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ADDITIVE PROCESSES ON NUCLEAR SPACES

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In this work we construct general additive processes on the nuclear spaces, and prove Khintchin's formula and Paul Lévy's decomposition for these processes. As applications, we construct some Ornstein-Uhlenbeck processes with jumps and solve some (stochastic) partial differential equations obtained from the transformations of these processes by a random diffeomorphism corresponding to a finite dimensional diffusion process.

Introduction. The processes with independent increments have been the guiding elements of the stochastic calculus. The main reasons for this are, firstly, that they can be constructed easily from their local characteristics and secondly, that almost all of them possess the necessary properties to be stochastic measures in the sense of [9]. Consequently, in order to show the validity of a stochastic calculus on the nuclear spaces, one has to show that such processes can be constructed on these spaces as easily as in the finite dimensional case and this is the essential motivation of this work.

In the first section we recall some results as the Prokhoroff's condition, nuclear space-valued martingales, etc. The second section is devoted to the definition of the additive processes and to the proof of the regularity of their trajectories using the general results of L. Schwartz about the Markov processes (cf. [11]). In the last part we prove the theorem of Khintchin and obtain the decomposition of Paul Lévy for the nonhomogeneous additive processes with jumps. Let us note that in [8], K. Itô has given the characterization of $\mathscr{S}'(\mathbb{R}^d)$-valued (i.e. the space of the tempered distributions) continuous additive processes using the canonical basis of this nuclear space. Our approach is different from his and works in any nuclear space encountered in applications, permitting us to handle also the discontinuous additive processes.

In the last section we give three applications: The first one deals with an Ornstein-Uhlenbeck type process with values in the space of the tempered distributions whose driving semimartingale is a homogeneous additive process. We calculate explicitly the infinitesimal generator of this Markov process on the cylindrical, twice-differentiable functions. The second application studies the image of the above process under a stochastic flow of diffeomorphisms generated by a finite dimensional diffusion process. The image process turns out to be the solution of a stochastic partial differential equation with random coefficients and its mathematical expectation gives the solution of an integro-differential type Cauchy problem with a functional integral. The third application deals with a stochastic partial differential equation whose solution is the Ornstein-Uhlenbeck process of free quantum field.

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I. Notations and preliminaries. Φ denotes a complete, separable nuclear space whose toplological dual Φ' is also nuclear under its strong topology $\beta(\Phi', \Phi)$, denoted by Φ'_β. For practical reasons we suppose that Φ'_β is a Souslin space. If U is an absolutely convex neighborhood (of zero) in Φ, we denote by $\Phi(U)$ the quotient set $\Phi/p_U^{-1}(0)$ completed with respect to the gauge function p_U of U and $k(U)$ represents the canonical mapping from Φ into $\Phi(U)$. If $V \subset U$ is another such neighborhood, $k(U, V)$: $\Phi(V) \to \Phi(U)$ is defined by $k(U) = k(U, V) \circ k(V)$. Let us recall that Φ is called nuclear if there exists a neighborhood base $\mathscr{U}$ in Φ such that, for any $U \in \mathscr{U}$, there exists some $V \in \mathscr{U}$, $V \subset U$, for which the mapping $k(U, V)$ from $\Phi(U)$ into $\Phi(U)$ is a nuclear mapping. If B is a bounded, absolutely convex subset of Φ, we note by $\Phi[B]$ the completion of the subspace (of Φ) spanned by B with respect to the norm p_B(i.e., the gauge function of B). It is well known that (c.f. [10]) in each nuclear space Φ, there exists a neighborhood base $\mathscr{U}_h(\Phi)$ such that, for any $U \in \mathscr{U}_h(\Phi)$, $\Phi(U)$ is a separable Hilbert space whose dual can be identified by $\Phi'[U^0]$, U^0 being the polar of U, and Φ is (a subspace of) the projective limit of

$$\{(\Phi(U), k(U, V)): V, U \in \mathscr{U}_h(\Phi), V \subset U\}.$$

We note by $\mathscr{K}_h(\Phi)$ the set

$$\{U^0: U \in \mathscr{U}_h(\Phi'_\beta)\}$$

and $\mathscr{K}_h(\Phi')$ is defined by interchanging Φ and Φ'_β.

