The Annals of Probability
1984, Vol. 12, No. 3, 647–702

Special Invited Paper

BOUNDARY VALUE PROBLEMS AND SHARP INEQUALITIES FOR MARTINGALE TRANSFORMS[1]

BY D. L. BURKHOLDER

University of Illinois

Let p^* be the maximum of p and q where $1 < p < \infty$ and $1/p + 1/q = 1$. If $d = (d_1, d_2, \cdots)$ is a martingale difference sequence in real $L^p(0, 1)$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots)$ is a sequence of numbers in $\{-1, 1\}$, and n is a positive integer, then

$$\| \sum_{k=1}^n \varepsilon_k d_k \|_p \le (p^* - 1) \| \sum_{k=1}^n d_k \|_p$$

and the constant $p^* - 1$ is best possible. Furthermore, strict inequality holds if and only if $p \neq 2$ and $\| \sum_{k=1}^n d_k \|_p > 0$. This improves an earlier inequality of the author by giving the best constant and conditions for equality. The inequality holds with the same constant if ε is replaced by a real-valued predictable sequence uniformly bounded in absolute value by 1, thus yielding a similar inequality for stochastic integrals. The underlying method rests on finding an upper or a lower solution to a novel boundary value problem, a problem with no solution (the upper is not equal to the lower solution) except in the special case $p = 2$. The inequality above, in combination with the work of Ando, Dor, Maurey, Odell, Olevskiĭ, Pełczyński, and Rosenthal, implies that the unconditional constant of a monotone basis of $L^p(0, 1)$ is $p^* - 1$. The paper also contains a number of other sharp inequalities for martingale transforms and stochastic integrals. Along with other applications, these provide answers to some questions that arise naturally in the study of the optimal control of martingales.

CONTENTS

Received October 1983.

[1] This paper is an expanded version of the hour talk "Optimal control of martingales" given at the 1982 Annual Meeting of the Institute of Mathematical Statistics, Cincinnati, Ohio, 17 August 1982. Our research was supported by NSF Grant MCS-8203602.

AMS 1980 *subject classifications.* Primary 60G42, 60H05; secondary 60G46, 46E30.

Key words and phrases. Martingale, martingale transform, stochastic integral, optimal control, unconditional constant, Haar system, monotone basis, contractive projection, biconvex function, boundary value problem, nonlinear partial differential equation.

1. Introduction. It must have been well known to Alexander Calder that it is possible to design a mobile that can be hung initially in a small room but which, if it is to move freely through all of its possible configurations, will have to be hung anew in an exceedingly large room. There is a close mathematical analogue. To each possible configuration of a mobile made with string, rods, and weights, there corresponds a martingale with a similar arrangement of successive centers of gravity and this martingale is a transform of the martingale corresponding to the initial configuration. It is easy to see, either by looking first at mobiles or directly at martingales, that there do exist small martingales with large transforms.

This paper contains more precise information about the relative sizes of martingales and their transforms. This information throws light on a number of problems in the optimal control of martingales. It also leads to the unconditional constant of any monotone basis of $L^p(0, 1)$ and to some sharp inequalities for stochastic integrals. Some of the underlying methods are of independent interest.

We recall some definitions, notation, and other background. Suppose that $f = (f_1, f_2, \cdots)$ is a sequence of real integrable functions on the Lebesgue unit interval $[0, 1)$ and $(d_1, d_2, \cdots)$ is its difference sequence: $f_n = \sum_{k=1}^n d_k$, $n \geq 1$. Then f is a *martingale* if d_{n+1} is orthogonal to $\varphi(d_1, \cdots, d_n)$ for all real bounded continuous functions φ on $\mathbb{R}^n$ and all $n \geq 1$. This orthogonality (equivalent, of course, to $E(f_{n+1} \mid f_1, \cdots, f_n) = f_n$ a.e., $n \geq 1$) is a convenient tool here and much of the paper requires little else. The one other property of martingales that we shall use frequently is a simple consequence of orthogonality: If Φ is a nonnegative convex function on $[0, \infty)$ with $\Phi(0) = 0$, then

$$E\Phi(|f_n|) \leq E\Phi(|f_{n+1}|), \quad n \geq 1. \tag{1.0}$$

(For some nondecreasing odd function ψ on $\mathbb{R}$, $\Phi(t) = \int_0^t \psi(s)\, ds$, $t \geq 0$, so that $\Phi(|f_{n+1}|) \geq \Phi(|f_n|) + \psi(f_n)d_{n+1}$. If ψ is bounded and continuous, (1.0) follows from the orthogonality of $\psi(f_n)$ and d_{n+1}. The general case follows by monotone convergence. For the usual proof of (1.0) and related background, see [10].)

If B is a Banach space, the definition of a B-valued martingale is similar: The integral of the product of the B-valued (strongly) integrable function d_{n+1} and the scalar-valued function $\varphi(d_1, \cdots, d_n)$, where φ is bounded and continuous on B^n, is equal to 0, the origin of B. Here, except in a few remarks, we shall consider only martingales with values in some Euclidean space, usually $\mathbb{R}$, but even to study real-valued martingales and their transforms, we shall find it convenient to introduce related martingales with values in $\mathbb{R}^2$ or $\mathbb{R}^3$.

If $v = (v_1, v_2, \cdots)$ is a sequence of scalar-valued measurable functions on the Lebesgue unit interval, then v is *predictable* (relative to f) if d_{n+1} is orthogonal to every real bounded continuous function of $v_1, \cdots, v_{n+1}, d_1, \cdots, d_n$ for all $n \geq 1$. The sequence $g = (g_1, g_2, \cdots)$ defined by $g_n = \sum_{k=1}^{n} v_k d_k$ is *the transform of* f *by* v. If, for example, the v_n are bounded, then g is also a martingale. In most of our work here, we shall be able to reduce consideration to the case $v \equiv \varepsilon$ where $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots)$ and ε_k is a number in $\{-1, 1\}$. The *maximal function* of g is defined by $g^*(s) = \sup_n |g_n(s)|$, $s \in [0, 1)$, and the *p-norm* of f by $\| f \|_p = \sup_n \| f_n \|_p$. The letter P will denote Lebesgue measure on $[0, 1)$. Any other nonatomic probability space would do and nonatomicity is not needed if the space is allowed to vary as well as f and g.

The following theorem is one of the key results of the paper and is proved in Section 5.

THEOREM 1.1. *Let* $1 < p < \infty$ *and* p^* *be the maximum of* p *and* q *where* $1/p + 1/q = 1$. *If* g *is the transform of a real martingale* f *by a real predictable sequence* v *uniformly bounded in absolute value by* 1, *then*

$$\| g \|_p \leq (p^* - 1) \| f \|_p \tag{1.1}$$

and the constant $p^* - 1$ *is best possible. If* $0 < \| f \|_p < \infty$, *then equality holds if and only if* $p = 2$ *and* $\sum_{k=1}^{\infty} v_k^2 d_k^2 = \sum_{k=1}^{\infty} d_k^2$ *almost everywhere.*

This sharpens one of the inequalities of [3] by giving the best constant and conditions for equality.

The following closely related theorem is proved in Section 4. The proof will illustrate in a simpler setting some of the methods to be used in the proof of Theorem 1.1.

THEOREM 1.2. *Let* $2 < p < \infty$. *If* f *is a real martingale with* $\| f \|_\infty \leq 1$ *and* g *is the transform of* f *by a real predictable sequence* v *uniformly bounded in absolute value by* 1, *then*

$$\| g \|_p^p < \Gamma(p + 1)/2 \tag{1.2}$$

and the constant $\Gamma(p + 1)/2$ *is best possible.*

The following theorem, proved in Section 8, is also closely related to Theorem 1.1.

THEOREM 1.3. *Let* $1 \leq p \leq 2$. *If* g *is the transform of a real martingale* f *by a real predictable sequence* v *uniformly bounded in absolute value by* 1, *then*

$$\sup_{\lambda>0} \lambda^p P(g^* \geq \lambda) \leq 2 \| f \|_p^p / \Gamma(p + 1) \tag{1.3}$$

and the constant $2/\Gamma(p + 1)$ *is best possible. Strict inequality holds if* $0 < \| f \|_p < \infty$ *and* $1 < p < 2$ *but equality can hold if* $p = 1$ *or* 2.

The easy case $p = 2$, in which the best constant is 1, and the less easy case $p = 1$, in which the best constant is 2, are already known. For the case $p = 1$, see [3] and [4]. For new light on both cases, see the first part of Section 8.

In Section 2, the proofs of Theorems 1.1 and 1.2 are reduced to the special case of f starting at 0 with $v \equiv \varepsilon$ where ε is a numerical sequence in $\{-1, 1\}$. In fact, ε can always be taken to be the sequence $(1, -1, 1, -1, \cdots)$. Later it will be clear that the constants appearing in all three of the theorems above are already best possible in this special case. Zigzag martingales, to be used to study martingale transforms in the case $v \equiv \varepsilon$, are introduced in Section 3. Theorem 1.2 is proved in Section 4 with their use and the use of a biconcave function defined on the closure of

$$D = \{(x, y) \in \mathbb{R}^2 \colon |x - y| < 2\}.$$

The proof is not difficult once the right function is found. It is the least biconcave function u on the closure of D such that if $|x - y| = 2$, then

$$u(x, y) \geq \left|\frac{x + y}{2}\right|^p.$$

Theorem 1.1 is proved in Section 5 with the use of a similar method. Martingales with values in $\mathbb{R}^3$ are used, however, and the biconcave function u is replaced by a function defined on a subset of $\mathbb{R}^3$. It satisfies a more subtle concavity condition but is again the upper solution to a boundary value problem. It is found by solving a system of five second order nonlinear partial differential equations and inequalities on several different subdomains and by checking that the several solutions fit together in the right way. Once this is done, Theorem 1.1 follows fairly quickly.

Theorem 1.2 is extended in Section 6 by replacing the power function of that theorem by a more general convex function Φ with a strictly convex first derivative. Theorem 6.2 identifies the upper and lower solutions of the related boundary value problem.

Suppose that α and β are real numbers and f is a real martingale with $f_1 \equiv \alpha$. What condition on f assures the existence of a predictable sequence $v = (1, v_2, v_3, \cdots)$ uniformly bounded in absolute value by 1 such that the transform g of f by v satisfies

$$P(\sup_n g_n \geq \beta) = 1? \tag{1.4}$$

That is, when can the martingale f be controlled to satisfy (1.4)? Theorem 7.3 gives a necessary condition:

$$\|f\|_1 \geq (\beta - \alpha) \vee |\alpha|, \tag{1.5}$$

the maximum of $\beta - \alpha$ and $|\alpha|$. This is best possible; in fact, there is always an f satisfying (1.5) with equality so that f can be controlled by $v \equiv (1, -1, 1, -1, \cdots)$ to satisfy (1.4).

The first part of Section 8 is devoted to a genuinely elementary proof of the inequality (1.3) in the case $p = 1$. The proof of Theorem 1.3, and somewhat more, is completed in the second part. Section 9 has additional information about the control of martingales and contains the final part of the proof that the functions U_p and L_p, defined there, are respectively the upper and lower solutions of the boundary value problem in $\mathbb{R}^3$ mentioned above.

Sections 10 to 17 contain diverse variations, extensions, and applications of some of the methods and results described above. For example, it has been known for about fifty years that the unconditional constant of the Haar system in $L^p(0, 1)$ is finite if $1 < p < \infty$ (see Paley [20] and Marcinkiewicz [16]). About ten years ago, Pelczyński and Rosenthal [21], and Dor and Odell [12] proved, more generally, that the unconditional constant of any monotone basis of $L^p(0, 1)$ is finite. With the help of Theorem 1.1, these results can be strengthened (see Section 15): The Haar system and, indeed, any monotone basis of $L^p(0, 1)$ has the unconditional constant $p^* - 1$. Here is a related result, an extension of Theorem 1.1 to an arbitrary positive measure space in the special case $v \equiv a$ where a is a sequence in $[-1, 1]$: If $P_0 = 0, P_1, P_2, \cdots$ is a nondecreasing sequence of contractive projections in L^p of an arbitrary positive measure space, then the series

$$\textstyle\sum_{k=1}^{\infty} a_k(P_k - P_{k-1})$$

converges in the strong operator topology to an operator having a norm no greater than $p^* - 1$ (see Theorem 15.3).

Section 16 contains some of the sharp inequalities for stochastic integrals that follow from Theorems 1.1, 1.2, and 1.3, and the other inequalities of the first nine sections of the paper.

Theorem 1.1 is analogous to the classical inequality of M. Riesz [23]. In Section 17, the best constants in these two inequalities are compared, not only in the real case but also in a broader setting.

2. Some reductions. It is enough to prove Theorem 1.1, Theorem 1.2, and Theorem 1.3 for martingales $f = (f_1, f_2, \cdots)$ starting at the origin: $f_1 \equiv 0$. For if f is a martingale and g is its transform by a predictable sequence v, let $F_1 \equiv 0$, $V_1 \equiv 1$,

$$\begin{aligned} F_{n+1}(s) &= f_n(2s) && \text{if} \quad s \in [0, \tfrac{1}{2}), \\ &= -f_n(2s - 1) && \text{if} \quad s \in [\tfrac{1}{2}, 1), \end{aligned}$$

and

$$\begin{aligned} V_{n+1}(s) &= v_n(2s) && \text{if} \quad s \in [0, \tfrac{1}{2}), \\ &= v_n(2s - 1) && \text{if} \quad s \in [\tfrac{1}{2}, 1). \end{aligned}$$

Then $F = (F_1, F_2, \cdots)$ is a martingale starting at the origin and $V = (V_1, V_2, \cdots)$ is predictable relative to F. Moreover, if G is the transform of F by V, then $\| F \|_p = \| f \|_p$, $\| G \|_p = \| g \|_p$, and

$$P(G^* \geq \lambda) = P(g^* \geq \lambda).$$

In addition, it will be enough to prove Theorems 1.1 and 1.2 for the special case in which v is a sequence of numbers in $\{-1, 1\}$. This will follow from the lemma below which expresses the nth term of g as the sum of a pointwise convergent series of the $2n$th terms of transforms by $(1, -1, 1, -1, \cdots)$.

LEMMA 2.1. *Let g be the transform of a martingale f by a real predictable sequence v uniformly bounded in absolute value by* 1 *and suppose that either $v_1 \equiv 1$ or that d_1 is orthogonal to every real bounded continuous function of v_1. Then there exist martingales $F^j = (F_1^j, F_2^j, \cdots)$ such that, for each positive integer n,*

$$f_n = F_{2n}^j, \quad j \geq 1, \tag{2.1}$$

$$g_n = \sum_{j=1}^{\infty} 2^{-j} G_{2n}^j, \tag{2.2}$$

where $G^j = (G_1^j, G_2^j, \cdots)$ is the transform of F^j by $(1, -1, 1, -1, \cdots)$.

PROOF. We begin with the case in which each function v_n, in addition to satisfying the above assumptions, has all of its values in $\{-1, 1\}$. Let $D_{2n-1} = (1 + v_n)d_n/2$ and $D_{2n} = (1 - v_n)d_n/2$. Then $D = (D_1, D_2, \cdots)$ is a martingale difference sequence. To see this, let φ be a real bounded continuous function on $\mathbb{R}^{2n-1}$ where $n > 1$, the other cases being even simpler. The orthogonality of D_{2n} and $\varphi(D_1, \cdots, D_{2n-1})$ follows from the orthogonality of d_n and

$$(1 - v_n)\varphi(D_1, \cdots, D_{2n-1}) = (1 - v_n)\varphi(D_1, \cdots, D_{2n-2}, 0). \tag{2.3}$$

To see that (2.3) holds, note that both sides vanish on the set $\{v_n = 1\}$ and D_{2n-1} vanishes on its complement.

Let F be the martingale corresponding to D and G its transform by $(1, -1, 1, -1, \cdots)$. It follows from the definition of D that $d_n = D_{2n-1} + D_{2n}$ and $v_n d_n = D_{2n-1} - D_{2n}$ so

$$f_n = F_{2n} = \sum_{k=1}^{2n} D_k, \tag{2.4}$$

$$g_n = G_{2n} = \sum_{k=1}^{2n} (-1)^{k-1} D_k. \tag{2.5}$$

Now consider the general case in which the functions v_n have values in the interval $[-1, 1]$. There exist measurable functions φ_j: $[-1, 1] \to \{-1, 1\}$ such that $t = \sum_{j=1}^{\infty} 2^{-j}\varphi_j(t)$ for $t \in [-1, 1]$. Let $v_n^j = \varphi_j(v_n)$. Since v is predictable relative to f, the same is true of $(v_1^j, v_2^j, \cdots)$. For each positive integer j, the simpler case discussed above now yields a martingale F^j, and its transform G^j by $(1, -1, 1 -1, \cdots)$, such that $f_n = F_{2n}^j$ and

$$\sum_{k=1}^{n} v_k^j d_k = G_{2n}^j.$$

Multiplying both sides by 2^{-j} and summing gives the desired decomposition and completes the proof of Lemma 2.1.

Once we prove that Theorem 1.1 holds for all pairs f and g such that f is a martingale starting at 0 and g is the transform of f by a sequence of numbers in $\{-1, 1\}$, we can conclude that Theorem 1.1 holds as stated. For if f and g are as in the statement of the theorem and f starts at 0, as we can assume by our earlier discussion, then f and g can be written as in Lemma 2.1 and

$$\| g \|_p \leq \sum_{j=1}^{\infty} 2^{-j} \| G^j \|_p \leq (p^* - 1) \sum_{j=1}^{\infty} 2^{-j} \| F^j \|_p$$
$$= (p^* - 1) \sum_{j=1}^{\infty} 2^{-j} \| f \|_p = (p^* - 1) \| f \|_p.$$

Here we have used the fact, which follows from (1.0) and (2.1), that $\| F^j_{2n-1} \|_p \leq \| F^j_{2n} \|_p = \| f_n \|_p$. Note that if f has a finite and positive p-norm and $p \neq 2$, then $\| G^j \|_p < (p^* - 1) \| f \|_p$ for every j and this implies that $\| g \|_p < (p^* - 1) \| f \|_p$. If $\| f \|_2$ is finite, then $(d_1, d_2, \cdots)$ is an orthogonal sequence so $\| f \|_2^2$ is the integral of $\sum_{k=1}^{\infty} d_k^2$, giving the last part of Theorem 1.1. Finally, the value of the constant of the inequality in Theorem 1.1 can be no smaller than the $p^* - 1$ of the special case.

Similarly, once we prove Theorem 1.2 in the special case of f a martingale starting at 0 with $\| f \|_\infty \leq 1$ and g the transform of f by a sequence of numbers in $\{-1, 1\}$, we can conclude that the theorem holds as stated.

REMARK 2.1. The above lemma and its proof carry over with no change to B-valued martingales, giving an alternative proof of Theorem 2.2 of [5]. The approach taken here has the advantage of preserving strict inequality in the transition from the special to the general case.

3. Zigzag martingales. There is another reduction or transformation of the kind of problem that we consider here that is convenient and provides additional intuition. Let $(x, y) \in \mathbb{R}^2$ and suppose that f is a real martingale starting at $(x + y)/2$ and g is the transform of f by a sequence $\varepsilon = (1, \varepsilon_2, \varepsilon_3, \cdots)$ of numbers in $\{-1, 1\}$. Let $Z_n = (X_n, Y_n)$ where $Z_1 \equiv (x, y)$ and, for $n > 1$,

$$X_n = x + \textstyle\sum_{k=2}^{n} (1 + \varepsilon_k) d_k, \tag{3.1}$$

$$Y_n = y + \textstyle\sum_{k=2}^{n} (1 - \varepsilon_k) d_k. \tag{3.2}$$

Then $Z = (Z_1, Z_2, \cdots)$ is a martingale with values in $\mathbb{R}^2$ such that Z starts at (x, y) and, for each positive integer n, either $X_{n+1} - X_n \equiv 0$ or $Y_{n+1} - Y_n \equiv 0$. We shall call such a martingale a *zigzag martingale*: If Z moves at all at the nth step ($n \geq 2$), it moves either horizontally or vertically, which way depending only on n.

Given a zigzag martingale Z starting at (x, y), a pair f and g as above can be constructed uniquely as follows (the ε is not necessarily unique but this is inessential): Let $\varepsilon_1 = 1$ and $d_1 \equiv (x + y)/2$. If $Y_2 - Y_1 \equiv 0$, let $\varepsilon_2 = 1$ and $d_2 = (X_2 - X_1)/2$. Otherwise, let $\varepsilon_2 = -1$ and $d_2 = (Y_2 - Y_1)/2$, and so on. Then

$$f_n = (X_n + Y_n)/2, \tag{3.3}$$

$$g_n - y = (X_n - Y_n)/2. \tag{3.4}$$

This one-to-one correspondence will help not only in the demonstration of a number of inequalities but also in the construction of examples showing that the inequalities are sharp.

4. A boundary value problem in two dimensions. The principal goal of this section is to illustrate some of the methods that will be used to prove Theorem 1.1. This we do by proving Theorem 1.2, a theorem with its own interest and connections.

