The Annals of Probability
1984, Vol. 12, No. 3, 918–921

ON THE CONVERGENCE OF SPECTRAL DENSITIES OF ARRAYS OF WEAKLY STATIONARY PROCESSES

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It is proved that the spectral densities of arrays of weakly stationary processes fulfilling a certain regularity condition converge uniformly to a continuous spectral density.

Let X_k, $k \in \mathbb{Z}$, be a weakly stationary sequence of square integrable real-valued random variables ($\equiv$ r.v.'s) which are centered at expectations. The existence of a continuous spectral density can be achieved by making further assumptions on X_k, $k \in \mathbb{Z}$, which we state in the following.

DEFINITION. For any weakly stationary sequence $X := (X_k)_{k\in\mathbb{Z}}$ of real-valued random variables define

$$(1) \qquad r(X, m) := \sup |\operatorname{Corr}(\textstyle\sum_{k=-K}^{0} a_k X_k, \sum_{j=m}^{J} b_j X_j)|, \quad m \in \mathbb{N},$$

where Corr denotes the correlation coefficient and the supremum is taken over all nonnegative integers K, J and real numbers a_k, b_j, $-K \le k \le 0$, $m \le j \le J$.

Now, by Lemma 2.2 of Ibragimov (1975), any weakly stationary sequence X of centered r.v.'s satisfying

$$(2) \qquad \textstyle\sum_{m\in\mathbb{N}} r(X, 2^m) < \infty$$

has a continuous spectral density. This follows from Theorem 1 of Kolmogorov and Rozanov (1960) and the well-known trick of applying Lemma 2.2 of Ibragimov (1975) to a stationary Gaussian sequence with the same covariance function as X.

We are now ready to formulate our main result which states the uniform convergence of spectral densities of arrays of certain weakly stationary processes fulfilling condition (2).

THEOREM. *Suppose that for each $n \in \mathbb{N}$, $X^{(n)} := (X_k^{(n)})_{k\in\mathbb{Z}}$ is a weakly stationary sequence of real-valued r.v.'s which are centered at expectations. Assume that, $\lim_{m\in\mathbb{N}}\sup_{n\in\mathbb{N}} \sum_{k=m}^{\infty} r(X^{(n)}, 2^k) = 0$. Denote by $s_n(t)$, $-\pi \le t \le \pi$, the (continuous and even) spectral density function of the sequence $X^{(n)}$, $n \in \mathbb{N}$, and suppose that for each $m \ge 0$, $c_m := \lim_{n\in\mathbb{N}} E(X_0^{(n)} X_m^{(n)})$ exists in $\mathbb{R}$. Then the functions s_n, $n \in \mathbb{N}$, are equicontinuous and converge uniformly as n tends to*

Received June 1983; revised October 1983.
AMS 1980 *subject classifications.* Primary 60G10.
Key words and phrases. Weakly stationary process, spectral density, Gaussian process.

infinity to a continuous (nonnegative and even) function $s(t)$, $-\pi \le t \le \pi$, *satisfying*

$$\int_{-\pi}^{\pi} \exp(imt)s(t)\,dt = c_{|m|}, \quad m \in \mathbb{Z}.$$

Notice that a weakly stationary sequence X with spectral density s (i.e. with autocovariances c_m, $m \in \mathbb{Z}$) automatically satisfies (2). Furthermore, the assumption $\lim_{m\in\mathbb{N}}\sup_{n\in\mathbb{N}} \sum_{k=m}^{\infty} r(X^{(n)}, 2^k) = 0$ can in general not be omitted as examples show.

PROOF. Denote by $F_n(t) := \int_{-\pi}^{t} s_n(x)\,dx$, $-\pi \le t \le \pi$, the spectral distribution function of the sequence $X^{(n)}$, $n \in \mathbb{N}$. Since F_n is continuous and $\sum_{k\in\mathbb{N}} k^{-1}r(X^{(n)}, k) < \infty$, as is shown below, we can conclude from Theorem 3.2 on page 474 of Doob (1953) that for $t \in [-\pi, \pi]$

$$\begin{aligned} F_n(t) = (2\pi)^{-1}\{&(t + \pi)E(X_0^{(n)2}) \\ &+ \textstyle\sum_{k=-\infty,k\neq 0}^{\infty} (ik)^{-1}[\exp(i\pi k) - \exp(-itk)]E(X_0^{(n)}X_k^{(n)})\}. \end{aligned} \tag{3}$$