If $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ is a completed probability space with a right continuous, complete filtration $(\mathscr{F}_t; t \geq 0)$, a stochastic process M with values in Φ' is called a martingale if, for any $U \in \mathscr{U}_h(\Phi'_\beta)$, $(k(U)(M_t); t \geq 0)$ has a modification M^U which is a martingale with values in the separable Hilbert space $\Phi'(U)$. In [13] we have showed that any such martingale has a modification with almost surely right continuous trajectories having left limits (in Φ'_β).

Suppose that X is a completely regular topological space and A is a set of Radon measures on X. We say that A satisfies Prokhoroff's condition if A is a bounded set in the variation-norm topology and if for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset X$ such that

$$\sup(|m|(X - K_\varepsilon); m \in A) < \varepsilon.$$

If A is a set of cylindrical probability measures on the nuclear space Φ'_β, then A satisfies Prokhoroff's condition if and only if the characteristic functions $\hat{m}$, $m \in A$, are equicontinuous at zero (c.f. [2], page 178).

II. Additive processes. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_t, t \geq 0)$ be a filtration of $\mathscr{F}$. A stochastic process X on $(\Omega, \mathscr{F}, P)$ with values in Φ' is called an additive process or a process with independent increments if

i) for any $t \geq 0$, X_t is $\mathscr{F}_t$-measurable,
ii) for any $0 \leq s < t$, $X_t - X_s$ is independent of $\mathscr{F}_s$
iii) the mapping $(t, \varphi) \to f_t(\varphi) = \mathrm{E}(\exp iX_t(\varphi))$ is continuous on $\mathbb{R}_+ \times \Phi$
iv) $X_0(\varphi) = 0$ a.s., for any $\varphi \in \Phi$.

Under these conditions the mapping

$$(s, t, \varphi) \mapsto E(\exp i(X_t(\varphi) - X_s(\varphi)))$$

is continuous on [0, T] × [0, T] × Φ for any $T > 0$. If A is a measurable subset of Φ', denote by $P_{st}(x, A)$, $x \in \Phi'$, the following quantity:

$$P_{st}(x, A) = P\{X_t - X_s + x \in A\}.$$

By (i) and the theorem of Minlos-Sazonov-Badrikian, $A \mapsto P_{st}(x, A)$ is a Radon measure on Φ'. Moreover, if $H \subset \Phi'$ is compact, then the set of measures

$$\{P_{st}(x, \cdot): s, t \in [0, T], s \leq t, x \in H\}$$

satisfies the condition of Prokhoroff, hence it is tight (cf. [2]). We have also:

PROPOSITION II.1. *For any compact set H in Φ' and $T > 0$, the mapping defined, from* $[0, T] \times [0, T] \times H$ *into the space of the* (*Radon*) *probability measures on* Φ', *by*

$$(s, t, x) \mapsto P_{st}(x, \cdot)$$

is weakly continuous.

PROOF. We know already that the set

$$\{P_{st}(x, \cdot): (s, t) \in [0, T]^2, x \in H\}$$

is tight. Let $\varepsilon > 0$; then by Prokhoroff's theorem, there exists a compact set $K_\varepsilon \subset \Phi'$ such that

$$P_{st}(x, K_\varepsilon) \geq 1 - \varepsilon$$

uniformly in s, t and $x \in H$. Let f be any bounded, continuous function on Φ'_β. Since Φ'_β is nuclear, it has the approximation property (cf. [10]), hence there exists a finite dimensional linear operator I_α on Φ' such that

$$\sup_{y \in K_\varepsilon} | f(y) - f(I_\alpha y) | < \varepsilon/4.$$

Denoting by $P_{st}(x, f)$ the integral of f with respect to $P_{st}(x, dy)$, we have

