n(\theta_n \notin G(a)) = \mu_n^n(\{x \in B^n: \textstyle\sum_{j=1}^n f(\|x_j\|) > na\} \le e^{-na}(\mu_n(e^f))^n \le e^{-na+n}. \tag{2.3}$$

Let $\Lambda(a) = \{\nu(\mathrm{id}): \nu \in C(a) \cap G(a)\}$. $C(a)$ is compact and $G(a)$ is closed in $\mathbb{P}$. Furthermore $\nu \to \nu(\mathrm{id})$ restricted to $G(a)$ is continuous. It follows that $\Lambda(a)$ is compact in B. Furthermore

$$\mu_n^{*n}(n\Lambda^c(a)) = \mu_n^n(\theta_n \notin \Lambda(a)) \le \exp(-na) + \exp(-na + n) \le 2\exp(-n(a-1)).$$

If A is closed in B with $h(A \mid \mu) < \infty$, then

$$\begin{aligned}\limsup_{n\to\infty} (1/n)\log \mu_n^{*n}(nA) &\le \limsup_{n\to\infty} (1/n)\log(\mu_n^{*n}(n(A \cap \Lambda(a))) + 2\exp(-n(a-1)))\\ &= \limsup_{n\to\infty} (1/n)\log \mu_n^{*n}(n(A \cap \Lambda(a))) \quad \text{if} \quad a > h(A) + 1\\ &\le -h(A \cap \Lambda(a) \mid \mu) \le -h(A \mid \mu).\end{aligned}$$

So (1.4) is proved.

3. The lower estimate.

LEMMA 5. *Let $A \subset B$ be open, $\varepsilon > 0$, $\mu \in \mathbb{P}$ with $\int \exp(t\|x\|)\mu(dx) < \infty$ for all t. Then there is a $\nu \in \mathbb{P}$ with a bounded continuous everywhere positive density g w.r.t. μ, such that $k(\nu \mid \mu) \le h(A \mid \mu) + \varepsilon$ and $\nu(\mathrm{id}) \in A$.*

PROOF. We may assume that $h(A \mid \mu) < \infty$. Then there is a $\nu' \in \mathbb{P}$ with $k(\nu' \mid \mu) \le h(A \mid \mu) + \varepsilon$ and $\nu'(\mathrm{id}) \in A$. Let $g' = d\nu'/d\mu$. If we put $g_n = (n \wedge g') \vee (1/n)$, then $\int g_n \log g_n d\mu \to k(\nu' \mid \mu)$, $\int g_n d\mu \to 1$ and $\int x\, g_n(x)\mu(dx) \to \nu'(\mathrm{id})$. By taking the densities $g_n/\int g_n\, d\mu$, we see that there is a bounded density g'', which is bounded away from 0, such that if $d\nu'' = g'' d\mu$, we have $k(\nu'' \mid \mu) \le h(A \mid \mu) + \varepsilon$, $\nu''(\mathrm{id}) \in A$. Approximating this density pointwise by bounded continuous densities which remain bounded away from 0, we arrive at the desired conclusion.

Let now $\mu_n \to \mu$ as in the statement of the theorem and let g be as in Lemma 5. We put $d\nu_n = g d\mu_n / \int g d\mu_n$. Then $\int g d\mu_n \to \int g d\mu = 1$ and therefore $k(\nu_n \mid \mu_n) \to k(\nu \mid \mu)$ and $\nu_n(\mathrm{id}) \to \nu(\mathrm{id})$.

Given $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that if $n \ge N$ $k(\nu_n \mid \mu_n) \le h(A \mid \mu) + \varepsilon$ and $\nu_n(\mathrm{id}) \in A$. If f is the function constructed after Lemma 4, we have $\sup_n \int \exp(f(\|x\|))\nu_n(dx) < \infty$ and therefore $\sup_n \int f(\|x\|)\nu_n(dx) < \infty$.