For example, what conditions on a martingale f and a class of predictable sequences v imply that $\| g \|_\infty \le 1$ for some transform g of f by a member of the class? Related problems abound. For a basic class of predictable sequences, the following lemma contains a necessary condition on f.

LEMMA 4.1. *Let $2 < p < \infty$. If f is a real martingale such that $\| g \|_\infty \le 1$ for some transform g of f by a sequence of numbers in $\{-1, 1\}$, then*

$$\| f \|_p^p < \Gamma(p+1)/2 \tag{4.1}$$

and the constant $\Gamma(p+1)/2$ is best possible.

If g is the transform of f by a sequence in $\{-1, 1\}$, then f is the transform of g by the same sequence ($\varepsilon_k^2 = 1$). Therefore, the lemma yields Theorem 1.2 for the special case in which v is a numerical sequence in $\{-1, 1\}$. In turn, by the results of Section 2, this special case yields the full theorem.

PROOF OF LEMMA 4.1. Let $D = \{(x, y) \in \mathbb{R}^2 : |x - y| < 2\}$ and ∂D denote the boundary of D. If $(x, y) \in D \cup \partial D$, let $\mathbf{F}(x, y)$ be the family of all real martingales f such that $f_1 \equiv (x+y)/2$ and, for some sequence $(1, \varepsilon_2, \varepsilon_3, \cdots)$ in $\{-1, 1\}$, the transform of g by this sequence satisfies

$$\sup_n \| g_n - y \|_\infty \le 1.$$

This family is nonempty: If $f_n \equiv (x+y)/2$ for all n, then $f \in \mathbf{F}(x, y)$. Let

$$U(x, y) = \sup\{\| f \|_p^p : f \in \mathbf{F}(x, y)\}. \tag{4.2}$$

Keeping in mind that we may assume $f_1 \equiv 0$ as in Section 2, we see that the least upper bound of $\| f \|_p^p$ for f as in Lemma 4.1 is $U(0, 0)$.

The following characterization of U will help lead to an explicit formula: U is the least biconcave function u on $D \cup \partial D$ satisfying

$$u(x, y) \ge \left| \frac{x+y}{2} \right|^p \quad \text{if} \quad (x, y) \in \partial D. \tag{4.3}$$

Or, to say the same thing another way, $U(\cdot, y)$ is concave on $[y-2, y+2]$, $U(x, \cdot)$ is concave on $[x-2, x+2]$, and U is the least such function satisfying (4.3).

Lemma 4.1 does not depend explicitly on this characterization of U so we defer its proof to Section 6. Here it will serve as a guide to our intuition.

If u is the least biconcave function on $D \cup \partial D$ satisfying (4.3), then

$$u(x, y) = u(y, x) = u(-x, -y). \tag{4.4}$$

Otherwise, replace $u(x, y)$ by the minimum of $u(x, y)$, $u(y, x)$, $u(-x, -y)$, and $u(-y, -x)$ to obtain a strictly smaller biconcave function.

A study of the analogous problem on $\mathbb{Z}^2 \cap (D \cup \partial D)$ and on grids with a finer mesh suggests that the least biconcave function u on $D \cup \partial D$ satisfying (4.3) must be of the form

$$(4.5)\qquad \begin{aligned} 2u(x, y) &= (1 + xy)A(1) && \text{if } |x| \vee |y| \le 1, \\ &= (y - x + 2)A(x) + (x - y)B(x) && \text{if } x \ge 1,\ x - 2 \le y \le x, \end{aligned}$$

where $B(x) = (x - 1)^p$. The definition of u on $D \cup \partial D$ is completed by using the symmetry property (4.4). This gives equality in the boundary condition (4.3). If A were twice differentiable on $(1, \infty)$, then, for $x > 1$ and $x - 2 < y < x$,

$$(4.6)\qquad 2u_x(x, y) = (y - x + 2)A'(x) - A(x) + (x - y)B'(x) + B(x),$$

$$(4.7)\qquad 2u_{xx}(x, y) = (y - x + 2)A''(x) - 2A'(x) + (x - y)B''(x) + 2B'(x),$$

and the biconcavity of u would imply the differential inequality $u_{xx}(x, x-) \le 0$ or, equivalently, $A''(x) - A'(x) + B'(x) \le 0$. Since u is extremal, it would be reasonable to expect that $u_{xx}(x, x-) = 0$ so that

$$(4.8)\qquad A''(x) - A'(x) + B'(x) = 0.$$

One solution of this equation on $(1, \infty)$ is the function A defined by

$$(4.9)\qquad A(x) = e^x \int_x^\infty B(t)e^{-t}\,dt = e^{x-1}\int_{x-1}^\infty t^p e^{-t}\,dt.$$

In fact, differentiating A on $(1, \infty)$ gives

$$(4.10)\qquad A'(x) - A(x) + B(x) = 0.$$

With this choice of $A(x)$, the function u defined by (4.5) and (4.4) is biconcave on $D \cup \partial D$: Trivially, u is biconcave on $D_0 \cup \partial D_0$ where

$$(4.11)\qquad D_0 = \{(x, y): |x| \vee |y| < 1\}.$$

It is also biconcave on $D_1 \cup \partial D_1$ where

$$(4.12)\qquad D_1 = \{(x, y): x > 1, x - 2 < y < x\}$$

and this can be seen as follows. On D_1 the derivative u_{yy} vanishes and, by (4.7) and (4.8),

$$(4.13)\qquad \begin{aligned} 2u_{xx}(x, y) &= (y - x)[A''(x) - B''(x)] \\ &= (y - x)e^x \int_x^\infty [B''(t) - B''(x)]e^{-t}\,dt < 0 \end{aligned}$$

since $B''(t) = p(p - 1)(t - 1)^{p-2}$ is strictly increasing in t for $t > 1$ and $p > 2$. It remains to show that u is biconcave on the whole of $D \cup \partial D$. By the symmetry and continuity of u, it will be enough to show that $u_x(\cdot, y)$ exists and is nonincreasing on $(y - 2, y + 2)$. In fact, it will be enough to show that $u_x(1-, y) = u_x(1+, y)$ for $-1 < y \le 1$, and $u_x(y-, y) = u_x(y+, y)$ for $y > 1$. If $-1 < y \le 1$, then, by (4.6) and (4.10),

$$\begin{aligned} 2u_x(1+, y) &= (y + 1)A'(1+) - A(1) \\ &= (y + 1)[A(1) - B(1)] - A(1) \\ &= yA(1) = 2u_x(1-, y). \end{aligned}$$

If $y > 1$, then, by the symmetry of u,

$$\begin{aligned} 2u_x(y+, y) &= 2A'(y) - A(y) + B(y) = A(y) - B(y) \\ &= 2u_y(y, y-) = 2u_x(y-, y). \end{aligned}$$

Thus u is biconcave on $D \cup \partial D$. Similarly, u_x and u_y not only exist on D but are also continuous there and have continuous extensions to $D \cup \partial D$.

Because u is biconcave on $D \cup \partial D$ and satisfies the boundary condition (4.3), u must also satisfy

$$\left|\frac{x + y}{2}\right|^p \le u(x, y), \quad (x, y) \in D \cup \partial D, \tag{4.14}$$

with strict inequality holding for $(x, y) \in D$: For fixed y, the difference between the left-hand side and the right-hand side is strictly convex in x for x in $[y - 2, y + 2]$.

The next step is to show that, for $f \in \mathbf{F}(0, 0)$,

$$\| f \|_p^p < u(0, 0). \tag{4.15}$$

Let $f \in \mathbf{F}(0, 0)$ and suppose that g is the transform of f by a sequence $\varepsilon = (1, \varepsilon_2, \varepsilon_3, \cdots)$ of numbers in $\{-1, 1\}$ such that $\| g \|_\infty \le 1$. We may assume without loss of generality that g is uniformly bounded in absolute value by 1. Let Z be the zigzag martingale determined by f and g as in Section 3. Then $| X_n - Y_n | = 2 | g_n | \le 2$ everywhere so Z_n has all of its values in $D \cup \partial D$. Therefore, by (3.3) and (4.14),

$$\| f_n \|_p^p = \left\| \frac{X_n + Y_n}{2} \right\|_p^p \le Eu(Z_n). \tag{4.16}$$

Furthermore,

$$Eu(Z_{n+1}) \le Eu(Z_n) \tag{4.17}$$

and strict inequality holds, for example, if

$$P(Z_n \in D_0 \cup \partial D_0, Z_{n+1} \notin D_0 \cup \partial D_0) > 0. \tag{4.18}$$

To prove (4.17), we shall use an inequality that follows at once from the biconcavity and smoothness of u: If both (x, y) and $(x + h, y + k)$ belong to the closure of D and either $h = 0$ or $k = 0$, then

$$u(x + h, y + k) \le u(x, y) + u_x(x, y)h + u_y(x, y)k. \tag{4.19}$$

Here u_x and u_y denote the continuous extensions to $D \cup \partial D$ of the respective first order derivatives on D. By (4.19) and the zigzag property of Z,

$$u(Z_{n+1}) \le u(Z_n) + u_x(Z_n)(X_{n+1} - X_n) + u_y(Z_n)(Y_{n+1} - Y_n). \tag{4.20}$$

Note that $X_{n+1} - X_n = (1 + \varepsilon_{n+1})d_{n+1}$ and $Y_{n+1} - Y_n = (1 - \varepsilon_{n+1})d_{n+1}$. The uniform boundedness of g implies the uniform boundedness of d, hence the boundedness of Z_n. Accordingly, there is a bounded continuous function φ: $\mathbb{R}^n \to \mathbb{R}$ such that $u_x(Z_n) = \varphi(d_1, \cdots, d_n)$. Since d is a martingale difference sequence, $X_{n+1} - X_n$ is orthogonal to $u_x(Z_n)$ and, similarly, $Y_{n+1} - Y_n$ is orthogonal to $u_y(Z_n)$. Therefore, integrating both sides of (4.20) gives (4.17). Moreover, if (4.18) holds, then, by (4.13), strict inequality holds in (4.20) on a set of positive measure. Both sides of (4.20) are bounded, hence integrable. Therefore, (4.18) implies that strict inequality holds in (4.17).

We can now complete the proof of (4.15) as follows. If (4.18) does not hold for any positive integer n, then

$$P(Z_n \in D_0 \cup \partial D_0, n \ge 1) = 1 \tag{4.21}$$

and

$$\| f_n \|_p^p \le 1 = ½[u(2, 0) + u(-2, 0)] < u(0, 0) \tag{4.22}$$

for all positive integers n. Otherwise (4.22) holds for $n \le m$ where m is the least integer n satisfying (4.18), and, for $n > m$,

$$\| f_n \|_p^p \le Eu(Z_n) \le Eu(Z_{m+1}) < Eu(Z_m) \le Eu(Z_1) = u(0, 0). \tag{4.23}$$

Therefore, $\| f \|_p^p < u(0, 0)$ always holds and, since $u(0, 0) = \Gamma(p + 1)/2$, so does the desired inequality (4.1).

The last step of the proof of Lemma 4.1 is to construct an example showing that the constant $\Gamma(p + 1)/2$ in (4.1) is best possible. It is clear from (4.22) and (4.23) that, if $\| f \|_p^p$ is to be near $u(0, 0)$, the martingale Z must move out of $D_0 \cup \partial D_0$ but in such a way that the two sides of (4.20) are not too far apart. Also, if the martingale is in D_1 after n steps, the next step should be vertical: $u_{yy} = 0$ on D_1 but $u_{xx} < 0$ there. Considerations of this kind lead to a candidate for Z and hence, by Section 3, a candidate for f. Here $f \in \mathbf{F}(1, 0)$ but this determines a companion example in $\mathbf{F}(0, 0)$ by the first paragraph of Section 2. Let $0 < \delta < 2$, $\gamma = 1/(2 + \delta)$, and $\beta = (2 - \delta)/(2 + \delta)$. Using the same notation for an interval $[a, b)$ and its indicator function, we set

$$d_1 = ½[0, 1), \quad d_2 = ½(1 + \delta)[0, \gamma) - ½[\gamma, 1),$$

$$d_3 = \delta[0, \beta\gamma) - (1 - \delta/2)[\beta\gamma, \gamma),$$

$$d_4 = \delta[0, \beta^2\gamma) - (1 - \delta/2)[\beta^2\gamma, \beta\gamma),$$

and so forth. This is a martingale difference sequence: d_{n+1} is orthogonal to constant functions and is supported by an interval on which $\varphi(d_1, \cdots, d_n)$ is constant. Consider the transform g of the associated martingale f by $(1, -1, 1, -1, \cdots)$. It is easy to see that $\| g \|_\infty = 1$. Hence $f \in \mathbf{F}(1, 0)$. Also, for $n \ge 1$,

$$f_{n+2} = [1 + (2n + 1)\delta/2][0, \beta^n\gamma) + \textstyle\sum_{k=1}^n k\delta[\beta^k\gamma, \beta^{k-1}\gamma), \tag{4.24}$$

which implies that

$$\| f \|_p^p \geq \gamma(1-\beta) \textstyle\sum_{k=1}^{\infty} (k\delta)^p \beta^{k-1}.$$

Set $\alpha = (-\log \beta)/\delta$. Then

$$\begin{aligned} \| f \|_p^p &\geq \gamma(1-\beta) \textstyle\sum_{k=1}^{\infty} (k\delta)^p e^{-\alpha(k-1)\delta} \\ &\geq \gamma(1-\beta) \textstyle\sum_{k=1}^{\infty} \displaystyle\int_{(k-1)\delta}^{k\delta} \frac{t^p e^{-\alpha t}\, dt}{\delta} = 2 \int_0^{\infty} \frac{t^p e^{-\alpha t}\, dt}{(2+\delta)^2}. \end{aligned}$$

Using $1/\beta = 1 + \delta/(1-\delta/2)$, we see that $\alpha \to 1$ as $\delta \to 0$ and, by Fatou's lemma, that

$$\liminf_{\delta \to 0} \| f \|_p^p \geq \int_0^{\infty} \frac{t^p e^{-t}\, dt}{2} = \frac{\Gamma(p+1)}{2}. \tag{4.25}$$

(By (4.1), the limit exists and equality holds.) This shows that the constant $\Gamma(p+1)/2$ is best possible and completes the proof of Lemma 4.1, hence the proof of Theorem 1.2.

5. A boundary value problem in three dimensions.

We shall now prove Theorem 1.1. The case $p = 2$ has already been treated at the end of Section 2.

LEMMA 5.1. *Let $2 < p < \infty$. If f is a real martingale such that $\| g \|_p \leq 1$ for some transform g of f by a sequence of numbers in $\{-1, 1\}$, then*

$$\| f \|_p < p - 1 \tag{5.1}$$

and the constant $p - 1$ is best possible.

This implies Theorem 1.1 in the case $p > 2$: See the paragraph immediately following Lemma 4.1.

PROOF OF LEMMA 5.1. Consider the domain

$$\Omega = \left\{ (x, y, t) \in \mathbb{R}^3 \colon \left| \frac{x-y}{2} \right|^p < t \right\} \tag{5.2}$$

and note that the section $\{(x, t) \colon (x, y, t) \in \Omega\}$ of Ω determined by y is convex as is the section of Ω determined by x. If $(x, y, t) \in \Omega \cup \partial\Omega$, let $\mathbf{F}(x, y, t)$ denote the family of all real martingales f such that $f_1 \equiv (x+y)/2$ and, for some sequence $(1, \varepsilon_2, \varepsilon_3, \cdots)$ in $\{-1, 1\}$, the transform g of f by this sequence satisfies

$$\sup_n \| g_n - y \|_p^p \leq t. \tag{5.3}$$

Just as in Section 4, this family is nonempty. Let U denote here the function on $\Omega \cup \partial\Omega$ defined by

$$U(x, y, t) = \sup\{ \| f \|_p^p \colon f \in \mathbf{F}(x, y, t)\}.$$

By the first paragraph of Section 2, the least upper bound of $\| f \|_p^p$ for f as in Lemma 5.1 is $U(0, 0, 1)$.

As in Section 4, the function U is the upper solution of a nonclassical boundary value problem. Here U is the least function u on $\Omega \cup \partial\Omega$ such that

(5.4) the mapping $(x, t) \to u(x, y, t)$ is concave on the section of $\Omega \cup \partial\Omega$ determined by y,

(5.5) the mapping $(y, t) \to u(x, y, t)$ is concave on the section of $\Omega \cup \partial\Omega$ determined by x,

and

$$u(x, y, t) \geq \left|\frac{x+y}{2}\right|^p \quad \text{if} \quad (x, y, t) \in \partial\Omega. \tag{5.6}$$

The proof of this characterization is given in Section 9 or, for a slightly different approach, see the proof of Theorem 3.3 in [5]. Here, as in Section 4, this characterization of U is a guide to our intuition and is not used directly.

Let us consider the above boundary value problem. Suppose that there does exist a function $u\colon \Omega \cup \partial\Omega \to \mathbb{R}$ satisfying (5.4), (5.5), and (5.6), and that u is the least such function. Then u must also satisfy the symmetry property

$$u(x, y, t) = u(y, x, t) = u(-x, -y, t). \tag{5.7}$$

Otherwise, replace $u(x, y, t)$ by the minimum of $u(x, y, t)$, $u(y, x, t)$, $u(-x, -y, t)$ and $u(-y, -x, t)$ to obtain a strictly smaller function satisfying (5.4), (5.5), and (5.6). Also,

$$u(x, y, t) = \lambda^{-p}u(\lambda x, \lambda y, \lambda^p t), \quad \lambda > 0. \tag{5.8}$$

For if $u_\lambda(x, y, t)$ denotes the right-hand side, the function u_λ satisfies (5.4), (5.5), and (5.6), so $u \leq u_\lambda$ by the minimality of u. But this inequality implies the reverse inequality $u_{1/\lambda} \leq u$. Both inequalities hold for all $\lambda > 0$. Therefore, the homogeneity property (5.8) holds and, for $(x, y, t) \in \Omega \cup \partial\Omega$ with $t > 0$,

$$u(x, y, t) = tF(xt^{-1/p}, yt^{-1/p}) \tag{5.9}$$

where $F(x, y) = u(x, y, 1)$. Note that F is defined on $D \cup \partial D$ where

$$D = \{(x, y) \in \mathbb{R}^2\colon |x - y| < 2\}.$$

We shall be able to obtain a formula for F, hence a formula for u.

If u is twice continuously differentiable on a neighborhood of some point $(x_0, y_0, 1) \in \Omega$, then (5.4) and (5.5) imply that, on the same neighborhood, $u_{xx} \leq 0$, $u_{yy} \leq 0$, $u_{tt} \leq 0$, $u_{xx}u_{tt} - u_{xt}^2 \geq 0$, and $u_{yy}u_{tt} - u_{yt}^2 \geq 0$. These lead, by (5.9), to the following system of differential inequalities for F on a neighborhood of $(x_0, y_0) \in D$:

$$F_{xx} \leq 0, \tag{5.10}$$

$$F_{yy} \leq 0, \tag{5.11}$$

$$x^2F_{xx} + 2xyF_{xy} + y^2F_{yy} - (p-1)[xF_x + yF_y] \leq 0, \tag{5.12}$$

$$(p-1)[xF_x - yF_y]F_{xx} - [(p-1)F_x - yF_{xy}]^2 + y^2F_{xx}F_{yy} \geq 0, \tag{5.13}$$

(5.14) $$(p-1)[yF_y - xF_x]F_{yy} - [(p-1)F_y - xF_{xy}]^2 + x^2F_{xx}F_{yy} \geq 0.$$

If $T_0(F)$, $T_1(F)$, and $T_2(F)$ denote the functions on the left-hand side of (5.12), (5.13), and (5.14), respectively, then

(5.15) $$u_{xx}(x, y, t) = t^{1-2/p}F_{xx}(xt^{-1/p}, yt^{-1/p}),$$

(5.16) $$u_{yy}(x, y, t) = t^{1-2/p}F_{yy}(xt^{-1/p}, yt^{-1/p}),$$

(5.17) $$u_{tt}(x, y, t) = p^{-2}t^{-1}T_0(F)(xt^{-1/p}, yt^{-1/p}),$$

(5.18) $$u_{xx}(x, y, t)u_{tt}(x, y, t) - u_{xt}^2(x, y, t) = p^{-2}t^{-2/p}T_1(F)(xt^{-1/p}, yt^{-1/p}),$$

(5.19) $$u_{yy}(x, y, t)u_{tt}(x, y, t) - u_{yt}^2(x, y, t) = p^{-2}t^{-2/p}T_2(F)(xt^{-1/p}, yt^{-1/p}),$$

for all (x, y, t) in any subdomain of Ω on which u is twice continuously differentiable.