$$\begin{aligned}
| P_{st}(x, f) - P_{ru}(z, f) | &\leq \left| \int_{K_\varepsilon} f(y) P_{st}(x, dy) - \int_{K_\varepsilon} f(y) P_{ru}(z, dy) \right| \\
&\quad + \left| \int_{K_\varepsilon^c} f(y) P_{st}(x, dy) - \int_{K_\varepsilon^c} f(y) P_{ru}(z, dy) \right| \\
&\leq \left| \int_{K_\varepsilon} (f(y) - f(I_\alpha y)) P_{st}(x, dy) \right| \\
&\quad + \left| \int_{K_\varepsilon} f(I_\alpha y) P_{st}(x, dy) - \int_{K_\varepsilon} f(I_\alpha y) P_{ru}(z, dy) \right|
\end{aligned}$$

$$+ \left| \int_{K_\varepsilon} (f(I_\alpha y) - f(y)) P_{ru}(z, dy) \right|$$

$$+ \left| \int_{K_\varepsilon^c} f(y) P_{st}(x, dy) - \int_{K_\varepsilon^c} f(y) P_{ru}(z, dy) \right|$$

$$= A + B + C + D.$$

By the choice of I_α, we have $A \le \varepsilon/4$; B also can be majorated by $\varepsilon/4$ when (x, s, t) approaches to (y, r, u) as we know from the finite dimensional case (cf. [3], [7]). From the approximation property, C is smaller than $\varepsilon/4$ and by the tightness D is smaller than $\varepsilon \, \| f \|$. □

The following result is a corollary of Proposition II.1 and Theorem 8.3 of [11]:

COROLLARY II.1. *Denote by $\tilde{\Omega}$ the following set*

$$\tilde{\Omega} = \{\omega \in \Omega : X(\omega) \text{ is right continuous with left limits}\}.$$

Then the outer probability measure of $\tilde{\Omega}$ is equal to one.

It is well known that P has a unique extension P' to $\mathscr{F} \vee \{\tilde{\Omega}^c\}$ (i.e., the σ-algebra generated by $\mathscr{F}$ and $\{\tilde{\Omega}^c\}$) in such a way that $\tilde{\Omega}^c$ is a P'-negligible set (cf. [5]). Consequently we may and shall suppose that X is almost surely right continuous with left limits. Let us also note that such a procedure is unnecessary when Φ'_β is metrizable, since, in this case $\tilde{\Omega}^c$ is a P-negligible set. If $(\mathscr{F}_t;\, t \ge 0)$ is the canonical filtration of X, then, as in the finite dimensional case, it is right continuous. Hence, in the following we shall suppose that $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ satisfies the "usual" conditions (cf. [5]).

III. Lévy-Khintchin Formula. In this section we shall study in detail the trajectories of the additive process X and the representation of the Fourier transform of the law of X_t, $t \in \mathbb{R}_+$.

The following result is a simple consequence of the regularity of the trajectories of X:

LEMMA III.1. *Let A be any measurable subset of Φ' such that $0 \notin \bar{A}$. Then we have*

$$N_t(\omega, A) = \textstyle\sum_{s \le t} 1_A(\Delta X_s(\omega)) < +\infty \quad \text{a.s.},$$

for any $t \ge 0$, where $\Delta X_s = X_s - X_{s-}$.

Denote by μ_t the set function defined by

$$\mu_t(A) = E(N_t(\cdot, A)).$$

Obviously, for any countable, increasing, measurable subsets (A_n), one has

$$\lim_n \mu_t(A_n) = \mu_t(\lim_n A_n).$$

Since, for any $U \in \mathscr{U}_h(\Phi'_\beta)$, $k(U)(X) = X^U$ is an additive process, one has the following Lévy's decomposition (cf. [3], [7]):

$$X_t^U = a_t^U + W_t^U + \int_{U_0^c} x^U N_t^U(dx^U) + \int_{U_0} x^U (N_t^U - \mu_t^U)(dx^U),$$

where U_0 denotes the closed unit ball of $\Phi'(U)$, W^U is a continuous Gaussian martingale with values in $\Phi'(U)$, a^U is a continuous, deterministic process in $\Phi'(U)$, the first integral converges almost surely and the second integral converges in $L^2(\Omega, \mathscr{F}, P; \Phi'(U))$ (i.e., $\Phi'(U)$-valued, norm-square integrable random variables). Furthermore, the second integral is a discontinuous martingale and

$$N_t^U(\omega, B) = \sum_{s \le t} 1_B(\Delta X_s^U(\omega)), \quad \mu_t^U(B) = E(N_t^U(\cdot, B)).$$

Obviously we have $\mu_t^U = k(U)(\mu_t)$ and $N_t^U = k(U)(N_t)$ (a.s.).