PROOF OF (1.5). Let $b: \mathbb{P} \to B$ be defined by $b(\nu) = \nu(\mathrm{id})$, whenever it exists.

Then

$$\mu_n^{*n}(nA) = \mu_n^n(\theta_n \in b^{-1}(A)) \geq \mu_n^n(\theta_n \in b^{-1}(A) \cap G(a))$$

for any $a > 0$. As b is continuous on $G(a)$ and $\nu \in G(a)$ for sufficiently large a, we have a weak neighbourhood U of ν in $\mathbb{P}$, such that $b^{-1}(A) \cap G(a) \supset U \cap G(a)$.

$$\mu_n^n(\theta_n \in b^{-1}(A) \cap G(a)) \geq \mu_n^n(\theta_n \in U) - \mu_n^n(\theta_n \notin G(a))$$
$$\geq \mu_n^n(\theta_n \in U) - \exp(-n(a-1)).$$

Let U be of the form

$$U = \{\pi \in \mathbb{P} : |\pi(f_1) - \nu(f_1)| < \varepsilon, \cdots, |\pi(f_k) - \nu(f_k)| < \varepsilon\}$$

where $f_1, \cdots, f_k$ are bounded continuous functions on B. As μ_n is equivalent to ν_n, we have

$$\mu_n^n(\theta_n \in U) = \int_{A_n} \exp\left(-\sum_{j=1}^n \log\left(\frac{d\nu_n}{d\mu_n}(x_j)\right)\right)\nu_n^n(d\underline{x}),$$

where $A_n = \{\underline{x} \in B^n: |(1/n)\sum_{j=1}^n f_i(x_j) - \nu(f_i)| < \varepsilon, 1 \leq i \leq k\}$.

$$\mu_n^n(\theta_n \in U) = \exp(-nk(\nu \mid \mu)) \int_{A_n} \exp\left(-\sum_{j=1}^n \left(\log \frac{d\nu_n}{d\mu_n}(x_j) - k(\nu \mid \mu)\right)\right)\nu_n^n(d\underline{x})$$
$$\geq \int_{A_n \cap B_n(\delta)} \exp(-n\delta)\exp(-nk(\nu \mid \mu))\nu_n^n(d\underline{x})$$
$$= \exp(-n(k(\nu \mid \mu) + \delta))\ \nu_n^n(A_n \cap B_n(\delta))$$

where

$$B_n(\delta) = \left\{\underline{x} \in B^n: \left|\frac{1}{n}\sum_{j=1}^n \log\frac{d\nu_n}{d\mu_n}(x_j) - k(\nu \mid \mu)\right| < \delta\right\}$$

and $\delta > 0$ is arbitrary. $\operatorname{Log}(d\nu_n/d\mu_n)$ is bounded and has mean $k(\nu_n \mid \mu_n)$ when integrated over $d\nu_n$. As $k(\nu_n \mid \mu_n) \to k(\nu \mid \mu)$ and $\nu_n(f_j) \to \nu(f_j)$, $1 \leq j \leq k$. We obtain by an application of the Tchebychev inequality that $\nu_n^n(A_n \cap B_n(\delta)) \to 1$ as $n \to \infty$. Therefore, we have

$$\liminf_{n\to\infty} (1/n) \log \mu_n^{*n}(nA) \geq -\min(k(\nu \mid \mu) + \delta, a - 1).$$

As a is arbitrary large, δ arbitrary small and $k(\nu \mid \mu)$ arbitrary close to $h(A \mid \mu)$, (1.5) follows.