The minimality of u suggests that F should satisfy

(5.20) $$F(x, y) = \left|\frac{x+y}{2}\right|^p, \quad (x, y) \in \partial D,$$

and that, on some subdomains of D, equality should hold in at least one of the above differential inequalities, which one depending on the subdomain. Later, in this section, when we return to the martingale origins of our boundary value problem, we shall construct an example suggesting that, if $x > 0$, then

(5.21) $$F(x, x) = (w-1)^p$$

where $w > p$ is the unique positive solution of

(5.22) $$x^p + pw^{p-1} - w^p = 0.$$

This example is suggested in part by the example in Section 4.

All of this and additional study suggests that

(5.23) $$F(x, y) = (w-1)^p$$

on the subdomain

(5.24) $$D_1 = \{(x, y) \in D: x > 0, (1 - 2/p)x < y < x\}$$

with $w = w(x, y) > p$ being here the unique positive solution of

(5.25) $$x^p[1 - p(x-y)/2x] + pw^{p-1} - w^p = 0,$$

and that

(5.26) $$F(x, y) = \left|\frac{x+y}{2}\right|^p + \left[1 - \left|\frac{x-y}{2}\right|^p\right](p-1)^p$$

on the subdomain

(5.27) $$D_2 = \{(x, y) \in D: x > 0, -x < y < (1 - 2/p)x\}.$$

Indeed, all of this is true and we have the following lemma which we shall use when we return to the proof of Lemma 5.1.

LEMMA 5.2. *Suppose that* $2 < p < \infty$.

(i) *Let F be the continuous function on* $D \cup \partial D$ *given by* (5.23) *on the subdomain* D_1, *by* (5.26) *on* D_2, *and satisfying*

$$F(x, y) = F(y, x) = F(-x, -y). \tag{5.28}$$

Such a function exists, satisfies the boundary condition (5.20), *and has continuous first partial derivatives on D. On* D_1 *the function F has continuous second partial derivatives and satisfies* (5.10)–(5.13) *with strict inequality and* (5.14) *with equality. On* D_2 *it has continuous second partial derivatives and satisfies* (5.10)–(5.11) *with strict inequality and* (5.12)–(5.14) *with equality.*

(ii) *Let u be the function on* $\Omega \cup \partial\Omega$ *with* $u(x, x, 0) = |x|^p$ *and such that, for* $(x, y, t) \in \Omega \cup \partial\Omega$ *with* $t > 0$.

$$u(x, y, t) = tF(xt^{-1/p}, yt^{-1/p}). \tag{5.29}$$

Then u is continuous on $\Omega \cup \partial\Omega$ *and satisfies the concavity conditions* (5.4) *and* (5.5). *It also satisfies the boundary condition* (5.6) *with equality.*

PROOF OF LEMMA 5.2. We note first that

$$0 < \frac{x-y}{2} < \frac{x}{p} \quad \text{and} \quad \frac{x}{q} < \frac{x+y}{2} < x \quad \text{on} \quad D_1, \tag{5.30}$$

$$0 < \frac{x+y}{2} < \frac{x}{q} \quad \text{and} \quad \frac{x}{p} < \frac{x-y}{2} < x \quad \text{on} \quad D_2, \tag{5.31}$$

and

$$\frac{x+y}{2} = \frac{x}{q} \quad \text{and} \quad \frac{x-y}{2} = \frac{x}{p} \quad \text{on} \quad \partial D_1 \cap \partial D_2. \tag{5.32}$$

Recall that q is conjugate to p: $1/p + 1/q = 1$.

Let H be the function on D_1 defined by

$$H(x, y) = x^p[1 - p(x-y)/2x] = (1 - p/2)x^p + px^{p-1}y/2. \tag{5.33}$$

The continuous extension of H to $D_1 \cup \partial D_1$, which we also denote by H, is strictly positive and infinitely differentiable on D_1 and vanishes on $\partial D_1 \cap \partial D_2$. The mapping $\lambda \longrightarrow \lambda^p - p\lambda^{p-1}$ from $[p, \infty)$ onto $[0, \infty)$ is strictly increasing and continuous. Let w be its inverse composed with H. Then w is continuous on $D_1 \cup \partial D_1$. Thus, (5.23) is well-defined on D_1 and has a continuous extension to $D_1 \cup \partial D_1$. Also, (5.26) has a continuous extension to $D_2 \cup \partial D_2$ and the two extensions agree on $\partial D_1 \cap \partial D_2$ where they are constant and have the value $(p-1)^p$. We can conclude that the function given by (5.23) on D_1 and by (5.26) on D_2 does have a continuous extension F to $D \cup \partial D$ satisfying the symmetry condition (5.28). It is also clear that F satisfies the boundary condition (5.20): If

$x \geq p$ and $x - y = 2$, then (5.25) becomes

$$x^p - px^{p-1} + pw^{p-1} - w^p = 0$$

so $w(x, y) = x$ and $F(x, y) = (x - 1)^p = |(x + y)/2|^p$. In case $1 \leq x \leq p$ and $x - y = 2$, the boundary condition follows directly from (5.26).

To calculate the first and second partial derivatives of F on D_1, we shall use the function Ψ defined on $(0, \infty)$ by

$$\Psi(\lambda) = (1 + \lambda^{1/p})^p - p(1 + \lambda^{1/p})^{p-1}.$$

Note that $\Psi((p - 1)^p) = 0$,

$$\Psi'(\lambda) = (1 + \lambda^{1/p})^{p-2}(2 - p + \lambda^{1/p})\lambda^{-1+1/p},$$

which is strictly positive for $\lambda > (p - 2)^p$, and

$$\Psi''(\lambda) = [(p - 1)(p - 2)/p](1 + \lambda^{1/p})^{p-3}\lambda^{-2+1/p},$$

which is strictly positive for all $\lambda > 0$. If $(x, y) \in D_1 \cup \partial D_1$, then, by the discussion above, $F(x, y) = (w(x, y) - 1)^p \geq (p - 1)^p$ so $\Psi'(F(x, y)) > 0$, $\Psi''(F(x, y)) > 0$, and

$$\Psi(F(x, y)) = H(x, y). \tag{5.34}$$

Now consider the first and second partial derivatives of F on D_1. By (5.34) we have $F_x = H_x/\Psi'(F)$ and this leads to expressions for F_{xx} and F_{xy}. It is convenient to let

$$M = 1/\Psi'(F) \quad \text{and} \quad N = \Psi''(F)/[\Psi'(F)]^2, \tag{5.35}$$

both strictly positive on $D_1 \cup \partial D_1$. Then, on D_1,

$$F_x = MH_x, \tag{5.36}$$

$$F_y = MH_y, \tag{5.37}$$

$$F_{xx} = MH_{xx} - MNH_x^2, \tag{5.38}$$

$$F_{yy} = MH_{yy} - MNH_y^2, \tag{5.39}$$

$$F_{xy} = MH_{xy} - MNH_xH_y. \tag{5.40}$$

These lead directly to the following expressions for $T_0(F)$, $T_1(F)$, and $T_2(F)$, the functions on the left-hand side of (5.12), (5.13), and (5.14), respectively:

$$T_0(F) = MT_0(H) - MN[xH_x + yH_y]^2, \tag{5.41}$$

$$T_1(F) = M^2T_1(H) - M^2NT_3(H), \tag{5.42}$$

$$T_2(F) = M^2T_2(H) - M^2NT_4(H). \tag{5.43}$$

Here $T_0(H)$ is the left hand side of (5.12) with F replaced by H, and $T_1(H)$ and $T_2(H)$ have a similar meaning. On the other hand, $T_3(H)$ is the function on D_1

defined by

$$T_3(H) = (p-1)[xH_x - yH_y]H_x^2 + 2y[(p-1)H_x - yH_{xy}]H_xH_y + y^2H_{xx}H_y^2 + y^2H_{yy}H_x^2 \tag{5.44}$$

and

$$T_4(H) = (p-1)[yH_y - xH_x]H_y^2 + 2x[(p-1)H_y - xH_{xy}]H_xH_y + x^2H_{xx}H_y^2 + x^2H_{yy}H_x^2. \tag{5.45}$$

For $x > 0$, the function $H(x, y) = (1 - p/2)x^p + px^{p-1}y/2$ is positively homogeneous of order p: $v(x, y, t) = tH(xt^{-1/p}, yt^{-1/p})$ is constant as a function of $t > 0$. Therefore, $v_t(x, y, t) = 0$ so, by the analogues of (5.17)–(5.19) for v and H,

$$T_0(H) = T_1(H) = T_2(H) = 0 \text{ on } D_1. \tag{5.46}$$

The first and second partial derivatives of H on D_1 are given by

$$2H_x(x, y) = p(2-p)x^{p-1} + p(p-1)x^{p-2}y, \tag{5.47}$$

$$2H_{xx}(x, y) = -p(p-1)(p-2)x^{p-3}(x-y), \tag{5.48}$$

$$2H_y(x, y) = px^{p-1}, \tag{5.49}$$

$$2H_{yy}(x, y) = 0, \tag{5.50}$$

$$2H_{xy}(x, y) = p(p-1)x^{p-2}. \tag{5.51}$$

Combining (5.38)–(5.51), we obtain the following information about the function F on D_1:

$$F_{xx} \le MH_{xx} < 0, \tag{5.52}$$

$$F_{yy} = -MNH_y^2 < 0, \tag{5.53}$$

$$\begin{aligned} T_0(F) &= -MN[xH_x + yH_y]^2 \\ &= -p^2MNH^2 < 0, \end{aligned} \tag{5.54}$$

$$\begin{aligned} T_1(F) &= -M^2NT_3(H) \\ &= p^3(p-1)(p-2)x^{p-3}(x-y)M^2NH^2/2 > 0, \end{aligned} \tag{5.55}$$

$$T_2(F) = -M^2NT_4(H) = 0. \tag{5.56}$$

Now consider the function F on D_2:

$$2^pF(x, y) = (x+y)^p + [2^p - (x-y)^p](p-1)^p.$$

If $(xt^{-1/p}, yt^{-1/p}) \in D_2$, then $u(x, y, t) = tF(xt^{-1/p}, yt^{-1/p})$ satisfies $u_{xt}(x, y, t) = u_{yt}(x, y, t) = u_{tt}(x, y, t) = 0$ so that, by (5.17)–(5.19),

$$T_0(F) = T_1(F) = T_2(F) = 0 \text{ on } D_2. \tag{5.57}$$

Using (5.31), we also see that

$$2^p F_x(x, y) = p(x + y)^{p-1} - p(x - y)^{p-1}(p - 1)^p, \tag{5.58}$$

$$2^p F_y(x, y) = p(x + y)^{p-1} + p(x - y)^{p-1}(p - 1)^p, \tag{5.59}$$

and

$$\begin{aligned} 4F_{xx}(x, y) &= 4F_{yy}(x, y) \\ &= p(p - 1)\left[\left|\frac{x + y}{2}\right|^{p-2} - \left|\frac{x - y}{2}\right|^{p-2}(p - 1)^p\right] \\ &< p(p - 1)[(x/q)^{p-2} - (x/p)^{p-2}(p - 1)^p] \\ &= p(p - 1)q^{-p}x^{p-2}[q^2 - p^2] < 0. \end{aligned} \tag{5.60}$$

To see that F_x and F_y exist and are continuous on all of D, recall the following elementary fact: If δ is a positive number, φ is a continuous function from the interval $(-\delta, \delta)$ to $\mathbb{R}$, and φ' exists and is continuous on $(-\delta, 0) \cup (0, \delta)$ with $\varphi'(0-) = \varphi'(0+)$, then φ' exists and is continuous on $(-\delta, \delta)$. Note that F_x and F_y are continuous on $D_1 \cup D_2$ and hence, by the symmetry condition (5.28), on the set of S of all $(x, y) \in D$ satisfying $x^2 \neq y^2$, $y \neq (1 - 2/p)x$, and $x \neq (1 - 2/p)y$. It is easy to check that F_x, for example, has a continuous extension G from S to D, even to $D \cup \partial D$. If $(x, y) \in D \backslash S$, then there is a positive number δ such that $(x + \alpha, y) \in S$ for $0 < |\alpha| < \delta$. Thus F_x exists at (x, y) and $F_x(x, y) = G(x, y)$.

This completes the proof of part (i) of Lemma 5.2. To prove part (ii), we shall use the following additional property of F: There exist positive real numbers a_p and b_p such that, for all $(x, y) \in D \cup \partial D$,

$$\left|\frac{x + y}{2}\right|^p \leq F(x, y) \leq \left|\frac{x + y}{2}\right|^p + a_p\left|\frac{x + y}{2}\right|^{p-1} + b_p. \tag{5.61}$$

By the continuity and symmetry of F, it is enough to prove this on $D_1 \cup D_2$. Provided $b_p \geq (p - 1)^p$, as we can assume, the inequality (5.61) does hold on D_2 as is clear from (5.26). So it is enough to prove that (5.61) holds on D_1. If $(x, y) \in D_1$, then

$$H(x, y) \leq \left|\frac{x + y}{2}\right|^p. \tag{5.62}$$

To see this, let $x > 0$ and observe that φ, defined by

$$\varphi(y) = \left|\frac{x + y}{2}\right|^p - H(x, y) = \left|\frac{x + y}{2}\right|^p - x^p + px^{p-1}(x - y)/2,$$

is convex on the section of $D_1 \cup \partial D_1$ determined by x, and $\varphi(x) = \varphi'(x-) = 0$, so $\varphi \geq 0$ and (5.62) holds. It is clear from the definition of Ψ that

$$\lim_{\lambda\to\infty} \Psi(\lambda)/\lambda = 1,$$

implying that, for a suitable choice of b_p and all $\lambda > 0$,

$$\lambda \leq (1 + \lambda^{1/p})^p = \Psi(\lambda) + p(1 + \lambda^{1/p})^{p-1}$$
$$\leq \Psi(\lambda) + 2p[\Psi(\lambda)]^{(p-1)/p} + b_p.$$

Recalling (5.34), we see that

$$F(x, y) \leq H(x, y) + 2p[H(x, y)]^{(p-1)/p} + b_p$$

and obtain the right-hand inequality of (5.61). To prove the left-hand inequality on D_1, we check it first on ∂D_1. By (5.20), it holds at least on $\partial D_1 \cap \partial D$. It also holds on the upper part of the boundary: If $x > 0$, then $w = w(x, x) > p$ satisfies

$$x^p = w^p - pw^{p-1} \leq (w - 1)^p = F(x, x).$$

On $\partial D_1 \cap \partial D_2$,

$$\left|\frac{x + y}{2}\right|^p = (x/q)^p \leq (p/q)^p = (p - 1)^p = F(x, y).$$

Therefore, the left-hand inequality holds on ∂D_1 and, by (5.52) or (5.53) and the argument leading to (4.14), the left-hand inequality holds on D_1. This completes the proof of (5.61).

The proof of part (ii) of Lemma 5.2 follows easily. By (5.20) and its definition, u satisfies the boundary condition (5.6) with equality. The next step is to show that u is continuous on $\Omega \cup \partial\Omega$. Except possibly at points of the form $(x_0, x_0, 0)$, this is clear from the continuity of F. But the continuity of u at $(x_0, x_0, 0)$ follows at once from an immediate consequence of (5.61): If $(x, y, t) \in \Omega \cup \partial\Omega$, then

$$\left|\frac{x + y}{2}\right|^p \leq u(x, y, t) \leq \left|\frac{x + y}{2}\right|^p + a_p t^{1/p}\left|\frac{x + y}{2}\right|^{p-1} + b_p t. \tag{5.63}$$

To show that u satisfies the concavity conditions (5.4) and (5.5) on $\Omega \cup \partial\Omega$, we need to show only, because of the continuity and symmetry of u, that the mapping $(x, t) \to u(x, y, t)$ is concave on the section of Ω determined by y. Let (x, y, t) be a point in Ω and δ its distance to $\partial\Omega$. Let $(\xi, \tau) \in \mathbb{R}^2$ satisfy $\xi^2 + \tau^2 < \delta^2$. Consider the function φ defined on $(-1, 1)$ by

$$\varphi(\alpha) = u(x + \alpha\xi, y, t + \alpha\tau).$$

It suffices to show that φ is concave. We do this by showing that

$$\varphi'(\alpha) = \xi u_x(x + \alpha\xi, y, t + \alpha\tau) + \tau u_t(x + \alpha\xi, y, t + \alpha\tau)$$

is nonincreasing on $(-1, 1)$. By the existence and continuity of F_x and F_y on D, the derivative φ' exists and is continuous on $(-1, 1)$. Suppose for the moment that $\xi \neq 0$. Then, by part (i),

$$\varphi''(\alpha) = \xi^2 u_{xx} + 2\xi\tau u_{xt} + \tau^2 u_{tt}$$

exists and is nonpositive for all but a finite set of α's in $(-1, 1)$. Thus $(-1, 1)$ is the union of a finite number of intervals such that φ' is nonincreasing on the interior of each interval. Since φ' is continuous on $(-1, 1)$, φ' is nonincreasing on $(-1, 1)$. If $\xi = 0$, φ' is nonincreasing because it is the limit of nonincreasing functions: $\varphi'(\alpha) = \lim_{n\to\infty}\varphi_n'(\alpha)$, where, for large n, φ_n is defined on $(-1, 1)$ by

$$\varphi_n(\alpha) = u(x + \alpha/n, y, t + \alpha\tau).$$

This completes the proof of Lemma 5.2.

Before returning to the proof of Lemma 5.1, we note that u_x and u_y extend continuously to $\Omega \cup \partial\Omega$: By (5.29) and the first-order differentiability of F,

$$u_x(x, y, t) = t^{1/q}F_x(xt^{-1/p}, yt^{-1/p})$$

for all $(x, y, t) \in \Omega$. There is a similar expression for u_y. At the least, since F_x and F_y extend continuously to $D \cup \partial D$, both u_x and u_y extend continuously to the set

$$\Omega_+ = (\Omega \cup \partial\Omega)\backslash\{(x, x, 0): x \in \mathbb{R}\}. \tag{5.64}$$

But u_x and u_y are also well-behaved near any point $(x_0, x_0, 0)$ with $x_0 \in \mathbb{R}$. For example, let $x_0 > 0$ and (x, y, t) be a point near $(x_0, x_0, 0)$ with $t > 0$ and $(xt^{-1/p}, yt^{-1/p}) \in D_1$. Then, by (5.36) and (5.47),

$$u_x(x, y, t) = H_x(x, y)/\Psi'(F(xt^{-1/p}, yt^{-1/p})). \tag{5.65}$$

Using $\lim_{\lambda\to\infty}\Psi'(\lambda) = 1$ and

$$F(xt^{-1/p}, yt^{-1/p}) \geq t^{-1}\left|\frac{x + y}{2}\right|^p, \tag{5.66}$$

which follows from (5.61), we see that $u_x(x, y, t)$ must be near $H_x(x_0, x_0) = px_0^{p-1}/2$. For the case $x_0 = 0$ with (x, y, t) as above, the same result holds since in this case $H_x(x_0, x_0) = 0$ and (5.66) may be replaced by

$$F(xt^{-1/p}, yt^{-1/p}) \geq (p - 1)^p$$

so the denominator in (5.65) is bounded away from 0. If (x, y, t) is a point near $(0, 0, 0)$ with $t > 0$ and $(xt^{-1/p}, yt^{-1/p}) \in D_2$, then $u_x(x, y, t) = F_x(x, y)$ where F_x is given by (5.58). So here also, $u_x(x, y, t)$ is near $H_x(x_0, x_0)$. These results also hold if (x, y, t) is near $(x_0, x_0, 0)$ with $t > 0$ and $(xt^{-1/p}, yt^{-1/p}) \in \partial D_1$ or ∂D_2. Similar results for u_y, and symmetry, lead to the conclusion that u_x and u_y have continuous extensions to $\Omega \cup \partial\Omega$. The same symbols u_x and u_y will be used to denote these extensions.

The function u of Lemma 5.2 also satisfies the following inequality: There is a positive real number c_p such that, for all $(x, y, t) \in \Omega \cup \partial\Omega$,

$$|u_x(x, y, t)| \leq c_p[t + u(x, y, t)]^{1/q} \tag{5.67}$$

and a similar inequality holds for u_y. This follows at once from the fact that

$$|F_x(x, y)|/[1 + F(x, y)]^{1/q} \tag{5.68}$$

is bounded on $D \cup \partial D$. For example, if x is positive and large and $x - 2 < y < x$, then, by (5.61), $F(x, y)/x^p$ is near 1 implying that $\Psi'(F(x, y))$ is near 1. Also, $H_x(x, y)/x^{p-1}$ is near $p/2$ so, by (5.36), the expression (5.68) is near $p/2$. So the boundedness of (5.68) near infinity is clear; elsewhere the boundedness follows from the continuity of (5.68).

The mapping u_t does not have a continuous extension to $\Omega \cup \partial\Omega$. However, the mapping $(x, y, t) \to tu_t(x, y, t)$ on Ω does have a continuous extension ψ to $\Omega \cup \partial\Omega$ given by

$$\psi(x, y, t) = u(x, y, t) - p^{-1}[xu_x(x, y, t) + yu_y(x, y, t)]. \tag{5.69}$$

If $(x, y, t) \in \Omega$, then $tu_t(x, y, t) = \psi(x, y, t)$ by differentiation and the continuity of ψ on $\Omega \cup \partial\Omega$ follows from the continuity of u, u_x, and u_y. In particular, u_t has a continuous extension to the set Ω_+ defined in (5.64). Although $u_t(x, x, 0)$ has not been defined, we adopt the convention that $tu_t(x, x, t) = 0$ if $t = 0$. By (5.67), $\psi(x, x, 0) = 0$ so $tu_t(x, y, t) = \psi(x, y, t)$ holds for all $(x, y, t) \in \Omega \cup \partial\Omega$.