PROPOSITION III.1. *There exists an absolutely convex, compact set K in Φ' such that $\mu_t(K^c) < +\infty$, for any $t \in [0, T]$.*

PROOF. Let us denote by θ_T the law of X_T. Since, for any $U \in \mathscr{U}_h(\Phi'_\beta)$, $k(U)(\theta_T) = \theta_T^U$ is the law of X_T^U, θ_T^U is an infinitely divisible probability measure on $\Phi'(U)$, hence θ_T is infinitely divisible (cf. [6], Satz 1.9), consequently there exists a uniquely defined Lévy measure G_T on Φ'_β, a compact, absolutely convex set K in Φ' with $G_T(K^c) < +\infty$, a positive, continuous quadratic form $\mathfrak{G}$ on Φ and $a_K \in \Phi'$ (depending on K) such that the Fourier transform $\hat{\theta}_T$ of θ_T can be expressed as

$$\theta_T(\varphi) = \exp\left(i a_K(\varphi) - \frac{1}{2}\mathfrak{G}(\varphi, \varphi) + \int (\exp ix(\varphi) - 1 - ix(\varphi)1_K(x))G_T(dx)\right).$$

By the uniqueness of G_T we obtain that

$$k(U)(G_T) = \mu_T^U,$$

i.e., G_T and μ_T are equal on the cylindrical sets. Since Φ is separable, there exists a dense set $(\varphi_i; i \in \mathbb{N})$ in K^0 (the polar of K) and we have

$$K = \cap_{n=1} \{x \in \Phi' : \sup_{i \le n} |x(\varphi_i)| \le 1\},$$

hence K^c can be written as an increasing union of cylindrical sets (C_n). Consequently we have

$$\mu_T(K^c) = \mu_T(\lim_n C_n) = \lim_n \mu_T(C_n)$$
$$= \lim_n G_T(C_n) = G_T(K^c) < +\infty.$$

Theorem III.1. *For any $T > 0$, $(X_t; t \in [0, T])$ can be represented as*

$$X_t = a_t + W_t + \int_{K^c} xN_t(dx) + \int_K x(N_t - \mu_t)(dx),$$

where the first integral is taken in $L^0(\Omega, \mathscr{F}, P)$(even almost surely), the second one converges in $\Phi'_\beta \tilde{\otimes} L^2(\Omega, \mathscr{F}, P)$(i.e., the projective tensor product topology), $W = (W_t; t \in [0, T])$ is a Gaussian martingale with continuous trajectories and $(a_t; t \in [0, T])$ is a continuous, deterministic process with values in Φ'_β, depending on the choice of the compact set K.

Proof. Let K be an absolutely convex, compact set such that $\mu_T(K^c) < \infty$, then $\mu_t(K^c) < +\infty$ for any $t \le T$ and the restriction of μ_t to the set K^c is a Radon measure on Φ' for any $t \le T$. Since

$$\textstyle\sum_{s \le T} 1_{K^c}(\Delta X_s)$$

is almost surely finite, the additive process Y defined by

$$Y_t = X_t - \textstyle\sum_{s \le t} \Delta X_s 1_{K^c}(\Delta X_s)$$

has bounded jumps for almost all $\omega \in \Omega$. Therefore, for any $\varphi \in \Phi$, $Y(\varphi)$ is an additive process with finite moments of all orders (cf. [7]). Define $a_t(\varphi)$ as

$$a_t(\varphi) = E(Y_t(\varphi)).$$

By the closed graph theorem $\varphi \mapsto \{a_t(\varphi); t \le T\}$ defines a continuous mapping from $[0, T]$ into Φ'_β which we denote again by $a = (a_t; t \in [0, T])$. Let L be the martingale (cf.[13])