4. The use of dominating points in convex sets. We shall discuss here how (1.1), (1.2), (1.4), (1.5) can be improved if A is convex. This can be achieved by the use of so-called dominating points (see P. Ney [11]). Let $A \subset B$ be closed and convex. By using (2.1) and the well-known strong convexity of $k(\nu \mid \mu)$ in ν

one sees that $h(a \mid \mu)$ is strongly convex. Therefore, if $h(A \mid \mu) < \infty$, there exists a unique $a_0 \in A$ with $h(a_0 \mid \mu) = h(A \mid \mu)$ and then a unique $\nu_0 \in \mathbb{P}$ with $k(\nu_0 \mid \mu) = h(A \mid \mu)$ and $\nu_0(\mathrm{id}) = a_0$. From Csiszar [5] (2.8) and Theorem 2.2 it follows that if $\nu \in \mathbb{P}$ is such that $k(\nu \mid \mu) < \infty$ and $\nu(\mathrm{id}) \in A$, then

$$\nu(\log(d\nu_0/d\mu)) \geq k(\nu_0 \mid \mu). \tag{3.1}$$

$$\nu \ll \nu_0. \tag{3.2}$$

Let $A_n = \{\underline{x} = (x_1, \cdots, x_n) \in B^n\colon (1/n) \sum_{j=1}^n x_j \in A\}$ and μ^n, ν_0^n be the n-fold product probabilities on B^n of μ resp. ν_0.

PROPOSITION 1.
a) $\mu^{*n}(nA) = \int_{A_n} \exp(-\sum_{j=1}^n \log(d\nu_0/d\mu)(x_j))\, \nu_0^n(d\underline{x})$,
b) $\sum_{j=1}^n \log(d\nu_0/d\mu)(x_j) \geq nk(\nu_0 \mid \mu)\ \mu^n -$ a.s. *on* A_n.

PROOF. a) $\mu^{*n}(nA) = \mu^n(A_n)$. It therefore suffices to show that for $1 \leq j \leq n$: $\mu^n(A_n \cap \{\underline{x}\colon (d\nu_0/d\mu)(x_j) = 0\}) = 0$. It suffices to take $j = 1$. Let $\Gamma = A_n \cap \{\underline{x} \in B^n\colon (d\nu_0/d\mu)(x_1) = 0\}$. If $\mu^n(\Gamma) > 0$, we define a probability measure ρ on $(B, \mathscr{B})$ by $\rho(C) = \int_\Gamma (1/n) \sum_{j=1}^n 1_C(x_j)\mu^n(d\underline{x})/\mu^n(\Gamma)$. Then $d\rho/d\mu \leq (\mu^n(\Gamma))^{-1}$ and $\rho(\mathrm{id}) \in A$. Therefore, using (3.2), one has $\rho << \nu_0$. Let $N = \{x \in B\colon (d\nu_0/d\mu)(x) = 0\}$. Then

$$\rho(N) = \int_\Gamma \frac{1}{n} \sum_{j=1}^n 1_N(x_j)\mu^n(d\underline{x})/\mu^n(\Gamma)$$
$$\geq \frac{1}{n} \int_\Gamma 1_N(x_1)\mu^n(d\underline{x})/\mu^n(\Gamma) = \frac{1}{n}$$

which is a contradiction.

PROOF OF b). Let $\Gamma = \{\underline{x} \in B^n\colon (1/n) \sum_{j=1}^n \log(d\nu_0/d\mu)(x_j) < k(\nu_0 \mid \mu)\} \cap A_n$. *If* $\mu^n(\Gamma) > 0$, we define ρ as in a). Then again $d\rho/d\mu \leq (\mu^n(\Gamma))^{-1}$ and $\rho(\mathrm{id}) \in A$. From (3.1) it follows that $\rho(\log(d\nu_0/d\mu)) \geq k(\nu_0 \mid \mu)$. This contradicts

$$\rho(\log d\nu_0/d\mu) = \int_\Gamma \frac{1}{n} \sum_{j=1}^n \log \frac{d\nu_0}{d\mu} (x_j)\mu^n(d\underline{x})/\mu^n(\Gamma) < k(\nu_0 \mid \mu).$$