All of this leads to the following inequality: Let (x, y, t) and $(x + h, y + k, t + r)$ belong to $\Omega \cup \partial\Omega$ where either $h = 0$ or $k = 0$ and where $r = 0$ if $t = 0$. Then

$$\begin{aligned} u(x + h, y + k, t + r) \le u(x, y, t) &+ u_x(x, y, t)h \\ &+ u_y(x, y, t)k + u_t(x, y, t)r \end{aligned} \tag{5.70}$$

and strict inequality holds, for example, if $x = 0$, $y = 0$, $t > 0$, and either $h \neq 0$ or $k \neq 0$. If both (x, y, t) and $(x + h, y + k, t + r)$ belong to Ω, then (5.70) follows from the concavity conditions (5.4) and (5.5) and the existence of the first-order derivatives. If $t > 0$, the more general result follows by continuity. If $t = 0$ and $r = 0$, then $x = y$ and $x + h = y + k$ so that $h = k = 0$ and (5.70) holds in this case also. The statement about strict inequality follows from the strict concavity of $u(\cdot, 0, t)$ and $u(0, \cdot, t)$ for $t > 0$.

We now return to the proof of Lemma 5.1 and show that

$$\| f \|_p^p < u(0, 0, 1) \tag{5.71}$$

if $f \in \mathbf{F}(0, 0, 1)$ and u is the function of Lemma 5.2.

Fix f and g as in the definition of $\mathbf{F}(0, 0, 1)$ and note, in particular, that $\| g \|_p \le 1$. Let Z be the zigzag martingale determined by f and g as in Section 3. We shall also use a martingale $W = (W_1, W_2, \cdots)$ with values in $\mathbb{R}^3$ defined as follows: $W_n = (Z_n, T_n) = (X_n, Y_n, T_n)$ with

$$T_n = E(| g_\infty |^p \mid \mathscr{A}_n)$$

where g_∞ is the limit in L^p of g (since g is an L^p-bounded martingale, such a limit exists [10]) and $\mathscr{A}_n$ is the smallest σ-field with respect to which $d_1, d_2, \cdots, d_n$ are measurable. Jensen's inequality for conditional expectations gives

$$| g_n |^p = | E(g_\infty \mid \mathscr{A}_n) |^p \le E(| g_\infty |^p \mid \mathscr{A}_n) = T_n$$

except possibly on a set of measure zero. We can and do assume without loss of generality that $\| g \|_p = 1$ and $| g_n |^p \le T_n$ everywhere so that

$$\left| \frac{X_n - Y_n}{2} \right|^p \le T_n$$

and W is a martingale starting at $(0, 0, 1)$ with values in $\Omega \cup \partial\Omega$. By (3.3) and

(5.63),

$$\|f_n\|_p^p = \left\| \frac{X_n + Y_n}{2} \right\|_p^p \le Eu(W_n). \tag{5.72}$$

The next step is to show that

$$Eu(W_{n+1}) \le Eu(W_n) \tag{5.73}$$

and that strict inequality holds if

$$P(Z_n = 0,\, Z_{n+1} \ne 0) > 0. \tag{5.74}$$

The zigzag property of Z and the fact that $T_{n+1} = 0$ almost everywhere on the set where $T_n = 0$ imply, by (5.70), that almost everywhere

$$\begin{aligned} u(W_{n+1}) \le u(W_n) &+ u_x(W_n)(X_{n+1} - X_n) \\ &+ u_y(W_n)(Y_{n+1} - Y_n) + u_t(W_n)(T_{n+1} - T_n). \end{aligned} \tag{5.75}$$

Integrating both sides of this inequality gives (5.73) since, as we shall see,

$$Eu_x(W_n)(X_{n+1} - X_n) = 0, \tag{5.76}$$

$$Eu_y(W_n)(Y_{n+1} - Y_n) = 0, \tag{5.77}$$

$$Eu_t(W_n)(T_{n+1} - T_n) = 0. \tag{5.78}$$

These depend on the integrability of $u(W_n)$: By (5.63),

$$0 \le u(W_n) \le |f_n|^p + a_p T_n^{1/p} |f_n|^{p/q} + b_p T_n$$

where $ET_n = 1$ and $\|f\|_p \le \sum_{k=1}^n \|d_k\|_p \le 2n \|g\|_p = 2n$. By Hölder's inequality, the middle term on the right is also integrable so $u(W_n) \in L^1$. This and (5.67) give $u_x(W_n) \in L^q$. Since $X_{n+1} - X_n = (1 + \varepsilon_{n+1})d_{n+1} \in L^p$, Hölder's inequality implies that $u_x(W_n)(X_{n+1} - X_n)$ is integrable. Now W is a martingale so

$$E[(u_x(W_n) \wedge j) \vee (-j)][W_{n+1} - W_n] = 0.$$

If we consider the first component of this vector in $\mathbb{R}^3$, let $j \to \infty$, and use the dominated convergence theorem, we obtain (5.76). The proof of (5.77) is similar. Turning to u_t, we see by (5.69) and what we have already proved that $T_n u_t(W_n) = \psi(W_n)$ is integrable. If j is a positive integer let $\varphi_j \colon \mathbb{R}^3 \to \mathbb{R}$ be a bounded continuous extension of the bounded continuous mapping $(x, y, t) \to u_t(x, y, t \vee j^{-1}) \wedge j$ on $\Omega \cup \partial\Omega$. Since $u(x, y, \cdot)$ is concave and nonnegative on its interval of definition, $u_t(x, y, \cdot)$ is nonnegative and nonincreasing on its interval of definition so we may assume that $0 \le \varphi_j \le \varphi_{j+1}$. If $(x, y, t) \in \Omega \cup \partial\Omega$ and $t > 0$, then $\lim_{j\to\infty} \varphi_j(x, y, t) = u_t(x, y, t)$. By the martingale property of W,

$$E\varphi_j(W_n)(T_{n+1} - T_n) = 0.$$

Therefore, by the monotone convergence theorem,

$$Eu_t(W_n)T_{n+1} = \int_{\{T_n>0\}} u_t(W_n)T_{n+1} = \lim_{j\to\infty} E\varphi_j(W_n)T_{n+1}$$

$$= \lim_{j\to\infty} E\varphi_j(W_n)T_n = Eu_t(W_n)T_n$$

which, in view of the integrability of $u_t(W_n)T_n$, gives (5.78) and completes the proof of (5.73). To check the statement about strict inequality, assume that (5.74) holds. Since

$$0 < \left|\frac{X_{n+1} - Y_{n+1}}{2}\right|^p \leq T_{n+1}$$

on the set where $Z_n = 0$ and $Z_{n+1} \neq 0$, we have, by (5.74), that

$$P(Z_n = 0,\ Z_{n+1} \neq 0,\ T_n > 0) > 0.$$

Therefore, by (5.70), strict inequality holds in (5.75) on a set of positive measure implying that strict inequality holds in (5.73).

To complete the proof of (5.71), we note first that if (5.74) does not hold for any positive integer n, then

$$P(Z_n = 0,\ n \geq 1) = 1$$

so $\|f\|_p^p = 0 < u(0, 0, 1)$. On the other hand, if (5.74) does hold for some positive integer n, let m denote the least such integer. Then, using (5.72) and (5.73), we see that, for $n \leq m$, $\|f_n\|_p^p = 0$ and, for $n > m$,

$$\|f_n\|_p^p \leq Eu(W_n) \leq Eu(W_{m+1})$$

$$< Eu(W_m) \leq Eu(W_1) = u(0, 0, 1).$$

This completes the proof of (5.71) and, since $u(0, 0, 1) = (p-1)^p$, the proof of (5.1).

To show that the constant $p - 1$ in (5.1) is best possible, let $x > 0$ and denote by $w = w(x, x)$ the unique number $w > p$ satisfying (5.22). Set $\theta = 1 - 1/w$ and

$$\pi_n = \left[\frac{x}{x + (n-1)\delta}\right]^w$$

where $\delta > 0$. Define $f = (f_1, f_2, \cdots)$ on $[0, 1)$ by

(5.79)
$$\begin{aligned} f_n(s) &= x + (n-1)\delta \quad &&\text{if } 0 \leq s < \pi_n,\\ &= \theta x s^{\theta-1} &&\text{if } \pi_n \leq s < 1. \end{aligned}$$

It is easy to check that $Ef_n = x$ so that d_{n+1} is orthogonal to every constant function. Also, d_{n+1} is supported by the interval $[0, \pi_n)$ on which every function of the form $\varphi(d_1, \cdots, d_n)$ is constant. Therefore, f is a martingale. Note that f starts at x, then increases in steps of size δ until it drops to the value $\theta x s^{\theta-1}$ where it stays ever after. Let g be the transform of f by $(1, -1, 1, -1, \cdots)$. We shall show that

$$\lim_{\delta\to 0}\sup_n \| g_n - x \|_p = 1, \tag{5.80}$$

$$\lim_{\delta\to 0} \| f \|_p = w - 1, \tag{5.81}$$

which gives immediately that

$$\lim_{x\to 0}\lim_{\delta\to 0} \| g \|_p = 1, \tag{5.82}$$

$$\lim_{x\to 0}\lim_{\delta\to 0} \| f \|_p = p - 1. \tag{5.83}$$

In turn, (5.82) and (5.83) imply that $p - 1$ is the best constant in (5.1). By (5.79),

$$\| f_n \|_p^p = [x + (n-1)\delta]^p \pi_n + \int_{\pi_n}^1 (\theta x s^{\theta-1})^p \, ds.$$

Since $w > p$, the first term on the right approaches 0 as $n \to \infty$. By (5.22), the second term approaches

$$\int_0^1 (\theta x s^{\theta-1})^p \, ds = (w - 1)^p. \tag{5.84}$$

This implies (5.81). To prove (5.80), note that if $\pi_{n+1} \le s < \pi_n$, then $| g_n(s) - x | \le \delta$,

$$\begin{aligned} | g_{n+1}(s) - g_n(s) | &= f_n(s) - f_{n+1}(s) \\ &= x + (n-1)\delta - \theta x s^{\theta-1} \le x(1-\theta)s^{\theta-1}, \end{aligned}$$

and $g_{n+1}(s) = g_{n+2}(s) = \cdots$. Consequently, if $0 \le s < 1$, then

$$\sup_n | g_n(s) - x | \le \delta + x(1-\theta)s^{\theta-1}$$

and, by (5.84),

$$\sup_n \| g_n - x \|_p \le \delta + (w-1)(1-\theta)/\theta = \delta + 1.$$

This implies that the left-hand side of (5.80) is not greater than 1 and the reverse inequality is obtained similarly. This completes the proof of Lemma 5.1.

The martingale in the above example may be slightly modified so that each term takes on only a finite number of values; see [6].

We finish the proof of Theorem 1.1 by studying the case $1 < p < 2$. One approach is to use duality as in [3] but to take care to check strict inequality if the p-norm of f is finite and positive. This is easy to do. An approach similar to the one used above to prove the case $2 < p < \infty$ is also possible and gives additional information. We sketch it here.

LEMMA 5.3. *Let $1 < p < 2$. If f is a real martingale such that $\| g \|_p \le 1$ for some transform g of f by a sequence of numbers in $\{-1, 1\}$, then*

$$\| f \|_p < q - 1 \tag{5.85}$$

and the constant $q - 1$ is best possible.

The proof has the same pattern as that of Lemma 5.1. Everything is the same as in that proof up to (5.21). However, here

$$F(x, y) = (w - 1)^p \tag{5.86}$$

on the subdomain

$$D_3 = \{(x, y) \in D: x > 0, -x < y < (1 - 2/q)x\}, \tag{5.87}$$

where $w = w(x, y) < q$ is here the unique positive solution to

$$x^p[1 - q(x - y)/2x] + qw^{p-1} - w^p = 0,$$

and

$$F(x, y) = \left|\frac{x + y}{2}\right|^p + \left[1 - \left|\frac{x - y}{2}\right|^p\right](q - 1)^p \tag{5.88}$$

on the subdomain

$$D_4 = \{(x, y) \in D: x > 0, (1 - 2/q)x < y < x\}. \tag{5.89}$$

LEMMA 5.4. *Suppose that* $1 < p < 2$.

(i) *Let* F *be the continuous function on* $D \cup \partial D$ *given by* (5.86) *on the subdomain* D_3, *by* (5.88) *on* D_4, *and satisfying the symmetry condition* (5.28). *Such a function exists, satisfies the boundary condition* (5.20), *and has continuous first partial derivatives on* D. *On* D_3 *the function* F *has continuous second partial derivatives and satisfies* (5.10)–(5.13) *with strict inequality and* (5.14) *with equality. On* D_4 *it has continuous second partial derivatives and satisfies* (5.10)–(5.11) *with strict inequality and* (5.12)–(5.14) *with equality.*

(ii) *Let* u *be defined on* $\Omega \cup \partial\Omega$ *using this* F *as in Lemma* 5.2. *Then* u *satisfies the concavity conditions* (5.4) *and* (5.5), *and the boundary condition* (5.6) *with equality.*

The proof and application of this lemma are similar to those of Lemma 5.2.

Finally, (5.82) and (5.83) remain valid for $1 < p < 2$. So we have that $(q - 1)f$ is the transform of the martingale $(q - 1)g$ by $(1, -1, 1, -1, \cdots)$,

$$\lim_{x\to 0}\lim_{\delta\to 0}\| (q - 1)g \|_p = q - 1, \tag{5.90}$$

and, using $(p - 1)(q - 1) = 1$, we have

$$\lim_{x\to 0}\lim_{\delta\to 0}\| (q - 1)f \|_p = 1. \tag{5.91}$$

Thus $q - 1$ is the best constant in (5.85).

This completes the proof of Theorem 1.1.

6. Boundary data with Φ' strictly convex. Let Φ be an increasing convex function on $[0, \infty)$ with $\Phi(0) = 0$ and

$$\int_0^\infty \Phi(t)e^{-t}\,dt < \infty. \tag{6.1}$$

Also, assume throughout this section that Φ is twice differentiable on $(0, \infty)$ with a strictly convex first derivative satisfying $\Phi'(0+) = 0$. (Some examples: $\Phi(t) = t^p$ for $p > 2$; $\Phi(t) = e^{\alpha t} - 1 - \alpha t$ for $0 < \alpha < 1$.)

The following extends Theorem 1.2.

THEOREM 6.1. *If f is a real martingale with $\|f\|_\infty \le 1$ and g is the transform of f by a real predictable sequence v uniformly bounded in absolute value by* 1, *then*

$$\sup_n E\Phi(|g_n|) < \frac{1}{2}\int_0^\infty \Phi(t)e^{-t}\,dt \tag{6.2}$$

and the constant on the right is best possible.

As before, the constant is already best possible if it is assumed that v does not vary over all possibilities but is simply taken to be the numerical sequence $(1, -1, 1, -1, \cdots)$.

The proof has exactly the same pattern as the proof of Theorem 1.2. In the proof of the analogue of Lemma 4.1, the function u is again defined on $D \cup \partial D$ by (4.4) and (4.5) but here, for $x \ge 1$, $B(x) = \Phi(x - 1)$ and

$$A(x) = e^x \int_x^\infty B(t)e^{-t}\,dt = e^{x-1}\int_{x-1}^\infty \Phi(t)e^{-t}\,dt. \tag{6.3}$$

With this choice of u, we have the following extension of the main result of Section 4. Note that to prove Theorem 6.1 only the case $x = y = 0$ is needed.

LEMMA 6.1. *Suppose that $(x, y) \in \mathbb{R}^2$ and $|x - y| < 2$. If f is a real martingale starting at $(x + y)/2$ and, for some sequence $(1, \varepsilon_2, \varepsilon_3, \cdots)$ in $\{-1, 1\}$, the transform g of f by this sequence satisfies $\sup_n \|g_n - y\|_\infty \le 1$, then*

$$\sup_n E\Phi(|f_n|) < u(x, y) \tag{6.4}$$

and the bound on the right-hand side is best possible. Equality holds if $|x - y| =$ 2.

PROOF. Let $D = \{(x, y) \in \mathbb{R}^2 : |x - y| < 2\}$ as before. The martingales f and g determine, as in Section 3, a zigzag martingale Z starting at (x, y) with values in $D \cup \partial D$. If $|x - y| = 2$, these properties of Z imply that almost everywhere $Z_n = (x, y)$ and $\Phi(|f_n|) = u(Z_n) = u(x, y)$, and the last statement of the lemma follows.

Now assume that $(x, y) \in D$. As in Section 4,

$$E\Phi(|f_n|) \le Eu(Z_n) \le \cdots \le Eu(Z_1) = u(x, y) \tag{6.5}$$

so the left-hand side of (6.4) is not greater than $u(x, y)$. To show that it is strictly less than $u(x, y)$, we shall need to show only, since $E\Phi(|f_n|)$ is nondecreasing in n, that

$$\lim_{n\to\infty} E\Phi(|f_n|) = u(x, y) \tag{6.6}$$

does not hold.

Assume, on the contrary, that (6.6) does hold. Then (6.5) implies, since $u(Z_n) - \Phi(|f_n|)$ is nonnegative, that

$$\lim_{n\to\infty} \| u(Z_n) - \Phi(|f_n|) \|_1 = 0 \tag{6.7}$$

and

$$Eu(Z_{n+1}) = Eu(Z_n), \quad n \geq 1. \tag{6.8}$$

To see that (6.7) and (6.8) lead to a contradiction, consider first the case in which (x, y) in D satisfies $|x| \vee |y| \leq 1$. Then (6.8) implies that (4.21) holds. So $|f_n| = |(X_n + Y_n)/2|$ converges to 0 in probability: By (6.7), for large n, the probability is near 1 that $u(Z_n) - \Phi(|f_n|)$ is near 0 and, with $Z_n \in D_0 \cup \partial D_0$, this is possible if and only if Z_n is near $(1, -1)$ or $(-1, 1)$. Therefore, by the dominated convergence theorem and the fact that here $\Phi(|f_n|) \leq \Phi(1)$ almost everywhere, we have that

$$\lim_{n\to\infty} E\Phi(|f_n|) = \Phi(0) = 0 < u(x, y),$$

a contradiction of (6.6).

To handle the other cases, we let $Q(x, y)$ be the set of all points $(x + h, y + k)$ in $D \cup \partial D$ where either $h = 0$ or $k = 0$ and equality holds in

$$u(x + h, y + k) \leq u(x, y) + u_x(x, y)h + u_y(x, y)k. \tag{6.9}$$

For example, if $x > 1$ and $x - 2 < y < x$, then

$$Q(x, y) = \{(x, y + k): x - 2 \leq y + k \leq x\} \tag{6.10}$$

and $Q(x, x) = Q(x, x-) \cup Q(x-, x)$.

Since Z starts at (x, y) in $Q(x, y)$ either

$$P(Z_n \in Q(x, y), n \geq 1) = 1 \tag{6.11}$$

or, for some positive integer n,

$$P(Z_n \in Q(x, y), Z_{n+1} \notin Q(x, y)) > 0. \tag{6.12}$$

Let $y = x > 1$. If (6.11) holds, then $P(Z_n = (x, x)) = 1$, since (x, x) is an extreme point of $Q(x, x)$, so

$$E\Phi(|f_n|) = \Phi(x) < u(x, x),$$

a contradiction of (6.6). If (6.12) holds, then $Eu(Z_{n+1}) < Eu(Z_n)$, a contradiction of (6.8).

Now let $x > 1$ and $x - 2 < y < x$. If (6.11) holds, then (6.7) implies that Y

converges in probability to $x - 2$ and, by the uniform boundedness of Y,

$$y = EY_1 = \cdots = EY_n \to x - 2,$$

a contradiction of $y > x - 2$. On the other hand if (6.12) holds, then at least one of the following three probabilities must be positive:

$$P(X_n = x,\ x - 2 < Y_n < x,\ Z_{n+1} \notin Q(x, y)) \tag{6.13}$$

$$P(Z_n = (x, x),\ Z_{n+1} \neq Z_n), \tag{6.14}$$

$$P(Z_n = (x, x - 2),\ Z_{n+1} \neq Z_n). \tag{6.15}$$

Since Z is a zigzag martingale with values in $D \cup \partial D$, the last probability vanishes. Therefore, using the fact that (x, x) is an extreme point of $Q(x, x)$, we have that

$$P(Z_n \in Q(x, y) \cap D,\ Z_{n+1} \notin Q(Z_n)) > 0.$$

But this gives $Eu(Z_{n+1}) < Eu(Z_n)$, a contradiction of (6.8).

All the other cases follow by symmetry. Therefore strict inequality holds in (6.6), and (6.4) follows.