$$L_t = Y_t - a_t.$$

For any $\varphi \in \Phi$, $L(\varphi)$ can be decomposed as

$$L_t(\varphi) = M^c(\varphi)_t + \tilde{M}^d(\varphi)_t$$

where $M^c(\varphi)$ is a continuous Gaussian martingale and $\tilde{M}^d(\varphi)$ is a purely discontinuous martingale (in the sense of martingales) and this decomposition is unique (cf. [7], [9]). Consequently $\varphi \mapsto M^c(\varphi)$ and $\varphi \mapsto \tilde{M}^d(\varphi)$ define linear mappings from Φ into the space of the square integrable martingales. The closed graph theorem implies that these mappings are continuous, Φ being nuclear, they are also nuclear. Hence (cf. [13]) there exists an absolutely convex, compact set S in $\mathscr{K}_h(\Phi')$ such that L is concentrated in $\Phi'[S]$ as a martingale (cf. [13]), a continuous Gaussian martingale $\hat{W}$ and a purely discontinuous martingale $\hat{M}^d$, both with values in $\Phi'[S]$, such that, for any $\varphi \in \Phi$, one has

$$M^c(\varphi) = \langle i_S(\hat{W}), \varphi \rangle, \quad \hat{M}^d(\varphi) = \langle i_S(\hat{M}^d), \varphi \rangle$$

up to an evanescent process, where i_S denotes the injection $\Phi'[S] \hookrightarrow \Phi'$. Furthermore we can choose S such that $S \supset K$ and as in the finite dimensional case $\hat{M}^d$ can be expressed as

$$\hat{M}_t^d = \int_{\Phi'[S]\cap K} x(N_t - \mu_t)(dx),$$

where the integral converges in $L^2(\Omega, \mathscr{F}, P; \Phi'[S])$ and injecting it into Φ' by i_S we find:

$$L_t = M_t^d + W_t, \quad M_t^d = i_S(\hat{M}_t^d) = \int_K x(N_t - \mu_t)(dx)$$

$$W_t = i_S(\hat{W}_t). \quad \square$$

REMARK. Since $\hat{M}_t^d$ belongs to $L^2(\Omega, \mathscr{F}, P; \Phi'[S])$, we have

$$\int_K p_S(x)^2 \mu_t(dx) < +\infty,$$

where p_S is the gauge function of S. Since S is compact in Φ'_β, we have

$$\int_K p(x)^2 \mu_t(dx) < +\infty,$$

for any continuous seminorm on Φ'_β and $t \in [0, T]$ (cf. [10]).

We have also the following result:

COROLLARY III.1. *Suppose that we are given a functional f_t on Φ for $t \in [0, T]$, $T > 0$, of the following form*:

$$f_t(\varphi) = \exp\left(ia_t(\varphi) - \frac{1}{2}\mathfrak{G}_t(\varphi, \varphi) + \int (\exp ix(\varphi) - 1 - ix(\varphi))\mu_t(dx)\right)$$

such that

i) *$t \mapsto a_t$ is a continuous mapping from $[0, T]$ into Φ'_β.*
ii) *$t \mapsto \mathfrak{G}_t$ is a continuous mapping from $[0, T]$ into the space of the continuous bilinear mappings on Φ, equipped with the strong topology and, for any $t \in [0, T]$, $\mathfrak{G}_t$ is nonnegative-definite on Φ.*
iii) *For any $t \in [0, T]$, μ_t is an abstract, positive set function on the measurable subsets of Φ', continuously increasing with t and, there exists an absolutely convex compact set K in Φ' such that μ_t restricted to K^c is a Radon measure for any $t \in [0, T]$ with*

$$\int_K p(x)^2 \mu_t(dx) < +\infty,$$

for any continuous seminorm p on Φ'_β.