The propositions may be applied to get upper bounds in the following way: If A is closed and convex, then

$$\mu^{*n}(nA) \leq e^{-nh(A\mid\mu)} \int_\Gamma \exp\left(- \sum_{j=1}^n \left(\log \frac{d\nu_0}{d\mu} (x_j) - h(A \mid \mu)\right)\right)\nu_0^n(d\underline{x}) \tag{3.3}$$

where $\Gamma = \{\underline{x} \in B^n\colon \sum_{j=1}^n \log(d\nu_0/d\mu)(x_j) \geq h(A \mid \mu)\}$ and this is

$$\leq e^{-nh(A\mid\mu)} \int \exp\left(- \left| \sum_{j=1}^n \log \frac{d\nu_0}{d\mu} (x_j) - h(A \mid \mu) \right|\right)\nu_0^n(d\underline{x}).$$

As an application, we prove the following result. Let

$$J = \{a \in B\colon h(a \mid \mu) < \infty\}. \quad x_0 = \mu(\mathrm{id}) \in J \quad \text{and}$$

$$J' = \{\lambda x_0 + (1 - \lambda)a\colon a \in J, \lambda \in (0, 1]\} \subset J.$$

THEOREM 3. *If A is closed and convex, $x_0 \notin A$ and $h(A \cap J' \mid \mu) < \infty$, then*

$$\mu^{*n}(nA)\exp(nh(A \mid \mu)) = O\left(\int \frac{1}{n} \left| \sum_{j=1}^n \left(\log \frac{d\nu_0}{d\mu}(x_j) - h(A \mid \mu)\right)\right| \nu_0^n(d\underline{x})\right) = o(1).$$

REMARK. It seems likely that the condition $h(A \cap J' \mid \mu) < \infty$ is satisfied in all reasonable cases where $h(A \mid \mu) < \infty$ although a proof eludes me. It is certainly satisfied if $h(\mathrm{int}\, A) < \infty$ or if $J - x_0$ is a linear subspace of B, as is true for Gaussian measures.

PROOF OF THEOREM 3. $\mathrm{Log}(d\nu_0/d\mu)$ has expectation $h(A \mid \mu)$ under ν_0. If $h(A \cap J' \mid \mu) < \infty$ is satisfied, there exists a $\nu' \in \mathbb{P}$ with $k(\nu' \mid \mu) < \infty$, $\nu' \sim \mu$ and ν' (id) $\in A$. Therefore, it follows from (3.2) that $\nu_0 \gg \nu' \sim \mu$ and therefore $\nu_0 \sim \mu$. If $\log(d\nu_0/d\mu) = h(A)$ ν_0 – a.s. it follows that $d\nu_0/d\mu = 1$ contradicting $x_0 \notin A$. Therefore we have $\nu_0(\log(d\nu_0/d\mu) = h(A)) < 1$. The theorem then follows from (3.3) and the following.

LEMMA 6. *Let $Y_1, Y_2, \cdots$ be an* i.i.d. *sequence of real valued random variables with $E \mid Y_i \mid < \infty$, $EY_i = 0$, $P(Y_i = 0) < 1$. Then there is a constant $c > 0$, such that*

$$E(\exp(-\mid \textstyle\sum_{j=1}^n Y_j \mid)) \le c\, E \mid (1/n) \sum_{j=1}^n Y_j \mid \quad \textit{for all} \quad n.$$

PROOF. Let $f(x) = \mathrm{sign}(x)(1 - e^{-|x|})(\mathrm{sign}(0) = 1)$. Then $f'(x) = e^{-|x|}$. Let $S_n = \sum_{j=1}^n Y_j$. Then

$$E(S_n f(S_n)) = n\, E(Y_n f(S_n)) = n\, E(Y_n(f(S_n) - f(S_{n-1})))$$
$$\ge n\, E(Y_n^2 f'(S_{n-1} + \theta Y_n);\ \mid Y_n \mid \le \beta)$$

for all $\beta > 0$, where θ is a random variable with $0 \le \theta \le 1$. Now $\exp(-\mid x + t \mid) \ge \exp(-\mid x \mid)e^{-\beta}$ if $\mid t \mid \le \beta$. Therefore

$$E \mid S_n \mid \ge E(S_n f(S_n)) \ge ne^{-\beta}E(\exp(-\mid S_{n-1} \mid))E(Y_n^2;\ \mid Y_n \mid \le \beta)$$

$E(Y_n^2;\ \mid Y_n \mid \le \beta)$ is a constant, which is > 0 if β is large enough. So the lemma follows.