It remains to show that the bound $u(x, y)$ in (6.4) is best possible. The example f in (4.24) satisfies

$$\lim_{\delta \to 0} \sup_n E\Phi(|f_n|) = \frac{1}{2}\int_0^\infty \Phi(t)e^{-t}\,dt = u(1, 0)$$

exactly as in Section 4. So the bound is best possible for $(x, y) = (1, 0)$, hence also for $(x, y) = (0, 0)$. Related examples for all the other cases can be constructed easily as follows. Let $0 < \delta < 2$ and $Z_1(s) = (x, y)$ for $0 \le s < 1$. If $Z_1, \cdots, Z_n$ have been defined on $[0, 1)$ and the set

$$\{s: Z_n(s) = (x', y')\}$$

is an interval $[a, b)$, define Z_{n+1} on $[a, b)$ as follows. If $-1 < x' < 1$ and $-1 \le y' \le 1$, let

$$\begin{aligned} Z_{n+1}(s) &= (-1, y') \quad \text{if } a \le s < c, \\ &= (1, y') \quad \text{if } c \le s < b \end{aligned}$$

where $c \in [a, b]$ is chosen so that

$$\int_a^b [Z_{n+1}(s) - Z_n(s)]\,ds = 0. \tag{6.16}$$

If $x' \ge 1$ and $x' - 2 \le y \le x'$, let

$$\begin{aligned} Z_{n+1}(s) &= (x', x' + \delta) \quad \text{if } a \le s < c, \\ &= (x', x' - 2) \quad \text{if } c \le s < b, \end{aligned}$$

where $c \in [a, b]$ is again chosen so that (6.16) holds. Similarly, if $y' > 1$ and

$y' - 2 \le x' < y'$, let

$$Z_{n+1}(s) = (y' + \delta, y') \quad \text{if } a \le s < c,$$
$$= (y' - 2, y') \quad \text{if } c \le s < b$$

with c chosen as before. The other cases are handled symmetrically. Note that if δ is small, then $Z_{n+1}(s)$ is either in or near $Q(x', y')$. Clearly, $Z = (Z_1, Z_2, \cdots)$ is a zigzag martingale. Let f be the martingale defined by $f_n = (X_n + Y_n)/2$ and g the transform of f by the sequence $\varepsilon = (1, \varepsilon_2, \varepsilon_3, \cdots)$ defined in Section 3. Then f and g are as in the statement of Lemma 6.1. It is easy to see by considering separately the two main cases that

$$\lim_{\delta \to 0} \sup_n E\Phi(|f_n|) = u(x, y). \tag{6.17}$$

This completes the proof of Lemma 6.1.

Let U_Φ and L_Φ be functions on the closure of $D = \{(x, y) \in \mathbb{R}^2 : |x - y| < 2\}$ satisfying the symmetry condition (4.4) and such that $2U_\Phi(x, y)$ is

$$(1 + xy) \int_0^\infty \Phi(t) e^{-t}\, dt \tag{6.18}$$

if $|x| \vee |y| \le 1$, and is

$$(y - x + 2)e^x \int_x^\infty \Phi(t - 1)e^{-t}\, dt + (x - y)\Phi(x - 1) \tag{6.19}$$

if $x \ge 1$ and $x - 2 \le y \le x$, and $2L_\Phi(x, y)$ is

$$(y - x + 2)e^{-y}\left[\Phi(1) + \int_0^y \Phi(t + 1)e^t\, dt\right] + (x - y)\Phi(y + 1) \tag{6.20}$$

if $0 \le y \le x \le y + 2$, and is

$$(y - x + 2)\left[\Phi(1) + \int_y^0 \frac{\Phi(t + 1)}{(t + 1)^2}\, dt\right] + (x + y)\frac{\Phi(y + 1)}{y + 1} \tag{6.21}$$

if $-1 < y \le 0$ and $-y \le x \le y + 2$, with $L_\Phi(-1, 1) = 0$. The function U_Φ is simply the function u of Lemma 6.1 and will be shown to be the upper solution of a boundary value problem. The function L_Φ is the lower solution. The information that L_Φ contains about the control of martingales will become clear in Sections 7 and 11.

THEOREM 6.2. *The function U_Φ is the least biconcave function u on $D \cup \partial D$ such that*

$$u(x, y) \ge \Phi\left(\left|\frac{x + y}{2}\right|\right) \quad \text{if} \quad (x, y) \in \partial D. \tag{6.22}$$

The function L_Φ is the greatest biconvex function u on $D \cup \partial D$ such that

$$u(x, y) \le \Phi\left(\left|\frac{x + y}{2}\right|\right) \quad \text{if} \quad (x, y) \in \partial D. \tag{6.23}$$

PROOF. We know from Lemma 6.1 that

$$U_\Phi(x, y) = \sup_Z \sup_n E\Phi(|f_n|) \tag{6.24}$$

where $f_n = (X_n + Y_n)/2$, as usual, and $Z = (X, Y)$ is a zigzag martingale starting at (x, y) with values in $D \cup \partial D$. In fact, by (6.17), each term of Z can be assumed to have only a finite number of values.

We already know that U_Φ is biconcave and satisfies the boundary condition (6.22). It remains to show that it is the least such function. Let u be any biconcave function on $D \cup \partial D$ satisfying (6.22). Then $E\Phi(|f_n|) \le Eu(Z_n)$ as before. To show that $Eu(Z_{n+1}) \le Eu(Z_n)$ we may assume that $Z_{n+1} = (X_{n+1}, Y_n)$. (The other case, $Z_{n+1} = (X_n, Y_{n+1})$, is similar.) By the concavity of $u(\cdot, y)$ and Jensen's inequality,

$$E[u(Z_{n+1}) \mid Z_n] \le u(E(X_{n+1} \mid Z_n), Y_n) = u(Z_n) \tag{6.25}$$

and this gives $Eu(Z_{n+1}) \le Eu(Z_n)$. Thus $E\Phi(|f_n|) \le Eu(Z_n) \le \cdots \le Eu(Z_1) = u(x, y)$ so $U_\Phi(x, y) \le u(x, y)$. This completes the proof of the first half of the theorem. The proof of the second half has the same pattern.

7. Boundary data with Φ' strictly concave. We assume here, as in Section 6, that Φ is an increasing convex function on $[0,\infty)$ such that Φ is twice differentiable on $(0, \infty)$ and $\Phi(0) = \Phi'(0+) = 0$. In this section, however, we assume that Φ' is strictly concave. (Examples: $\Phi(t) = t^p$ for $1 < p < 2$; $\Phi(t) = t\log(t + 1)$; $\Phi(t) = t - \log(t + 1)$.) In this case (6.1) is automatically satisfied since, for $t > x > 0$,

$$\Phi(t) \le \Phi(x) + \Phi'(x)(t - x) + \tfrac{1}{2}\Phi''(x)(t - x)^2.$$

The following is the dual of Theorem 6.1.

THEOREM 7.1. *If f is a real martingale such that, for some real predictable sequence v uniformly bounded in absolute value by* 1, *the transform g of f by v satisfies $g^* \ge 1$ almost everywhere, then*

$$\frac{1}{2}\int_0^\infty \Phi(t)e^{-t}\,dt < \sup_n E\Phi(|f_n|) \tag{7.1}$$

and the constant on the left is best possible.

Again u is defined on $D \cup \partial D$ by (4.4) and (4.5) where $B(x) = \Phi(x - 1)$ and $A(x)$ is given by (4.9). Here, however, u is biconvex on $D \cup \partial D$: Note, for example, that the strict concavity of Φ' implies that $u_{xx}(x, y) > 0$ if $(x, y) \in D_1$ as is clear from (4.13).

LEMMA 7.1. *Suppose that $(x, y) \in \mathbb{R}^2$ and $|x - y| < 2$. If f is a real martingale starting at $(x + y)/2$ and, for some real predictable sequence $v = (1, v_2, v_3, \cdots)$ uniformly bounded in absolute value by* 1, *the transform g of f by v satisfies*

$\sup_n |g_n - y| \geq 1$ *almost everywhere, then*

$$(7.2) \qquad u(x, y) < \sup_n E\Phi(|f_n|)$$

and the bound on the left-hand side is best possible. If $|x - y| = 2$, *then* $u(x, y) \leq \sup_n E\Phi(|f_n|)$ *and equality holds if and only if* $f = ((x + y)/2, (x + y)/2, \cdots)$ *almost everywhere.*

By the first reduction of Section 2, Theorem 7.1 follows at once from the special case $x = y = 0$.

PROOF OF LEMMA 7.1. If $|x - y| = 2$, then $u(x, y) = E\Phi(|f_1|)$. The conditions on Φ imply that the mapping $t \longrightarrow \Phi(|t|)$ is strictly convex. Therefore, by Jensen's inequality, $u(x, y) = \Phi(|Ef_n|) < E\Phi(|f_n|)$ unless $f_n = (x + y)/2$ a.e. and the last statement of the lemma is proved.

To prove (7.2), we can assume that $\sup_n E\Phi(|f_n|)$ is finite so, by the inequality $t \leq 1 + \Phi(t)/\Phi(1)$, $t \geq 0$, the martingale f is L^1-bounded. Let f_∞ denote its almost everywhere limit [10] and g_∞ the almost everywhere limit of g [3].

We can also assume in the proof that

$$(7.3) \qquad |g_\infty - y| \geq 1 \text{ a.e.}$$

For if (7.3) is not satisfied, we can replaced g by $g^\tau = \{g_{\tau\wedge n}, n \geq 1\}$ where $\tau = \inf\{n: |g_n - y| \geq 1\}$. On the set $\{\tau < \infty\}$,

$$|(g^\tau)_\infty - y| = |g_\tau - y| \geq 1,$$

and, on the set $\{\tau = \infty\}$, the inequality $|g_n - y| < 1$ holds for all n but $\sup_n |g_n - y| \geq 1$ a.e., so g^τ satisfies (7.3) a.e. The martingale f^τ starts at $(x+y)/2$ and g^τ is the transform of f^τ by v. Furthermore, $E\Phi(|f_{\tau\wedge n}|) \leq E\Phi(|f_n|)$, an immediate consequence of the martingale property of f; see [10]. So we can and do assume in the proof of (7.2) that (7.3) holds.

Let $Z = (Z_1, Z_2, \cdots)$, where $Z_n = (X_n, Y_n)$, be the martingale with values in $\mathbb{R}^2$ defined by

$$(7.4) \qquad X_n = x + \textstyle\sum_{k=2}^n (1 + v_k)\, d_k,$$

$$(7.5) \qquad Y_n = y + \textstyle\sum_{k=2}^n (1 - v_k)\, d_k.$$

Note that (3.3) and (3.4) are satisfied but that here

$$(7.6) \qquad (X_{n+1} - X_n)(Y_{n+1} - Y_n) \geq 0.$$

Also note that $Z_\infty = (X_\infty, Y_\infty) \notin D$ a.e., where D has the same meaning as in Section 4, since $|X_\infty - Y_\infty|/2 = |g_\infty - y| \geq 1$ a.e. The argument leading to (7.3) allows us to assume that

$$(7.7) \qquad Z_{n+1} = Z_n \text{ on the set } \{Z_n \notin D\}.$$

We now extend to $\mathbb{R}^2$ the function u in the statement of the lemma by defining

$$u(x, y) = \Phi\left(\left|\frac{x+y}{2}\right|\right)$$

for $(x, y) \notin D \cup \partial D$. This extension, also denoted by u, is biconvex on the whole of $\mathbb{R}^2$. For example, if $x > 1$, then

$$\begin{aligned} 2u_x(x+, x-2) - 2u_x(x-, x-2) &= \Phi'(x-1) - [-A'(x) + 2B'(x)] \\ &= A'(x) - B'(x) \\ &= e^x \int_x^\infty [B'(t) - B'(x)]e^{-t}\,dt > 0. \end{aligned}$$

We shall also use the fact that u is strictly convex along any line with positive slope: Let $h > 0$ and $k > 0$. Then φ, defined by $\varphi(\alpha) = u(x + \alpha h, y + \alpha k)$, has a strictly increasing right-hand derivative on $\mathbb{R}$. A key element in the proof of this is that, on any neighborhood of $(x + \alpha h, y + \alpha k)$ in which u is twice continuously differentiable, $u_{xy} > 0$ so that

(7.8) $$\varphi''(\alpha) = h^2 u_{xx} + 2hk u_{xy} + k^2 u_{yy} > 0.$$

Suppose that $h > k > 0$. (The case $h = k > 0$ is also easy and the remaining case is symmetric). As in Section 4, the derivatives u_x and u_y are continuous on D. Therefore, by (7.8), φ' is continuous and strictly increasing on the interval

$$\{\alpha \in \mathbb{R} : (x + \alpha h, y + \alpha k) \in D\}.$$

Also, for example, if $x > 1$ and $y = x - 2$, then

$$2\varphi'(0+) - 2\varphi'(0-) = [A'(x) - B'(x)](h - k) > 0.$$

Accordingly, for $(x, y) \in D$ and $(x + h, y + k) \in \mathbb{R}^2$ with $hk \geq 0$,

(7.9) $$u(x + h, y + k) \geq u(x, y) + u_x(x, y)h + u_y(x, y)k$$

and strict inequality holds if $hk > 0$ or, more generally, if $hk \geq 0$ and $(x + h, y + k) \notin Q(x, y)$, exactly the same set as in Section 6.

We now show that

(7.10) $$Eu(Z_{n+1}) \geq Eu(Z_n).$$

Let I denote the indicator function of D and J the indicator function of its complement. Then, by (7.7), $Eu(Z_{n+1})J(Z_n) = Eu(Z_n)J(Z_n)$ so it suffices to show that

(7.11) $$Eu(Z_{n+1})I(Z_n) \geq Eu(Z_n)I(Z_n).$$

But $u_x(Z_n)I(Z_n)(X_{n+1} - X_n)$ and $u_y(Z_n)I(Z_n)(Y_{n+1} - Y_n)$ are integrable, as we show below, so (7.11) follows at once from (7.9) and the fact that Z is a martingale satisfying (7.6).

The integrability of $u_x(Z_n)I(Z_n)(X_{n+1} - X_n)$ follows from

(7.12) $$|u_x(Z_n)I(Z_n)| \leq \tfrac{1}{2}\,\Phi'(|f_n|) + c,$$

(7.13) $$|X_{n+1} - X_n| \le 2|d_{n+1}| \le 2|f_{n+1}| + 2|f_n|,$$

(7.14) $$b\Phi'(a) \le 2\Phi(a) + 2\Phi(b), \quad a, b \ge 0,$$

and the integrability of $\Phi(|f_n|)$ and $\Phi(|f_{n+1}|)$. Here c is a suitably chosen positive constant and $\Phi'(0)$ denotes the vanishing right-hand derivative at 0. On D the derivative u_x is continuous up to the boundary so u_x is locally bounded on D. Therefore, by symmetry, it is enough to consider u_x for $(x, y) \in D$ with $x \ge 2$. Then, using the convexity of $u(\cdot, y)$, we have that

$$u_x(x, y) \le u_x(y + 2+, y) = ½\, \Phi'(y + 1).$$

Since Φ' is increasing but Φ'' is decreasing,

$$u_x(x, y) \le \frac{1}{2}\,\Phi'\!\left(\frac{x+y}{2} + 2\right) \le \frac{1}{2}\,\Phi'\!\left(\frac{x+y}{2}\right) + \Phi''(1),$$

which gives (7.12). To show (7.14), assume that $a > 0$. Then, since Φ' is concave, $\Phi'(a)/a \le \Phi'(t)/t$ for $0 < t \le a$. Therefore,

$$a\Phi'(a) = 2\int_0^a t\Phi'(a)/a\,dt \le 2\int_0^a \Phi'(t)\,dt = 2\Phi(a),$$

which, by monotonicity, implies (7.14). By symmetry, integrability for u_y also follows. This completes the proof of (7.10).

The next step is to observe that

(7.15) $$\lim_{n\to\infty} \| u(Z_n) - \Phi(|f_n|) \|_1 = 0.$$

Since u and Φ are continuous and $Z_\infty \notin D$ a.e., the integrand converges to $u(Z_\infty) - \Phi(|f_\infty|) = 0$ a.e., and by (7.18) below,

(7.16) $$0 \le u(Z_n) - \Phi(|f_n|) \le u(0, 0).$$

So, by the Lebesgue dominated convergence theorem, (7.15) holds.

By (7.10), (7.15), and the inequality

$$Eu(Z_n) \le E\Phi(|f_n|) + \| u(Z_n) - \Phi(|f_n|) \|_1,$$

we have that

(7.17) $$u(x, y) = Eu(Z_1) \le \cdots \le \lim_n Eu(Z_n) \le \lim_n E\Phi(|f_n|).$$

Furthermore, $Eu(Z_n) < Eu(Z_{n+1})$ must hold for some positive integer n. This is even simpler to prove than the corresponding result of Section 6 since here $Z_\infty \notin D$ a.e. and $Z_{n+1} = Z_n$ on $\{Z_n \notin D\}$. For example, here the probability in (6.15) must vanish since $(x, x - 2) \notin D$. Apart from the proof of (7.18), given below, this establishes the desired inequality (7.2). The example of Section 6 also shows here that the bound on the left-hand side of (7.2) is best possible. (Note, in this connection, that the α defined in the last paragraph of Section 4 is an increasing function of δ for $0 < \delta < 2$. Therefore, the monotone convergence theorem can be used in place of Fatou's Lemma to obtain the dual of inequalities such as (4.25) in which the inequality sign must be reversed.)

We now show that

$$0 \le u(x, y) - \Phi\left(\left|\frac{x+y}{2}\right|\right) \le u(0, 0) \tag{7.18}$$

for all $(x, y) \in \mathbb{R}^2$. This gives (7.16). Equality holds on the left if and only if $(x, y) \notin D$, and on the right if and only if $x = y = 0$. We can assume in the proof that $\Phi(1) = 1$. Then

$$\Phi(x) \ge x^2, \quad 0 \le x \le 1. \tag{7.19}$$

To see this, fix a number x in the open unit interval. If $\Phi'(x) \ge 2x$, then, by the concavity of Φ', we have that $\Phi'(t) \ge 2t$ for $0 < t \le x$, so

$$\Phi(x) = \int_0^x \Phi'(t)\, dt \ge \int_0^x 2t\, dt = x^2.$$

On the other hand, if $\Phi'(x) < 2x$, then $\Phi'(t) < 2t$ for $x \le t \le 1$, so

$$\Phi(x) = 1 - \int_x^1 \Phi'(t)\, dt > 1 - \int_x^1 2t\, dt = x^2.$$

We shall also need

$$u(0, 0) < [u(2, 0) + u(-2, 0)]/2 = \Phi(1) = 1 \tag{7.20}$$

and, for $x \ge 1$,

$$A'(x) = e^x \int_x^\infty B'(t)e^{-t}\, dt < B'(x+1) = \Phi'(x). \tag{7.21}$$

The latter inequality follows from Jensen's inequality and

$$e^x \int_x^\infty te^{-t}\, dt = x + 1.$$

Now consider the right-hand term minus the middle term in (7.18) in the case $x = y \ge 0$: Let $M(x) = \Phi(x) + u(0, 0) - u(x, x)$. Then $M(0) = 0$ and, for $0 < x \le 1$, we have, by (7.19) and (7.20), that

$$M(x) = \Phi(x) - u(0, 0)x^2 > \Phi(x) - x^2 \ge 0.$$

If $x > 1$, then $M'(x) = \Phi'(x) - A'(x)$, which, by (7.21), is positive, so $M(x) > 0$ here also. Now fix a nonnegative number x and consider

$$N(s) = \Phi(x) + u(0, 0) - u(x + s, x - s)$$

for $0 \le s < \infty$. Then, for $0 < s < 1$,

$$N'(s) = u_y(x + s, x - s) - u_x(x + s, x - s) = 2su(0, 0) \quad \text{if } x + s < 1,$$

$$N'(s) = s[A'(x + s) - B'(x + s)] \quad \text{if } x + s > 1.$$

In both of these cases $N'(s) > 0$. (If $x + s = 1$, the third possible case, then $N'(s) = sA(1)$ and N' is continuous on $(0, 1)$ as it should be.) Therefore, for $0 < s \le 1$,

$$N(s) > N(0) = M(x) \tag{7.22}$$

and, since $N(s) = N(1) = u(0, 0)$ for $s > 1$, the inequality (7.22) holds for all $s > 0$. By symmetry, the right-hand side of (7.18) follows and equality holds if and only if $x = y = 0$. In a similar way, the left hand side is a consequence of

$$u(x + s, x - s) - \Phi(x) = N(1) - N(s) > 0, \quad 0 \le s < 1.$$

This completes the proof of Lemma 7.1.

The effect of the concavity of Φ' is to interchange the definitions of L_Φ and U_Φ given in Section 6. Here L_Φ is the function u of Lemma 7.1, that is, $2L_\Phi$ is given by (6.18) and (6.19) while $2U_\Phi$ is given by (6.20) and (6.21).