Then there exists an additive process $X = (X_t; t \in [0, T])$ with right continuous trajectories having left limits whose law is uniquely defined by the fact that

$$E(\exp(i\langle X_t, \varphi\rangle)) = f_t(\varphi), \quad \varphi \in \Phi.$$

PROOF. Define $P_{st}(x, dy)$, $x \in \Phi'$, as the unique probability measure on whose Fourier transformation is given by (c.f. [2])

$$\hat{P}_{st}(x, \varphi) = \int_{\Phi'} \exp iy(\varphi) P_{st}(x, dy) = \exp ix(\varphi) \cdot f_t(\varphi)/f_s(\varphi),$$

$\varphi \in \Phi$, $s \leq t$. It is obvious that $(P_{st}; s \leq t < +\infty)$ satisfies Kolmogorov's relation, hence it is a Markov semigroup. Then construct the corresponding Markov process on $(\Phi')^{\mathbb{R}_+}$. By the continuity of the mapping $(t, \varphi) \to f_t(\varphi)$, we see that the set of right continuous trajectories with left limits has full outer measure. The rest of the proof follows from Proposition III.1 and Theorem III.1. □

REMARK. To see the difference between the Banach and nuclear spaces let us mention that the validity of this corollary for a Banach space B under the same hypothesis implies that B is finite dimensional. In fact, in this case the original topology of B coincides with the Sazonov topology (i.e. the coarsest topology under which the Hilbert-Schmidt operators are continuous), hence B becomes a nuclear space, but any nuclear Banach space is finite dimensional (cf. [10]).

EXAMPLES AND APPLICATIONS.

1) Let $\Phi' = \mathscr{S}'(\mathbb{R}^d)$, i.e., the space of the tempered distributions and suppose that X is a homogeneous additive process with values in $\mathscr{S}'(\mathbb{R}^d)$. Denote by $\xi = (\xi_t; t \geq 0)$ the unique solution of the following stochastic differential equation (cf. [17]):

$$d\xi_t = \tfrac{1}{2}\Delta\, \xi_t\, dt + dX_t, \quad \xi_0 = \xi \in \mathscr{S}'(\mathbb{R}^d),$$

where Δ denotes Laplace operator on $\mathscr{S}'(\mathbb{R}^d)$. Note that the equation is well defined since X is a semimartingale in $\mathscr{S}'(\mathbb{R}^d)$ (it is homogeneous, hence $t \mapsto a_t$ is of finite variation) (cf. [14]). Using the uniqueness of the solutions, it is easy to show that (ξ_t) is a strong Markov process (cf. [18]) whose infinitesimal generator, say A, can be expressed on the set of twice differentiable cylindrical functions with bounded derivatives ($\mathscr{S}'(\mathbb{R}^d)$ being nuclear; weakly, Hadamard or Fréchet differentiability is the same thing) as

$$\begin{aligned} AF(\xi) &= \tfrac{1}{2}\mathfrak{G}(D^2F(\xi)) + \tfrac{1}{2}\langle DF(\xi), \Delta\xi + 2a\rangle \\ &\quad + \int_K (F(x+\xi) - F(\xi) - \langle DF(\xi), x\rangle)\mu(dx) \\ &\quad + \int_{K^c} (F(x+\xi) - F(\xi))\mu(dx) \end{aligned}$$

where $(a, \mathfrak{G}, \mu, K)$ are defined by X (i.e. $a = a_1$, $\mathfrak{G} = \mathfrak{G}_1$, $\mu = \mu_1$) as in Corollary III.1, and D^iF denotes the derivative of F of order $i = 1, 2$.

2) Let $\Phi'_\beta = \mathscr{D}'(\mathbb{R}^d)$ and suppose that b and g are C^∞-mappings on $\mathbb{R}^d$ with

values respectively in $\mathbb{R}^d$ and $\mathbb{R}^d \otimes \mathbb{R}^d$ having bounded derivatives. Then there exists a semimartingale $(n_t;\, t \geq 0)$ with values in $\mathscr{E}(\mathbb{R}^d) \otimes \mathbb{R}^d$ (i.e. the projective tensor product of the nuclear Fréchet space of C^∞-functions on $\mathbb{R}^d$ with $\mathbb{R}^d$) such that, for any $x \in \mathbb{R}^d$, $(n_t(x);\, t \geq 0)$ satisfies the following stochastic differential equation (cf. [17], [19]):