Let $V := \sup_{n\in\mathbb{N}} E(X_0^{(n)2})$. For each $n \in \mathbb{N}$ we have

$$\begin{aligned} \textstyle\sum_{k\in\mathbb{N}} k^{-1} |E(X_0^{(n)}X_k^{(n)})| &\le \textstyle\sum_{k\in\mathbb{N}} k^{-1}Vr(X^{(n)}, k) \\ &= V \textstyle\sum_{j\in\mathbb{N}} \sum_{k=2^{j-1}}^{2^j-1} k^{-1}r(X^{(n)}, k) \\ &\le V \textstyle\sum_{j\in\mathbb{N}} r(X^{(n)}, 2^{j-1}) < \infty, \end{aligned}$$

since r is nonincreasing. It is now easy to check that $\lim_{n\in\mathbb{N}} F_n(t) = F(t)$ uniformly in $t \in [-\pi, \pi]$, where

$$F(t) := (2\pi)^{-1}\{(t + \pi)c_0 + \textstyle\sum_{k=-\infty,k\neq 0}^{\infty} (ik)^{-1}[\exp(i\pi k) - \exp(-itk)]c_k\}, \tag{4}$$

the last sum being absolutely convergent. Now Lemmata 2.2 and 2.3 of Ibragimov (1975) imply that the functions s_n are equicontinuous and $\lim_{n\in\mathbb{N}} s_n(t) = s(t) := F'(t)$ uniformly in $t \in [-\pi, \pi]$. Thus, the proof of our theorem is complete.

Since Theorem 3.2 on page 474 of Doob (1953) is formulated for complex-valued covariance functions, our theorem can be extended from real-valued to complex-valued r.v.'s with appropriate modifications (see the footnote on page 108 of Ibragimov and Rozanov, 1978, and Theorem 3.1, page 72, of Doob, 1953).

The following corollary shows as an example in which way our theorem can be applied.

COROLLARY. *Let* $i(n)$, $n \in \mathbb{N}$, *be a sequence of positive integers tending to infinity as n increases. Then under the assumptions and with the notations of the preceding theorem*:

(i) $\lim_{n\in\mathbb{N}} i(n)^{-1}\mathrm{Var}(X_1^{(n)} + \cdots + X_{i(n)}^{(n)}) = 2\pi s(0)$.

(ii) *If in particular* $c_m = \lim_{n\in\mathbb{N}} E(X_0^{(n)}X_m^{(n)}) = 0$, $m \in \mathbb{N}$, *then*

$$\lim_{\in\mathbb{N}} i(n)^{-1}\mathrm{Var}(X_0^{(n)} + \cdots + X_{i(n)}^{(n)}) = c_0 = \lim_{n\in\mathbb{N}} E(X_0^{(n)2}),$$

where Var *denotes the variance.*

Part (i) can be regarded to a certain extent as a generalization of Theorem 18.2.1, formula (18.2.3) of Ibragimov and Linnik (1971) to arrays of processes. Because of the earlier mentioned trick of considering Gaussian sequences, part (ii) of the preceding corollary is essentially a special case of Theorem 2 of Bradley (1983) which is formulated without the assumption of weak stationarity. It is worthwhile to notice that this particular result finds an interesting application in the theory of kernel density estimators, showing their asymptotic normality with the same mean and variance as if the observations were independent (see Bradley, 1983 for details).

Thus, certain statistical procedures based on kernel density estimators are robust against fairly general deviations from the assumption of independence of the observations. This approach towards robustness of statistical procedures can also be found in a recent article by Falk and Kohne (1982) in which a robustified version of the two sample sign test is studied in detail.

PROOF OF THE COROLLARY. To prove part (i), let $(X_k)_{k\in\mathbb{Z}}$ be a weakly stationary sequence with spectral density s. Then, by Ibragimov and Linnik (1971), formulae (18.2.2) and (18.2.5)

$$|\, i(n)^{-1}\mathrm{Var}(X_1^{(n)} + \cdots + X_{i(n)}^{(n)}) - i(n)^{-1}\mathrm{Var}(X_1 + \cdots + X_{i(n)})\,|$$

$$= \left|\, i(n)^{-1} \int_{-\pi}^{\pi} [\sin^2(i(n)t/2)/\sin^2(t/2)][s_n(t) - s(t)]\, dt \,\right|$$

$$\leq 2\pi \max_{-\pi\leq t\leq\pi} |\, s_n(t) - s(t)\,| \to_{n\in\mathbb{N}} 0,$$

and by formula (18.2.3) of the same reference, $i(n)^{-1}\mathrm{Var}(X_1 + \cdots X_{i(n)}) \to_{n\in\mathbb{N}} 2\pi s(0)$. Part (i) follows. Now formula (4) implies that in part (ii) we have $s \equiv c_0/(2\pi)$ and hence the assertion is immediate from (i).

Acknowledgment. The author would like to thank R. C. Bradley for stimulating ideas and the referee for his careful reading of the manuscript and valuable suggestions.

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