THEOREM 7.2. *The function L_Φ is the greatest biconvex function u on $D \cup \partial D$ such that*

$$u(x, y) \le \Phi\left(\left|\frac{x + y}{2}\right|\right) \quad \textit{if} \quad (x, y) \in \partial D. \tag{7.23}$$

Similarly U_Φ is the least biconcave function u on $D \cup \partial D$ such that

$$u(x, y) \ge \Phi\left(\left|\frac{x + y}{2}\right|\right) \quad \textit{if} \quad (x, y) \in \partial D. \tag{7.24}$$

PROOF. By Lemma 7.1 and its proof,

$$L_\Phi(x, y) = \inf_Z \sup_n E\Phi(|f_n|) \tag{7.25}$$

where $f_n = (X_n + Y_n)/2$ and $Z = (X, Y)$ is a zigzag martingale starting at (x, y) with the property that almost everywhere the distance of Z_n to the complement of D is not bounded away from 0. In fact, the examples showing that the bound in (7.2) is best possible can be slightly modified (move immediately to ∂D after a suitably large number of steps) so that, in (7.25), Z may be assumed to have values in $D \cup \partial D$ and a finite number of values altogether, so that, for some positive integer n,

$$Z_n = Z_{n+1} = \cdots = Z_\infty \in \partial D.$$

The remainder of the proof is parallel to that of Theorem 6.2.

REMARK 7.1. Suppose that Φ' is merely concave and not necessarily strictly concave. Then Theorem 7.2 holds without change but the possibilty of equality must be allowed in (7.1) and (7.2). A similar remark applies to Section 6.

Consider the example $\Phi(t) = 2t$, which will be of particular interest below. Let u be the continuous function defined on $\mathbb{R}^2$ by

$$\begin{aligned} u(x, y) &= 1 + xy \quad &&\text{if } |x| \vee |y| \le 1, \\ &= |x + y| \quad &&\text{if } |x| \vee |y| > 1. \end{aligned} \tag{7.26}$$

The analogue of (7.2) here is

$$u(x, y) \le 2\|f\|_1 \tag{7.27}$$

where f is a real martingale starting at $(x+y)/2$ with $(x, y) \in \mathbb{R}^2$ such that for some real predictable sequence $v = (1, v_2, v_3, \cdots)$ uniformly bounded in absolute value by 1, the transform g of f by v satisfies $\sup_n |g_n - y| \ge 1$ almost everywhere.

We sketch a direct and simple proof of this. It is elementary to check that, for $(x, y) \in \mathbb{R}^2$,

$$0 \le u(x, y) - |x + y| \le 1, \tag{7.28}$$

and that if $(x + h, y + k) \in \mathbb{R}^2$ with $hk \ge 0$, then

$$u(x + h, y + k) \ge u(x, y) + \varphi(x, y)h + \psi(x, y)k \tag{7.29}$$

where $\varphi(x, y) = y$ and $\psi(x, y) = x$ if $|x| \vee |y| \le 1$, and $\varphi(x, y) = \psi(x, y) = \operatorname{sgn}(x + y)$ elsewhere in $\mathbb{R}^2$. The functions φ and ψ are bounded and measurable so the martingale property of Z implies that

$$E\varphi(Z_n)(X_{n+1} - X_n) = 0$$

with a similar result for ψ. Therefore,

$$u(x, y) \le Eu(Z_1) \le \cdots \le Eu(Z_n). \tag{7.30}$$

Using (7.28) and the permissible assumption that $Z_\infty \notin D$ a.e., we have (7.15) and (7.17) here so (7.27) holds.

To obtain (7.27), we assumed that f could be controlled to satisfy $\sup_n |g_n - y| \ge 1$ a.e. In the following, we assume that f can be controlled to satisfy a one-sided condition: $\sup_n g_n \ge \beta$ a.e. The gambling interpretation of this is obvious.

THEOREM 7.3. *Suppose that α and β are real numbers. If f is a real martingale starting at α and for some real predictable sequence $v = (1, v_2, v_3, \cdots)$ uniformly bounded in absolute value by* 1, *the transform g of f by v satisfies*

$$P(\sup_n g_n \ge \beta) = 1, \tag{7.31}$$

then

$$\|f\|_1 \ge (\beta - \alpha) \vee |\alpha| \tag{7.32}$$

and the bound on the right is best possible. In fact, there is a real martingale f starting at α such that f satisfies (7.32) *with equality and the transform of f by* $(1, -1, 1, -1, \cdots)$ *satisfies* (7.31).

PROOF. Suppose that f is a real martingale starting at α and for some real predictable sequence $v = (1, v_2, v_3, \cdots)$ uniformly bounded in absolute value by 1, the transform g of f by v satisfies (7.31). Let x and y be defined by $\alpha = (x + y)/2$ and $\beta = y + 1$. For each positive integer j, let f^j and g^j be the martingales satisfying $f^j_n = 2f_n/(j + 1)$ and $g^j_n = 2g_n/(j + 1)$. Note that g^j is the transform of

f^j by the above predictable sequence v, that f^j starts at $(x_j + y_j)/2$ where

$$x_j = 2x/(j+1) + (j-1)/(j+1),$$
$$y_j = 2y/(j+1) - (j-1)/(j+1),$$

and that, by (7.31), $\sup_n(g_n^j - y_j) \geq 1$ a.e. Therefore, by (7.27),

$$u(x_j, y_j) \leq 2\|f^j\|_1 = 4\|f\|_1/(j+1). \tag{7.33}$$

Now multiply both sides of this inequality by $(j+1)/4$ and let $j \to \infty$ to obtain (7.32).

We now show that there does exist a real martingale f starting at α such that f satisfies (7.32) with equality and the transform of f by some predictable sequence $(1, v_2, v_3, \cdots)$ satisfies (7.31) where each v_k has values in $\{-1, 1\}$. This is not quite the assertion of the theorem but does imply it by the first part of the proof of Lemma 2.1; see (2.4) and (2.5).

There are four cases. (i) If $\beta \leq \alpha$, let $f \equiv (\alpha, \alpha \cdots)$. (ii) If $\alpha < \beta \leq 0$, let $v \equiv (1, 1, \cdots)$ and f be any nonpositive martingale starting at α that converges a.e. to 0. (iii) If $\beta > 0$ and $\beta/2 \leq \alpha < \beta$, let $v = (1, -1, -1, -1, \cdots)$ and f be the double-or-nothing martingale starting at α: f_n is $2^{n-1}\alpha$ times the indicator function of $[0, 1/2^{n-1})$. (iv) Now consider the remaining case: $\beta > 0$ and $\alpha < \beta/2$. Our example here uses the double-or-nothing idea twice. Let σ be a measurable function from $[0, 1)$ to $\{2, 3, \cdots\}$ and $\tau > \sigma$ a similar function such that

$$P(\sigma = m, \tau = n) = 1/2^{n-1} \quad \text{if} \quad 2 \leq m < n.$$

Define f and v on $[0, 1)$ by

$$\begin{aligned} f_n(s) &= 2^{n-1}(\alpha - \beta/2) + \beta/2 \quad \text{if } n < \sigma(s), \\ &= 2^{n-\sigma(s)}\beta/2 \quad \text{if } \sigma(s) \leq n < \tau(s), \\ &= 0 \quad \text{if } n \geq \tau(s), \end{aligned} \tag{7.34}$$

$$\begin{aligned} v_n(s) &= 1 \quad \text{if } n \leq \sigma(s), \\ &= -1 \quad \text{if } n > \sigma(s). \end{aligned} \tag{7.35}$$

It is easy to check that f is a martingale and v is predictable relative to f. Furthermore, f satisfies (7.32) with equality and the transform of f by v satisfies (7.31).

REMARK 7.2. Let u be the continuous function on $D \cup \partial D$ satisfying the symmetry condition (4.4) and such that

$$\begin{aligned} u(x, y) &= x + y + (y - x + 2)e^{-y} \quad \text{if } 0 \leq y \leq x \leq y + 2, \\ &= 2(1 + y) - (y - x + 2)\log(1 + y) \\ &\qquad \text{if } -1 < y \leq 0, -y \leq x \leq y + 2. \end{aligned} \tag{7.36}$$

This is the least biconcave function on $D \cup \partial D$ such that $|x + y| \leq u(x, y)$ if $(x, y) \in \partial D$, the upper solution U_Φ for $\Phi(t) = 2t$. Note that $u_{xx} = 0$ and $u_{yy} < 0$

on the set where $x + y > 0$ and $x - 2 < y < x$. This contrasts with the lower solution L_Φ given by the restriction of (7.26) to $D \cup \partial D$ and the extremal, or nearly extremal, zigzag martingales Z must be chosen accordingly.

8. Weak-type inequalities. To illustrate the method, we begin by giving a new and elementary proof of the following weak-type inequality from [3] and [4]: If f is a real martingale and g is the transform of f by a real predictable sequence v uniformly bounded in absolute value by 1, then

$$\lambda P(g^* \geq \lambda) \leq 2 \|f\|_1, \quad \lambda > 0. \tag{8.1}$$

To prove this we can assume that $f_1 \equiv 0$, Z is defined by (7.4) and (7.5) with $x = y = 0$, and (7.7) holds. Then $\{g_n^* \geq 1\} = \{Z_n \notin D\}$ where

$$g_n^*(s) = \sup_{1 \leq k \leq n} |g_k(s)|.$$

Let u be the continuous function defined on $\mathbb{R}^2$ by (7.26). Then, by (7.30),

$$1 = u(0, 0) = Eu(Z_1) \leq \cdots \leq Eu(Z_n). \tag{8.2}$$

Let I denote the indicator function of D. Then, by (8.2),

$$P(g_n^* \geq 1) = 1 - EI(Z_n) \leq E[u(Z_n) - I(Z_n)].$$

By (7.26) and (7.28), the latter integrand is not greater than $|X_n + Y_n| = 2|f_n|$ and this gives

$$P(g^* > 1) \leq 2 \|f\|_1.$$

Thus $\lambda P(g^* > \lambda) \leq 2 \|f\|_1$ for all $\lambda > 0$, which implies (8.1) and, indeed, gives (1.3) in the case $p = 1$.

In the case $p = 2$, the inequality (1.3) is especially elementary, of course, but can also be obtained by the above method with the use of the function u defined by $u(x, y) = 1 + xy$ for all $(x, y) \in \mathbb{R}^2$. Here $u(Z_n) - I(Z_n) \leq |f_n|^2$.

We now prove (1.3) in the case $1 < p < 2$ by proving a little more. As in Section 7, Φ is an increasing convex function on $[0, \infty)$ such that Φ is twice differentiable on $(0, \infty)$, Φ' is strictly concave, and $\Phi(0) = \Phi'(0+) = 0$.

THEOREM 8.1. *If f is a real martingale and g is the transform of f by a real predictable sequence v uniformly bounded in absolute value by* 1, *then, for all* $\lambda > 0$,

$$P(g^* \geq \lambda) \leq 2 \sup_n E\Phi(|f_n|/\lambda) \Big/ \int_0^\infty \Phi(t) e^{-t}\, dt. \tag{8.3}$$

The constant on the right is best possible. Equality holds if and only if $f = (0, 0, \cdots)$ almost everywhere.

PROOF. We can assume that $\lambda = 1$, $\sup E\Phi(|f_n|) < \infty$, $f_1 \equiv 0$, Z is defined by (7.4) and (7.5) with $x = y = 0$, and (7.7) holds. Then, as in Section 7, the almost everywhere limit of Z exists and satisfies

$$P(Z_\infty \notin D) = P(|g_\infty| \geq 1) = P(g^* \geq 1).$$

Let u be the function used in the proof of Lemma 7.1. Then, by (7.17),

$$u(0, 0) \leq Eu(Z_n).$$

Therefore, with I denoting the indicator function of D as before,

$$\begin{aligned} &u(0, 0)P(g^* \geq 1) \\ &\quad = u(0, 0) - u(0, 0)EI(Z_\infty) \\ &\quad \leq E[u(Z_n) - u(0, 0)I(Z_\infty)] \\ &\quad = E\Phi(|f_n|) + E[u(Z_n) - u(0, 0)I(Z_\infty) - \Phi(|f_n|)]. \end{aligned}$$

The last integrand is dominated from above by $u(0, 0)$, so, by Fatou's lemma,

$$\begin{aligned} &u(0, 0)P(g^* \geq 1) \\ &\quad \leq \sup_n E\Phi(|f_n|) + E[u(Z_\infty) - u(0, 0)I(Z_\infty) - \Phi(|f_\infty|)]. \end{aligned} \tag{8.4}$$

By (7.18) and the definition of u on the complement of D,

$$u(Z_\infty) - u(0, 0)I(Z_\infty) \leq \Phi(|f_\infty|)$$

with strict inequality holding on the set $\{Z_\infty \in D, Z_\infty \neq (0, 0)\}$. Therefore,

$$u(0, 0)P(g^* \geq 1) \leq \sup_n E\Phi(|f_n|), \tag{8.5}$$

which is (8.3) in the case $\lambda = 1$.

If $f = (0, 0 \cdots)$ almost everywhere, then both sides of (8.5) vanish and equality holds. To go the other way, assume that equality holds in (8.5). Then

$$P(Z_\infty \notin D \text{ or } Z_\infty = (0, 0)) = 1 \tag{8.6}$$

for otherwise the second term on the right-hand side of (8.4) would be negative. Also, (4.21) holds, otherwise, for some positive integer m and all $n \geq m$, $u(0, 0) < Eu(Z_m) \leq Eu(Z_n)$. By (4.21) and (8.6),

$$P(Z_\infty = (1, -1) \text{ or } (-1, 1) \text{ or } (0, 0)) = 1$$

so $f_\infty = 0$ a.e. Also, by (4.21), $|f_n| \leq 1$ a.e., so

$$\sup_n E\Phi(|f_n|) = \lim_n E\Phi(|f_n|) = E\Phi(|f_\infty|) = 0.$$

Since $\Phi(t) > 0$ for $t > 0$, we can conclude that $f = (0, 0, \cdots)$ a.e.

The constant $2/\int_0^\infty \Phi(t)e^{-t}\,dt$ is best possible since it is already best possible, by Theorem 7.1, in the special case $P(g^* \geq 1) = 1$.

This completes the proof of Theorem 8.1 and Theorem 1.3.

REMARK 8.1. There is a boundary value problem underlying the above proof (cf. Theorem 3.2 of [5]): Find the greatest function u defined on

$$\{(x, y, t) \in \mathbb{R}^3 : |x - y| \leq 2, 0 \leq t \leq 1\} \tag{8.7}$$

such that

(8.8) if $h, k \geq 0$, then the mapping $(s, t) \to u(x + sh, y + sk, t)$ is convex,

$$u(x, y, 1) \leq \Phi\left(\left|\frac{x+y}{2}\right|\right) \text{ if } |x - y| = 2, \tag{8.9}$$

$$u(x, y, 0) \leq \Phi\left(\left|\frac{x+y}{2}\right|\right). \tag{8.10}$$

The function u defined by

$$u(x, y, t) = u(0, 0)t + [u(x, y) - u(0, 0)],$$

where $u(x, y)$ is as above, satisfies (8.8), (8.9), and (8.10) but is not the greatest such function. However, on the set

$$\{(0, 0, t): 0 \leq t \leq 1\}$$

it agrees with the greatest such function and this is enough to obtain Theorem 8.1.

9. Best possible bounds in the L^p case. Our aim here is to extend Theorem 1.1 and to check that the function u discovered in the course of its proof is indeed the upper solution of the boundary value problem of Section 5. The lower solution, which is equal to the upper solution only in the elementary case $p = 2$, is an easy consequence.

Let U_p be the function on $\Omega \cup \partial\Omega$ defined as follows: If $1 < p < 2$, then U_p is the function u of Lemma 5.4; if $2 < p < \infty$, then U_p is the function u of Lemma 5.2; and

$$U_2(x, y, t) = t + xy.$$

Similarly, let L_p be the function on $\Omega \cup \partial\Omega$ with $L_p(x, x, 0) = |x|^p$ and such that, for all $(x, y, t) \in \Omega \cup \partial\Omega$ with $t > 0$,

$$L_p(x, y, t) = tF(xt^{-1/p}, yt^{-1/p})$$

with F here the continuous function on $D \cup \partial D$ satisfying the symmetry condition (5.28) and the following additional property: If $1 < p < 2$, the restriction of F to D_1 is given by (5.23) and the restriction of F to D_2 is given by (5.26); if $2 < p < \infty$, the restriction of F to D_3 is given by (5.86) and the restriction of F to D_4 is given by (5.88); and $F(x, y) = 1 + xy$ in case $p = 2$.

If $(x, y, t) \in \Omega \cup \partial\Omega$, these two functions are related by the following identity:

$$L_p(x, -y, U_p(x, y, t)) = t = U_p(x, -y, L_p(x, y, t)). \tag{9.1}$$

LEMMA 9.1. *Suppose that* $1 < p < \infty$ *and* $(x, y, t) \in \Omega \cup \partial\Omega$. *If* f *is a real martingale starting at* $(x + y)/2$ *and, for some sequence* $(1, \varepsilon_2, \varepsilon_3, \cdots)$ *in* $\{-1, 1\}$, *the transform* g *of* f *by this sequence satisfies* $\sup_n \| g_n - y \|_p^p = t$, *then, for* $p \neq 2$

and $(x, y, t) \in \Omega$,

$$L_p(x, y, t) < \|f\|_p^p < U_p(x, y, t). \tag{9.2}$$

If $p = 2$ *or* $(x, y, t) \in \partial\Omega$, *equality holds. The bounds in* (9.2) *are best possible.*

PROOF. We must check, among other things, that if $p > 2$ and $(x, y, t) \in \Omega$, then

$$\|f\|_p^p < u(x, y, t) \tag{9.3}$$

where u is the function of Lemma 5.2. Up to the proof of strict inequality, the reasoning is similar to that in the proof of (5.71): Here the zigzag martingale Z, generated by f and g as in Section 3, starts at (x, y) and the martingale $T = (T_1, T_2, \cdots)$, defined by

$$T_n = E(|g_\infty - y|^p \mid \mathscr{A}_n), \tag{9.4}$$

satisfies $T_1 \equiv E|g_\infty - y|^p = t$ so $W = (X, Y, T) = (Z, T)$ starts at (x, y, t). Also, both (5.72) and (5.73) hold as before so

$$\|f\|_p^p = \lim_{n\to\infty} \|f_n\|_p^p \le \lim_{n\to\infty} Eu(W_n) \le Eu(W_1) = u(x, y, t). \tag{9.5}$$

To prove strict inequality, we shall need to consider the set $Q(x, y, t)$ of all points $(x + h, y + k, t + r)$ in $\Omega \cup \partial\Omega$ where either $h = 0$ or $k = 0$, as usual, and equality holds in (5.70). Let

$$\Omega_i = \{(x, y, t) \in \Omega\colon (xt^{-1/p}, yt^{-1/p}) \in D_i\}$$

for $i = 1, 2$. If $(x, y, t) \in \Omega_1$, then

$$Q(x, y, t) = \left\{(x, y + k, t + r) \in \Omega_1 \cup \partial\Omega_1 : k = \frac{2xr}{pt}\left[1 - \frac{p(x - y)}{2x}\right]\right\} \tag{9.6}$$

and on this set

$$u(x, y + k, t + r) = \frac{t + r}{t}\, u(x, y, t). \tag{9.7}$$

If $(x, y, t) \in (\Omega_2 \cup \partial\Omega_2) \cap \Omega$, then

$$Q(x, y, t) = \left\{(x, y, t + r)\colon \left|\frac{x - y}{2}\right|^p \le t + r\right\} \tag{9.8}$$

and here

$$u(x, y, t + r) = u(x, y, t) + ru(0, 0, 1). \tag{9.9}$$

If $x > 0$ and $t > 0$, then

$$\begin{aligned} Q(x, x, t) = {} & \{(x + h, x, t + r)\colon (x, x + h, t + r) \in \Omega_1 \cup \partial\Omega_1 \text{ and } pth = 2xr\} \\ & \cup \{(x, x + k, t + r) \in \Omega_1 \cup \partial\Omega_1 \text{ and } ptk = 2xr\}. \end{aligned} \tag{9.10}$$

For example, to prove (9.7) we may assume by continuity that $(x, y+k, t+r) \in \Omega_1$. Under this assumption, (9.7) follows immediately from (5.29), (5.23), (5.25), and the fact that

$$t\left[1-\frac{p(x-y-k)}{2x}\right] = (t+r)\left[1-\frac{p(x-y)}{2x}\right]$$

for k and r as in (9.6). Also, under this assumption,

$$Q(x, y+k, t+r) \subset Q(x, y, t) \tag{9.11}$$

and this follows from (9.6). On Ω_2,

$$u(x, y, t) = \left|\frac{x+y}{2}\right|^p + \left(t - \left|\frac{x-y}{2}\right|^p\right)(p-1)^p \tag{9.12}$$

and this implies (9.9). Now (9.7) and (9.9) together with the other properties of u described in Section 5 lead without difficulty to (9.6), (9.8), and (9.10). For example, the strict concavity of $u(\cdot, y, \cdot)$ on Ω_1 implies that, for $(x, y, t) \in \Omega_1$, the set $Q(x, y, t)$ does not contain any point $(x+h, y, t+r)$ with $(h, r) \neq (0, 0)$.