$$dz_t = b(z_t)\, dt + g(z_t)\, dB_t, \quad z_0 = x,$$

where $B = (B_t;\, t \geq 0)$ is the standard Wiener process with values in $\mathbb{R}^d$. Moreover, almost surely, for any $t \geq 0$, $x \mapsto n_t(x)$ is a diffeomorphism (of $\mathbb{R}^d$) whose (functional) inverse $(n_t^{-1};\, t \geq 0)$ is also a semimartingale with values in $\mathscr{E}(\mathbb{R}^d) \otimes \mathbb{R}^d$ (cf.[17]). Let us denote again by $\xi = (\xi_t)$ the image of the process constructed in the first example under the injection $\mathscr{S}'(\mathbb{R}^d) \hookrightarrow \mathscr{D}'(\mathbb{R}^d)$ and suppose that $X = (X_t;\, t \geq 0)$ and B are independent. For any $\varphi \in \mathscr{D}(\mathbb{R}^d)$, since X is a semimartingale and $(\varphi \circ n_s;\, s \geq 0)$ is a locally bounded, previsible process in $\mathscr{D}(\mathbb{R}^d)$, the following stochastic integral is well defined (cf. [13], [17]):

$$\int_0^t \langle dX_s, \varphi \circ n_s \rangle = Z_t(\varphi),$$

and the linear mapping $\varphi \mapsto (Z_t(\varphi);\, t \geq 0)$ defines a semimartingale in $\mathscr{D}'(\mathbb{R}^d)$ (cf. [14], [20]). Define y_t as

$$y_t(\varphi) = \langle \xi_t, \varphi \circ n_t \rangle,$$

then, using the integration by parts formula for the nuclear space-valued semimartingales (cf.[13]) one can show that $y = (y_t;\, t \geq 0)$ is a semimartingale with values in $\mathscr{D}'(\mathbb{R}^d)$ satisfying the following stochastic partial differential equation:

$$\begin{aligned} dy_t &= \tfrac{1}{2}(\partial^2/\partial x_i^2((\partial n_t/\partial x_i)^2 \circ n_t^{-1} \cdot y_t) - \partial/\partial x_i((\partial^2 n_t/\partial x_i^2) \circ n_t^{-1} \cdot y_t))\, dt \\ &\quad + L^* y_t\, dt + dZ_t - \partial/\partial x_i(g_{ij} y_t)\, dB_t^j, \\ y_0 &= \xi_0, \end{aligned}$$

where L^* is the adjoint of the infinitesimal generator of $(n_t(x);\, t \geq 0)$. Let us also note that, if one calculates

$$\langle X_t, \varphi \circ n_t \rangle$$

using the integration by parts formula and Theorem III.1, then the distribution u_t, defined by

$$u_t(\varphi) = E(\langle X_t, \varphi \circ n_t \rangle)$$

satisfies the following partial differential equation:

$$\frac{\partial u_t}{\partial t} = L^* u_t + \int_{K^c} e^{tL^*}\left(x + \frac{a}{\mu(K^c)}\right) \mu\,(dx);$$

of course in order that this equation makes sense one has to impose some conditions of integrability on X.

3) Let us suppose again that $\Phi'_\beta = \mathscr{S}'(\mathbb{R}^d)$ and $X_t = W_t$ for $t \geq 0$, where $\mathfrak{G}_t(\varphi, \varphi) = t \| \varphi \|^2_{L^2(dx)}$. Then the following equation has a unique solution

$$dL_t = -(-\Delta + m^2)^{1/2} L_t \, dt + dW_t, \quad m \neq 0, \quad L_0 \in \mathscr{S}'(\mathbb{R}^d).$$

In fact the uniqueness follows from the fact that $-\Delta + m^2$ is invertible on $\mathscr{S}'(\mathbb{R}^d)$. Since the semigroup T_t associated to $-(-\Delta + \mathrm{m}^2)^{1/2}$ operates on $\mathscr{S}(\mathbb{R}^d)$, the solution is given explicitly by (cf.[17])

$$L_t = T_t L_0 + \int_0^t T_{t-s} dW_s$$

where the stochastic integral is defined on each test function (cf. [13]). Then the nuclearity of $\mathscr{S}(\mathbb{R}^d)$ implies the existence of an $\mathscr{S}'(\mathbb{R}^d)$-valued semimartingale having almost surely continuous trajectories. This process is called the Ornstein-Uhlenbeck process of the free quantum field and it is used to construct the free field (cf.[1]).

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