REMARKS. In any case

$$\int \left| \frac{1}{n} \sum_{j=1}^n \left(\log \frac{d\nu_0}{d\mu}(x_j) - h(A \mid \mu)\right)\right| \nu_0^n(d\underline{x}) = o(1).$$

If one further knows that $\nu_0((\log(d\nu_0/d\mu))^2) < \infty$, it is $O(1/\sqrt{n})$. This is satisfied if A is the closure of an open convex set which is flat at the point at which satisfies $h(a \mid \mu) = h(A \mid \mu)$. We recall that a point $x \in \partial A$ is called a flat point if there is a unique closed hyperplane through x which has A on one side. If $y \in \text{int } A$ then $x \in \partial A$ is a flatpoint if and only if the function $q_y(z) = \inf\{\rho \geq o\colon z - y \in \rho(A - y)\}$ is Gâteaux-differentiable at x. This implies that if x is flat there is a $\varphi \in B^*$ such that for all z with $\varphi(x) = \varphi(z)$ and all $y \in \text{int } A$

$$\text{(3.4)} \quad \inf\{\lambda > 0\colon \lambda y + (1 - \lambda)(tz + (1 - t)x) \in A\} = o(t) \quad \text{as} \quad t \to 0.$$

If A is the closure of an open convex set B then $B = \text{int } A$ and if $x_0 = \mu(\text{id}) \notin A$ then it easily follows that $h(a \mid \mu) = h(B \mid \mu)$ and if this is smaller than infinity then the unique point a with $h(a \mid \mu) = h(A \mid \mu)$ belongs to ∂A.

THEOREM 4. *If in the above described situation a is a flat point then* $\mu^{*n}(nA)\exp(nh(A \mid \mu)) = O(1/\sqrt{n})$.

PROOF. As is mentioned above, $h(\text{int } A \mid \mu) < \infty$, so the conditions in Theorem 3 are satisfied. If $y \in B$ is any point with $h(y \mid \mu) < \infty$, and z satisfies $\varphi(z) = \varphi(a)$ (φ the above Gâteaux-derivative at a) then

$$\begin{aligned} h(a) &\leq \lambda(t)h(y) + t(1 - \lambda(t))h(z) + (1 - \lambda(t))(1 - t)h(a) \\ &\leq \lambda(t)h(y) + t(1 - \lambda(t))(h(z) - h(a)) + h(a) \end{aligned}$$

where $\lambda(t)$ is the infimum in (3.4). Using (3.4) and $h(y) < \infty$ we see that $h(z) \geq h(a)$. Therefore

$$h(a \mid \mu) = \inf\left\{k(\nu \mid \mu)\colon \int \varphi(x)\nu(dx) = \varphi(a)\right\}$$

and this infimum is attained at ν_0.

From Theorem 3.1 in [5] it follows that there is a $t \in \mathbb{R}$ with $d\nu_0/d\mu = \exp(t\varphi)/M(t\varphi)$. Therefore $\log(d\nu_0/d\mu)$ has moments of any order under ν_0 and so Theorem 4 follows from Theorem 3.

REMARK. In some Banach spaces the condition that all boundary points are flat is quite strong. E.g. balls have this property in L_p-space but not in $C[0, 1]$. On the other hand, even in $C[0, 1]$, balls have many flat boundary points, e.g. in the unit ball every f for which there is a unique $t \in [0, 1]$ with $|f(t)| = \|f\|_\infty = 1$ is a flat point (for this and the other facts on flat points used here, see Köthe [10], Section 26). So the estimate in Theorem 4 might be useful even in such spaces.

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