To complete the proof of (9.3), we must show that strict inequality holds somewhere in (9.5). If, on the contrary, $\|f\|_p^p = u(x, y, t)$, then

$$\lim_{n\to\infty} \|u(W_n) - |f_n|^p\|_1 = 0 \tag{9.13}$$

and

$$Eu(W_{n+1}) = Eu(W_n), \quad n \geq 1. \tag{9.14}$$

But it is impossible for both (9.13) and (9.14) to hold as can be seen by using the sets $Q(x, y, t)$ and an argument similar to that used in Section 6.

The proof of the right-hand side of (9.2) for $1 < p < 2$ is similar. The proof of the left-hand side in the case $p \neq 2$ can be based on the identity (9.1).

So the strict inequality (9.2) holds for $p \neq 2$ and $(x, y, t) \in \Omega$. On the other hand, suppose that $p = 2$. Then d is an orthogonal sequence in L^2 so that

$$\begin{aligned}\|f_n\|_2^2 &= \left|\frac{x+y}{2}\right|^2 + \textstyle\sum_{k=2}^n Ed_k^2 = xy + \left\|\frac{x-y}{2} + \textstyle\sum_{k=2}^n \varepsilon_k d_k\right\|_2^2 \\ &= \|g_n - y\|_2^2 + xy.\end{aligned}$$

Taking the supremum of both sides with respect to n, we obtain

$$\|f\|_2^2 = t + xy = L_2(x, y, t) = U_2(x, y, t).$$

If $1 < p < \infty$ but $(x, y, t) \in \partial\Omega$, then

$$t = \left|\frac{x-y}{2}\right|^p = \|g_1 - y\|_p^p \leq \sup_n \|g_n - y\|_p^p = t$$

so that $E|g_n - y|^p = |E(g_1 - y)|^p$. Thus, $g_n = g_1$ a.e. for all $n \geq 2$ so that $d_2 =$

$d_3 = \cdots = 0$ a.e. implying that

$$\|f\|_p^p = \left|\frac{x+y}{2}\right|^p = L_p(x, y, t) = U_p(x, y, t).$$

It remains to show that the bounds in (9.2) are best possible. Consider the case $2 < p < \infty$. By homogeneity and continuity, the example of Section 5 implies that, for $x = y \geq 0$ and $t > 0$, the function $u = U_p$ yields the best possible upper bound $u(x, x, t)$. If $(x, y, t) \in \Omega_1$ and (x, x, t_0) and (x, y_∞, t_∞) are the endpoints of the line segment $Q(x, y, t)$ as above, we can proceed as follows. Let $\delta > 0$ and (f^0, g^0) be a pair of martingales satisfying the assumptions of Lemma 9.1 with respect to (x, x, t_0) and such that

$$\lim_n Eu(W_n^0) = \|f^0\|_p^p > u(x, x, t_0) - \delta$$

where W^0 is the martingale with values in $\Omega \cup \partial\Omega$ determined by f^0 and g^0. Choose α so that

$$(x, y, t) = \alpha(x, x, t_0) + (1 - \alpha)(x, y_\infty, t_\infty)$$

and let $W_1 \equiv (x, y, t)$ and, for $n \geq 1$,

$$\begin{aligned} W_{n+1}(s) &= W_n^0(s/\alpha) && \text{if } 0 \leq s < \alpha, \\ &= (x, y_\infty, t_\infty) && \text{if } \alpha \leq s < 1. \end{aligned}$$

Then W is a martingale starting at (x, y, t) with values in $\Omega \cup \partial\Omega$. It has the form (Z, T) where Z is a zigzag martingale. Let f and g be generated by Z as in Section 3. Using (9.7), we have

$$\begin{aligned} Eu(W_{n+1}) &= \int_0^\alpha u(W_n^0(s/\alpha))\, ds + \int_\alpha^1 u(x, y_\infty, t_\infty)\, ds \\ &= \alpha Eu(W_n^0) + (1 - \alpha)u(x, y_\infty, t_\infty) \\ &> \alpha u(x, x, t_0) + (1 - \alpha)u(x, y_\infty, t_\infty) - \delta \\ &= u(x, y, t) - \delta. \end{aligned}$$

Similarly, $\sup_n \|g_n - y\|_p^p = t$. Therefore, f and g satisfy the assumptions of Lemma 9.1 and

$$\|f\|_p^p = \lim_n Eu(W_n) \geq u(x, y, t) - \delta$$

showing that the upper bound $u(x, y, t)$ is best possible. If $(x, y, t) \in \Omega_2 \cup \partial\Omega_2$, the proof is similar: In this case, let $y_\infty = y$, $t_\infty = |x - y|^p/2^p$, and t_0 be so large that $F(xt_0^{-1/p}, yt_0^{-1/p})$ is near $F(0, 0)$.

This completes the proof of Lemma 9.1.

THEOREM 9.1. *Suppose that* $(x, y, t) \in \Omega$ *and either* $1 < p < 2$ *or* $2 < p < \infty$. *If* f *is a real martingale starting at* $(x + y)/2$ *and, for some real predictable sequence* $v = (1, v_2, v_2, \cdots)$ *uniformly bounded in absolute value by* 1, *the transform* g *of* f

by this sequence satisfies $\sup_n \| g_n - y \|_p^p \geq t$, *then*

$$L_p(x, y, t) < \| f \|_p^p. \tag{9.15}$$

On the other hand, if v *is uniformly bounded away from the origin by* 1 *and* $\sup_n \| g_n - y \|_p^p \leq t$, *then*

$$\| f \|_p^p < U_p(x, y, t). \tag{9.16}$$

By Lemma 9.1, these bounds are best possible. The case $p = 2$ is elementary.

PROOF. To prove (9.15), we shall use the decomposition of g given in Lemma 2.1. Let $t_j = \sup_n \| G_n^j - y \|_p^p$. Then

$$t \leq \sup_n \| g_n - y \|_p^p = \sup_n \| \textstyle\sum_{j=1}^\infty 2^{-j}(G_{2n}^j - y) \|_p^p$$

$$\leq [\textstyle\sum_{j=1}^\infty 2^{-j} t_j^{1/p}]^p \leq \sum_{j=1}^\infty 2^{-j} t_j.$$

Since $v_1 \equiv 1$, both F^j and G^j start at $(x + y)/2$ so $G_1^j - y = (x - y)/2$ and $(x, y, t_j) \in \Omega \cup \partial\Omega$ for all j. But

$$\left| \frac{x - y}{2} \right|^p < t \leq \textstyle\sum_{j=1}^\infty 2^{-j} t_j$$

so $(x, y, t_j) \in \Omega$ for some j. Hence $L_p(x, y, t_j) \leq \| F^j \|_p^p = \| f \|_p^p$ for all j with strict inequality holding for some j. Here we have used (2.1) and Lemma 9.1. Therefore, by the monotonicity and convexity of $L_p(x, y, \cdot)$,

$$L_p(x, y, t) \leq \textstyle\sum_{j=1}^\infty 2^{-j} L_p(x, y, t_j) < \sum_{j=1}^\infty 2^{-j} \| f \|_p^p = \| f \|_p^p.$$

To prove (9.16), note that $(f_1 - y, f_2 - y, \cdots)$ is the transform of $(g_1 - y, g_2 - y, \cdots)$ by $(1, 1/v_2, 1/v_3, \cdots)$. If (9.16) does not hold, then

$$\sup_n \| (f_n - y) - (-y) \|_p^p \geq U_p(x, y, t)$$

and, by (9.1) and (9.15),

$$t = L_p(x, -y, U_p(x, y, t)) < \sup_n \| g_n - y \|_p^p,$$

a contradiction. This completes the proof of Theorem 9.1.

Here is another consequence of Lemma 9.1.

THEOREM 9.2. *Let* $1 < p < \infty$. *The function* L_p *is the greatest function* u *on* $\Omega \cup \partial\Omega$ *such that*

(9.17) *the mapping* $(x, t) \to u(x, y, t)$ *is convex on the section of* $\Omega \cup \partial\Omega$ *determined by* y,

(9.18) *the mapping* $(y, t) \to u(x, y, t)$ *is convex on the section of* $\Omega \cup \partial\Omega$ *determined by* x, *and*

$$u(x, y, t) \leq \left| \frac{x + y}{2} \right|^p \text{ if } (x, y, t) \in \partial\Omega. \tag{9.19}$$

Similarly, the function U_p is the least function u on $\Omega \cup \partial\Omega$ satisfying the dual conditions (5.4), (5.5), *and* (5.6).

PROOF. By Theorem 9.1,

$$U_p(x, y, t) = \sup_W \|f\|_p^p \tag{9.20}$$

where $f_n = (X_n + Y_n)/2$, $Z = (X, Y)$ is a zigzag martingale starting at (x, y), and $W = (X, Y, T) = (Z, T)$ is a martingale starting at (x, y, t) with values in $\Omega \cup \partial\Omega$. In view of the examples showing that the bounds in Lemma 9.1 are best possible, each term of W can be assumed to have only a finite number of values.

Suppose that u is a function on $\Omega \cup \partial\Omega$ satisfying (5.4), (5.5), and (5.6). If $W_{n+1} = (X_{n+1}, Y_n, T_{n+1})$, then almost everywhere

$$E[u(W_{n+1}) \mid W_n] \le u(E(X_{n+1} \mid W_n),\ Y_n,\ E(T_{n+1} \mid W_n)) = u(W_n)$$

and a similar inequality holds if $W_{n+1} = (X_n, Y_{n+1}, T_{n+1})$. Taking expectations, we have $Eu(W_{n+1}) \le Eu(W_n)$. Thus, $\|f_n\|_p^p \le Eu(W_n) \le \cdots \le u(x, y, t)$, which implies that $U_p(x, y, t) \le u(x, y, t)$.

The proof of the statement about L_p is similar; see the proof of Theorem 7.2.

10. The dyadic case. For many martingale inequalities, the best possible bounds in the dyadic case differ from those of the general case; here, however, the best possible bounds in the two cases are the same. Let f be a martingale with values in B where $B = \mathbb{R}$ or any other Banach space. Recall that f is *dyadic* if its difference sequence d satisfies $d_1 \equiv b_1$ for some $b_1 \in B$ and, for $n \ge 2$ and any nonempty set of the form

$$\{d_1 = b_1,\ \cdots,\ d_{n-1} = b_{n-1}\},$$

the restriction of d_n to this set either vanishes identically or has its values in $\{-b_n, b_n\}$ for some $b_n \in B$ with $b_n \ne 0$. Note that if f is a dyadic martingale and g is its transform by a sequence ε in $\{-1, 1\}$, then the corresponding zigzag martingale Z is also dyadic. The converse also holds; see Section 3.

Maurey [17] has shown that the best possible constant, whatever it might be, in the inequality

$$\|\textstyle\sum_{k=1}^n \varepsilon_k d_k\|_p \le c_p \|\sum_{k=1}^n d_k\|, \quad 1 < p < \infty,$$

is also best possible in the dyadic case. Our approach is quite different and rests on the simple fact that a nonnegative midpoint concave function is concave.

To illustrate our method in the setting of Section 6, we let U_Φ^0 be the function defined on $D \cup \partial D$ by

$$U_\Phi^0(x, y) = \sup_Z \sup_n E\Phi(|f_n|)$$

where Z is any dyadic zigzag martingale starting at (x, y) with values in $D \cup \partial D$. We claim that

$$U_\Phi^0(x, y) = U_\Phi(x, y), \tag{10.1}$$

which implies that the bounds in Lemma 6.1, Theorem 6.1, and Theorem 1.2 are already best possible in the dyadic case.

The inequality $U_\Phi^0 \le U_\Phi$ is an immediate consequence of (6.24). To prove the reverse inequality, note first that U_Φ^0 satisfies the boundary condition (6.22). The next step is to show that U_Φ^0 is biconcave, for example, that $U_\Phi^0(\cdot, y)$ is concave on $[y-2, y+2]$. But this follows at once from the fact that $U_\Phi^0(\cdot, y)$ is midpoint concave: If x_1 and x_2 belong to $[y-2, y+2]$ and $x = (x_1 + x_2)/2$, let Z^i be a dyadic zigzag martingale starting at (x_i, y) with values in $D \cup \partial D$, $i = 1, 2$. We can assume that Z^1 and Z^2 move vertically and horizontally together. Let $Z_1 \equiv (x, y)$, $Z_{n+1}(s) = Z_n^1(2s)$ if $s \in [0, 1/2)$ and $Z_{n+1}(s) = Z_n^2(2s-1)$ if $s \in [1/2, 1)$. Then $Z = (Z_1, Z_2, \cdots)$ is a dyadic zigzag martingale starting at (x, y) with values in $D \cup \partial D$. Therefore

$$\tfrac{1}{2}E\Phi(|f_n^1|) + \tfrac{1}{2}E\Phi(|f_n^2|) = E\Phi(|f_{n+1}|) \le U_\Phi^0(x, y),$$

which implies that $U_\Phi^0(\cdot, y)$ is midpoint concave. Therefore, U_Φ^0 is biconcave and $U_\Phi \le U_\Phi^0$ follows from Theorem 6.2.

11. A method for some general boundary value problems. We shall illustrate the method in $\mathbb{R}^2$. A set $S \subset \mathbb{R}^2$ is *biconvex* if each horizontal and vertical section of S is convex. Let S be a biconvex set of the form $S = D \cup B$ where B is nonempty. (Some examples in the case that D is a nonempty biconvex domain with a nonempty complement D^c: $B = \partial D$, $B = D \cup \partial D$, and $B = D^c$.) Let $\beta: B \to \mathbb{R}$. The problem is: If there exists at least one biconcave function u on S such that $u \ge \beta$ on B, find the least such function. There is of course the dual problem for biconvex functions. Note that we do not assume that β is measurable.

The solution is implicit in what we have already proved. Let $(x, y) \in S$ and $\mathbf{Z}(x, y)$ denote the set of all zigzag martingales with values in S such that $Z_1 \equiv (x, y)$, each term of Z takes on only a finite number of values, and, for some positive integer n, $Z_n = Z_{n+1} = \cdots$ with the pointwise limit Z_∞ having all of its values in B. We assume, and this is usually easy to check, that $\mathbf{Z}(x, y)$ is nonempty for all $(x, y) \in S$. Let

$$U_\beta(x, y) = \sup\{E\beta(Z_\infty): Z \in \mathbf{Z}(x, y)\}, \tag{11.1}$$

$$L_\beta(x, y) = \inf\{E\beta(Z_\infty): Z \in \mathbf{Z}(x, y)\}. \tag{11.2}$$

THEOREM 11.1. *The function U_β is the least biconcave function u on S such that $u \ge \beta$ on B provided at least one such function exists. Similarly, the function L_β is the greatest biconvex function u on S such that $u \le \beta$ on B provided at least one such function exists.*

PROOF. If u is a biconcave function on S and $u \ge \beta$ on B, then, for all $Z \in \mathbf{Z}(x, y)$, we have $\beta(Z_\infty) \le u(Z_\infty)$ so that

$$E\beta(Z_\infty) \le Eu(Z_\infty) \le \cdots \le Eu(Z_1) = u(x, y),$$

the monotonicity of $Eu(Z_n)$ following from Jensen's inequality as in the proof of Theorem 6.2. Therefore $U_\beta \le u$. By taking $Z_n \equiv (x, y)$, $n \ge 1$, we see that $U_\beta \ge \beta$ on B. By the splicing argument used in the proof of (10.1), U_β is biconcave.

This completes the proof of the first half of Theorem 11.1 and the proof of the second half is similar.

REMARK 11.1. Although the boundary theory needed in this paper is different from classical boundary theory, there are many analogies and connections. Biconvexity is analogous to subharmonicity. If a function is both biconvex and biconcave, it is *biaffine.* Since there are so few biaffine functions, the Dirichlet problem is usually not solvable. For example, consider the simple case of D the open unit disk of $\mathbb{R}^2$ and β a continuous function on ∂D. Then U_β and L_β are continuous on $D \cup \partial D$, with $L_\beta = \beta = U_\beta$ on ∂D, but the upper and lower solutions are not equal throughout D unless β is the restriction to ∂D of some function of the form

$$(x, y) \rightarrow a_0 + a_1 x + a_2 y + a_3 xy.$$

Let H_β be the classical Dirichlet solution. Then, on D,

$$L_\beta \le H_\beta \le U_\beta. \tag{11.3}$$

Note that U_β is superharmonic (simply check the averaging property of any biconcave function) on D, hence belongs to the upper class of β.

There are many other analogies and connections with the classical theory. For example, if Z is a zigzag martingale with values in a biconvex set S and u is a bounded biconvex function on S, then $\{u(Z_n), n \ge 1\}$ is a submartingale; see the proof of (6.25). This is analogous to the classical result for Brownian motion and subharmonic functions due to Doob [11].

12. Differential subordination. Let f and g be real martingales such that $(f, g) = \{(f_n, g_n), n \ge 1\}$ is a martingale. We shall say that g is *differentially subordinate* to f if its difference sequence e satisfies $|e_k| \le |d_k|$, $k \ge 1$. (Example: $e_k = v_k d_k$ where $|v_k| \le 1$.) Many of the results for martingale transforms carry over to such f and g; see [3]. (This is also true if f and g have their values in a space isomorphic to some Hilbert space but is not true for more general spaces [5].)

The best possible bounds and conditions for equality that have been obtained here for martingale transforms carry over to the differentially subordinate case.

For example, Lemma 7.1 carries over because (7.9) holds for $hk \ge 0$, not just for $hk = 0$, and (7.9) holds with strict inequality if $hk > 0$: Let (f, g) be a martingale with $f_1 = g_1 \equiv (x + y)/2$ such that g is differentially subordinate to f. Let $Z_n = (X_n, Y_n)$ where

$$X_n = x + \textstyle\sum_{k=2}^{n} (d_k + e_k), \tag{12.1}$$

$$Y_n = y + \textstyle\sum_{k=2}^{n} (d_k - e_k). \tag{12.2}$$

Then $Z = (Z_1, Z_2, \cdots)$ is a martingale and

$$(X_{n+1} - X_n)(Y_{n+1} - Y_n) = d_{n+1}^2 - e_{n+1}^2 \ge 0.$$

The proof of Lemma 7.1 carries over without further change.

If $hk \le 0$, the dual inequality (6.9) holds and, if $hk < 0$, (6.9) holds with strict inequality. So, in Lemma 6.1, we may suppose that (f, g) is a martingale such that f is differentially subordinate to g. Theorem 6.1 and its special case Theorem 1.2 then follow for g differentially subordinate to f.

Let $1 < p < \infty$. To see that Theorem 1.1 and Theorem 9.1 carry over, we need to show only that $u = U_p$ satisfies (5.70) with strict inequality if $hk < 0$ or, what is equivalent, that φ defined by

$$\varphi(\alpha) = u(x + \alpha h, y + \alpha k, t + \alpha r) \tag{12.3}$$

is strictly concave on the convex set

$$\{\alpha: (x + \alpha h, y + \alpha k, t + \alpha r) \in \Omega \cup \partial\Omega\}. \tag{12.4}$$

It would be difficult to show this directly but it can be checked easily using Theorem 9.1 and the splicing argument that has already been used several times, for example, in Section 10. Fix $\alpha = (\alpha_1 + \alpha_2)/2$ where $\alpha_1 \ne \alpha_2$ and the α_i belong to the set (12.4). We shall show that, for $p \ge 2$,

$$\tfrac{1}{2}[\varphi(\alpha_1) + \varphi(\alpha_2)] \le \varphi(\alpha) - \theta^p \tag{12.5}$$

where $\theta = |\alpha_1 - \alpha_2|(|h| \wedge |k|)/2$. A similar but slightly different inequality holds if $1 < p < 2$ so that in both cases

$$\tfrac{1}{2}[\varphi(\alpha_1) + \varphi(\alpha_2)] < \varphi(\alpha).$$

Since both sides of (12.5) are continuous in h and k, we can assume in the proof of (12.5) that

$$h + k \ne 0, \quad h + 3k \ne 0, \quad 3h + k \ne 0. \tag{12.6}$$

Let $\delta > 0$ and choose martingales $W^i = (X^i, Y^i, T^i) = (Z^i, T^i)$ starting at $(x + \alpha_i h, y + \alpha_i k, t + \alpha_i r)$ with values in $\Omega \cup \partial\Omega$ such that

$$\varphi(\alpha_i) - \delta \le \|f^i\|_p^p \tag{12.7}$$

where $f_n^i = (X_n^i + Y_n^i)/2$. Assume also that Z^1 and Z^2 are zigzag martingales moving horizontally and vertically together. All of this is possible by Lemma 9.1. Let W be the spliced martingale giving equal weight to W^1 and W^2 (see Section 10). The corresponding f and g have the following properties: $f_1 \equiv (x + \alpha h + y + \alpha k)/2$,

$$\sup_n \|g_n - y - \alpha k\|_p^p \le t + \alpha r,$$

and g is the transform of f by a sequence of the form $(1, a_2, a_3, \cdots)$ where $a_3, a_4, \cdots$ belong to $\{-1, 1\}$ but

$$|a_2| > 1. \tag{12.8}$$

In fact, $a_2 = (h - k)/(h + k)$ and (12.8) follows from the assumption that $hk < 0$. Note that, by (12.7),

$$\tfrac{1}{2}[\varphi(\alpha_1) + \varphi(\alpha_2)] - \delta \le \tfrac{1}{2}[\|f^1\|_p^p + \|f^2\|_p^p] = \|f\|_p^p. \tag{12.9}$$

Now g is also the transform of the martingale $M = (M_1, M_2, \cdots)$ with difference sequence

$$(d_1,\ |a_2|d_2,\ d_3,\ d_4,\ \cdots)$$

by $(1, \operatorname{sgn} a_2, a_3, a_4, \cdots)$. Furthermore, g is the transform of the martingale $N = (N_1, N_2, \cdots)$ with difference sequence

$$(d_1,\ (2-|a_2|)d_2,\ d_3,\ d_4,\ \cdots)$$

by the sequence $(1, a_2(2-|a_2|)^{-1}, a_3, a_4, \cdots)$. Note that $|a_2| \neq 2$ by (12.6) and that $|a_2(2-|a_2|)^{-1}| > 1$ by (12.8). Therefore, by Theorem 9.1,

$$\|M\|_p^p \leq \varphi(\alpha), \quad \|N\|_p^p \leq \varphi(\alpha).$$

Let $n \geq 2$. By Clarkson's inequality [8],

$$\|f_n\|_p^p = \left\| \frac{M_n + N_n}{2} \right\|_p^p \leq \frac{1}{2}(\|M_n\|_p^p + \|N_n\|_p^p) - \left\| \frac{M_n - N_n}{2} \right\|_p^p$$

$$\leq \varphi(\alpha) - \|(|a_2| - 1)d_2\|_p^p \leq \varphi(\alpha) - \theta^p.$$

The desired inequality now follows from (12.9).

13. Relaxation of the martingale condition. Recall that we have used the martingale condition of Lemma 4.1, for example, to obtain

$$E[u_x(Z_n)(X_{n+1} - X_n)] = E[u_y(Z_n)(Y_{n+1} - Y_n)] = 0. \tag{13.1}$$

Clearly, a weaker condition suffices. If $B = \mathbb{R}$ or any other Banach space and $f = (f_1, f_2, \cdots)$ is a sequence of integrable functions with values in B, then f is a *very weak martingale* if

$$E[\varphi(f_n)d_{n+1}] = 0 \tag{13.2}$$

for all real bounded continuous functions φ on B and all $n \geq 1$. (Equivalently, $E(d_{n+1} \mid f_n) = 0$ a.e.) Note that if $(f, g) = \{(f_n, g_n), n \geq 1\}$ is a very weak martingale starting at $((x+y)/2, (x+y)/2)$ with values in $\mathbb{R}^2$, then Z defined by (12.1) and (12.2) is a very weak martingale starting at (x, y) and this is enough to obtain (13.1) in the setting of Lemma 4.1. Thus, we have the following extension of Theorem 1.2.

THEOREM 13.1. *If (f, g) is a very weak martingale with values in $\mathbb{R}^2$ such that $\|f\|_\infty \leq 1$ and g is differentially subordinate to f, then*

$$\|g\|_p^p < \Gamma(p+1)/2, \quad 2 < p < \infty. \tag{13.3}$$

Theorem 6.1 extends in exactly the same way. Similarly, the following theorem extends Theorem 1.1. Of course, in these extensions, the constants remain best possible.

THEOREM 13.2. *Let $1 < p < \infty$. If (f, g) is a very weak martingale with values* *in $\mathbb{R}^2$ such that g is differentially subordinate to f, then*

$$\|g\|_p \le (p^* - 1)\|f\|_p. \tag{13.4}$$

If $0 < \|f\|_p < \infty$, then equality holds if and only if $p = 2$ and $\sum_{k=1}^{\infty} e_k^2 = \sum_{k=1}^{\infty} d_k^2$ almost everywhere.

The proof of the analogous extension of Lemma 5.1 requires a small change in the definition of $W = (X, Y, T)$. Let $S_n = |g_n|^p$. Simply let T_n be the almost everywhere limit of the almost everywhere nondecreasing sequence

$$(S_n,\ E(S_{n+1} \mid S_n),\ E[E(S_{n+2} \mid S_{n+1}) \mid S_n],\ \cdots).$$

It is easy to check that $T = (T_1, T_2, \cdots)$ is a very weak martingale satisfying $S_n \le T_n$ a.e. and $ET_n = \|S\|_1 = \|g\|_p^p = 1$, $n \ge 1$. Without loss of generality, assume that $S_n \le T_n$ everywhere. Then $W = (X, Y, T)$ is a very weak martingale starting at $(0, 0, 1)$ with values in $\Omega \cup \partial\Omega$.

Theorem 9.1 extends in a similar way. However, the results of Sections 7 and 8 do not carry over without substantial change. It is easy to see that there is a real very weak martingale f such that $\|f\|_1 < \infty$ but $f^* = \infty$ a.e. So (8.1), for example, does not hold in the very weak case even with $g = f$. To construct an example of such an f, let $\xi = (\xi_1, \xi_2, \cdots)$ be an independent sequence of functions each with the standardized Gaussian distribution. Then it is not hard to show there is a very weak martingale f satisfying

$$f_n = a_k\xi_k \quad \text{if} \quad n = n_k \tag{13.5}$$

for some sequence of positive integers $1 = n_1 < n_2 < \cdots$ and numbers $1 = a_1 < a_2 < \cdots \le 2$. If $n_k < n < n_{k+1}$, then f_n can be chosen to be a linear combination of ξ_k and ξ_{k+1}. (For the details, see the related example of the author that appears in [25].) Note that f is a sequence in a Guassian subspace of L^2 and therefore the simple orthogonality condition $E(f_n d_{n+1}) = 0$ implies that (13.2) holds. We see that $\|f\|_1 \le \|f\|_2 \le 2$ but, by (13.5) and the Borel-Cantelli lemma,

$$P(f^* > \lambda) \ge P(\xi^* > \lambda) = 1, \quad \lambda > 0.$$

The weak-type inequalities do carry over for the individual terms g_n. For example, if (f, g) is a very weak martingale and g is differentially subordinate to f, then

$$\lambda P(|g_n| \ge \lambda) \le 2\|f_n\|. \tag{13.6}$$

Only a slight modification in the proof of (8.1) is needed to see this.

14. The best constant in an inequality of R. E. A. C. Paley. This is an inequality that contains important information about the Haar system h on $[0, 1)$. To recall the definition of $h = (h_1, h_2, \cdots)$, we shall use the same notation for an interval $[a, b)$ and its indicator function. Then

$$h_1 = [0, 1), \quad h_2 = [0, \tfrac{1}{2}) - [\tfrac{1}{2}, 1),$$
$$h_3 = [0, \tfrac{1}{4}) - [\tfrac{1}{4}, \tfrac{1}{2}), \quad h_4 = [\tfrac{1}{2}, \tfrac{3}{4}) - [\tfrac{3}{4}, 1),$$
$$h_5 = [0, \tfrac{1}{8}) - [\tfrac{1}{8}, \tfrac{1}{4}), \quad h_6 = [\tfrac{1}{4}, \tfrac{3}{8}) - [\tfrac{3}{8}, \tfrac{1}{2}), \cdots.$$

Paley's inequality (see [20] and [16]) is the following: If $1 < p < \infty$, then there is a real number c_p such that

$$\| \textstyle\sum_{k=1}^{n} \varepsilon_k a_k h_k \|_p \le c_p \| \sum_{k=1}^{n} a_k h_k \|_p$$

for all $a_k \in \mathbb{R}$, $\varepsilon_k \in \{-1, 1\}$, and $n \ge 1$. Here we give the best constant and conditions for equality.

THEOREM 14.1. *Let $1 < p < \infty$ and p^* be the maximum of p and q where $1/p + 1/q = 1$. Then*

$$\| \textstyle\sum_{k=1}^{n} \varepsilon_k a_k h_k \|_p \le (p^* - 1) \| \sum_{k=1}^{n} a_k h_k \|_p \tag{14.1}$$

and the constant $p^* - 1$ is best possible. Furthermore, equality holds if and only if $p = 2$ or $(a_1, \cdots, a_n) = (0, \cdots, 0)$.

PROOF. Fix n, let f be the martingale with difference sequence

$$(a_1 h_1, \cdots, a_n h_n, 0, 0, \cdots),$$

and g the transform of f by $(\varepsilon_1, \varepsilon_2, \cdots)$. Note that f is indeed a martingale: If $k \ge 1$, then $Ed_{k+1} = 0$ and d_{k+1} is supported by a set on which $\varphi(d_1, \cdots, d_k)$ is constant, so d_{k+1} and $\varphi(d_1, \cdots, d_k)$ are orthogonal. Inequality (14.1) and the conditions for equality follow at once from Theorem 1.1.

It is easy to check by induction that a real dyadic martingale has the same distribution as some subsequence of $\{\sum_{k=1}^{n} a_k h_k, n \ge 1\}$ where the real coefficients a_k are suitably chosen. Accordingly, by Theorem 1.1 and the result of Maurey mentioned in Section 10, the constant $p^* - 1$ is best possible.

15. Unconditional constants and contractive projections. Let $1 < p < \infty$. If $e = (e_1, e_2, \cdots)$ is a sequence in real $L^p(0, 1)$, its *unconditional constant* is the least $K \in [1, \infty]$ such that if n is a positive integer and $a_1, \cdots, a_n$ are real numbers such that $\| \sum_{k=1}^{n} a_k e_k \|_p = 1$, then $\| \sum_{k=1}^{n} \varepsilon_k a_k e_k \|_p \le K$ for all choices of signs $\varepsilon_k \in \{-1, 1\}$.

THEOREM 15.1. *The unconditional constant of the Haar system in $L^p(0, 1)$ is $p^* - 1$.*

This result, announced in [6], is an immediate consequence of Theorem 14.1.

Recall that a sequence e in $L^p(0, 1)$ is a *basis* of $L^p(0, 1)$ if, for every $f \in L^p(0, 1)$, there is a unique sequence a in $\mathbb{R}$ such that $\| f - \sum_{k=1}^{n} a_k e_k \|_p \to 0$ as $n \to \infty$. Write $P_n f = \sum_{k=1}^{n} a_k e_k$. Then $(P_1, P_2, \cdots)$ is a nondecreasing sequence of projections in $L^p(0,1)$: P_n is linear and $P_mP_n = P_nP_m = P_m$ if $n \geq m \geq 1$. The basis e is a *monotone basis* if the P_n are contractions: $\| P_n \| \leq 1$ or, equivalently,

$$(15.1) \qquad \| \textstyle\sum_{k=1}^{n} a_k e_k \|_p \leq \| \sum_{k=1}^{n+1} a_k e_k \|_p$$

for all $a_k \in \mathbb{R}$ and $n \geq 1$. The Haar system is a basis [24] satisfying (15.1) so the following extends Theorem 15.1.

THEOREM 15.2. *The unconditional constant of a monotone basis of $L^p(0, 1)$ is $p^* - 1$.*

The finiteness of the unconditional constant is due independently to Pelczyński and Rosenthal [21], and Dor and Odell [12].

PROOF. Olevskiĭ ([18], [19]; also see [14] and [15]) proved that no basis of $L^p(0, 1)$ can have an unconditional constant smaller than the unconditional constant, whatever it might be, of the Haar system. Thus, by Theorem 15.1, no basis of $L^p(0, 1)$ can have an unconditional constant smaller than $p^* - 1$.

The other half of the proof rests on Theorem 1.1 and Ando's theorem [1] to the effect that, for $1 < p < \infty$, every nonvanishing contractive projection is isometrically equivalent to a conditional expectation. For the details, see [12] or [21], where our earlier (nonsharp) version of Theorem 1.1 is used.

Dor and Odell [12] show how the work of Ando [1], Tzafriri [26], and the author [3] can be used to establish the finiteness of the unconditional constant of a monotone decomposition of an arbitrary L^p space. With the help of Theorem 1.1, this constant can be shown to be no larger than $p^* - 1$. In fact, the direct-sum method needed to establish this (see [12]) gives the following extension of the first part of Theorem 1.1 in the special case $v = a$ where $a = (a_1, a_2, \cdots)$ is a numerical sequence in $[-1, 1]$. Let $1 < p < 2$ or $2 < p < \infty$ since the case $p = 2$, where equality can hold, is clear.

THEOREM 15.3. *Suppose that $(\Omega, \mathscr{A}, \mu)$ is a positive measure space (not necessarily σ-finite). Let $P_1, P_2, \cdots$ be any nondecreasing sequence of contractive projections in $L^p(\Omega, \mathscr{A}, \mu)$ and set $P_0 = 0$. If $f \in L^p(\Omega, \mathscr{A}, \mu)$ and $\|f\|_p > 0$, then*

$$(15.2) \qquad \| \textstyle\sum_{k=1}^{\infty} a_k(P_k - P_{k-1})f \|_p < (p^* - 1)\|f\|_p.$$

The operator $\sum_{k=1}^{\infty} a_k(P_k - P_{k-1})$, the limit in the strong operator topology of its partial sums, does not attain its norm if, as is possible, that norm is $p^* - 1$.

16. Some sharp inequalities for stochastic integrals. The inequalities of Section 1 and of Sections 4 to 9 imply similar sharp inequalities for stochastic integrals. Let $(\Omega, \mathscr{F}_\infty, P)$ be a complete probability space and $\mathscr{F} = (\mathscr{F}_t)_{t \geq 0}$ a nondecreasing right-continuous family of sub-σ-fields of $\mathscr{F}_\infty$ where $\mathscr{F}_0$ contains all of the sets A in $\mathscr{F}_\infty$ with $P(A) = 0$. Recall (see Dellacherie and Meyer [9] for further discussion) that $\mathscr{P}$, the predictable σ-field on $[0, \infty) \times \Omega$ is generated by all left-continuous processes adapted to $\mathscr{F}$. Let $V = (V_t)_{t \geq 0}$ be a real predictable process uniformly bounded in absolute value by 1, that is, the map $V\colon [0, \infty) \times \Omega \to [-1, 1]$ is measurable relative to $\mathscr{P}$. Assume that $X = (X_t)_{t \ge 0}$ is a real martingale adapted to $\mathscr{F}$ and that almost all of the paths of X are right-continuous on $[0, \infty)$ with left-limits on $(0, \infty)$. Now consider the stochastic integral Y of V with respect to X, an adapted r.c.l.l. process satisfying

$$Y_t = \int_{[0,t]} V\, dX \text{ a.s.}$$

(see Dellacherie and Meyer [9]). Let $\| X \|_p = \sup_t \| X_t \|_p$ and $Y^* = \sup_t | Y_t |$.

The inequalities of Section 1, for example, now lead to the following theorem.

THEOREM 16.1. *Let Y be the stochastic integral of V with respect to X as above. Then, for $1 < p < \infty$,*

(16.1) $$\| Y \|_p \le (p^* - 1) \| X \|_p.$$

If $\| X \|_\infty \le 1$ and $2 \le p < \infty$, then

(16.2) $$\| Y \|_p^p \le \Gamma(p + 1)/2.$$

If $1 \le p \le 2$, then

(16.3) $$\sup_{\lambda > 0} \lambda^p P(Y^* \ge \lambda) \le 2 \| X \|_p^p / \Gamma(p + 1).$$

Since these inequalities are sharp in the discrete case, they are also sharp here. In fact, they are sharp in the special case of the Itô integral: Simply embed in Brownian motion the examples that prove sharpness in the discrete case.

What about conditions for equality? These are undoubtedly analogous to those for the discrete case. For example, if $p > 2$, then strict inequality in (16.2) could be proved directly with the use of a general Itô formula provided the function u of Section 4 had continuous second order partial derivatives—which it does not. A slightly modified argument that would overcome this difficulty is a likely possibility.

PROOF. It will be convenient in the proof to assume that $X_0 \equiv 0$; see the first paragraph of Section 2.

Let $\mathbf{Z}$ be the collection of all processes $Z = (Z_t)_{t \ge 0}$ of the form

(16.4) $$Z_t = \sum_{k=1}^n a_k [X(\tau_k \wedge t) - X(\tau_{k-1} \wedge t)]$$

where n is a positive integer, the a_k belong to $[-1, 1]$, and the τ_k are bounded stopping times relative to $\mathscr{F}$ satisfying $0 \equiv \tau_0 \le \tau_1 \le \cdots \le \tau_n$. (As usual, $X(t) = X_t$.) Then, for all $Z \in \mathbf{Z}$,

(16.5) $$\| Z \|_p \le (p^* - 1) \| X \|_p.$$

To see this, let

(16.6) $$f = (X(\tau_1), \cdots, X(\tau_n), X(\tau_n), \cdots),$$

a martingale by Doob's optional sampling theorem, and let g be its transform by

$(a_1, \cdots, a_n, 0, 0, \cdots)$. By (1.1), for all $t \geq \| \tau_n \|_\infty$,

$$\| Z_t \|_p = \| g_n \|_p \leq (p^* - 1) \| f_n \|_p$$
$$= (p^* - 1) \| X(\tau_n) \|_p \leq (p^* - 1) \| X_t \|_p,$$

which implies (16.5). Similarly, if $\| X \|_\infty \leq 1$ and $2 \leq p < \infty$, then

$$\| Z \|_p^p \leq \Gamma(p + 1)/2. \tag{16.7}$$

Furthermore, if $1 \leq p \leq 2$, then

$$P(Z^* > 1) \leq 2 \| X \|_p^p / \Gamma(p + 1), \tag{16.8}$$

but here the method is slightly different. Let $\tau = \inf\{t: | Z_t | > 1\}$ and $\mu_k = \tau_k \wedge \tau$. Define f here by (16.6) but with the τ_k replaced by the μ_k. Then, by right-continuity,

$$P(Z^* > 1) \leq P(\tau \leq \tau_n) \leq P(| Z(\mu_n) | \geq 1)$$
$$= P(| g_n | \geq 1) \leq 2 \| f_n \|_p^p / \Gamma(p + 1).$$

Since $\| f_n \|_p \leq \| X \|_p$ as before, (16.8) follows.

In view of the following lemma, the desired inequalities (16.1), (16.2), and (16.3) are immediate consequences of (16.5), (16.7), and (16.8).

LEMMA 16.1. (i) *If* $1 < p < \infty$ *and* $\| X \|_p < \infty$, *there is a sequence* $(Z^j)_{j \geq 1}$ *in* $\mathbf{Z}$ *such that*

$$\lim_{j \to \infty} \| Z^j - Y \|_p = 0.$$

(ii) *If* $1 \leq p < \infty$ *and* $\| X \|_p < \infty$, *there is a sequence* $(Z^j)_{j \geq 1}$ *in* $\mathbf{Z}$ *such that*

$$\lim_{j \to \infty} (Z^j - Y)^* = 0 \text{ a.s.}$$

For a proof, see Bichteler [2].

Note that (16.2), for example, can be strengthened to

$$\sup_{t \geq 0} E\Phi(| Y_t |) \leq \frac{1}{2} \int_0^\infty \Phi(t) e^{-t}\, dt \tag{16.9}$$

where Φ is a convex function as in Section 6.

17. A comparison with the M. Riesz inequality. Theorem 1.1 is the martingale analogue of the M. Riesz inequality. To compare the best constants in the two inequalities, let $1 < p < \infty$ and B be a Banach space. Let $\alpha_p(B)$ be the least $\alpha \in [0, +\infty]$ with the property that if n is a positive integer and $a_0, \cdots, a_n$ and $b_1, \cdots, b_n$ belong to B, then the $L_B^p(0, 2\pi)$ norms of

$$f_n(\theta) = a_0/2 + \textstyle\sum_{k=1}^n (a_k \cos k\theta + b_k \sin k\theta)$$

and its conjugate

$$g_n(\theta) = \textstyle\sum_{k=1}^n (a_k \sin k\theta - b_k \cos k\theta)$$

satisfy

$$\|g_n\|_p \le \alpha \|f_n\|_p.$$

M. Riesz [23] discovered that $\alpha_p(\mathbb{R})$ is finite and used this to prove that the Hilbert transform is bounded in $L^p(\mathbb{R})$. Pichorides [22] showed that

$$\alpha_p(\mathbb{R}) = \cot(\pi/2p^*).$$

In addition, see the work of Cole as described in Gamelin [13].

Now let $\beta_p(B)$ be the least $\beta \in [0, +\infty]$ such that if $f = (f_1, f_2, \cdots)$ is a B-valued martingale and g is the transform of f by a numerical sequence ε in $\{-1, 1\}$, then

$$\|g\|_p \le \beta \|f\|_p.$$

By [5] or Remark 2.1, the same value of $\beta_p(B)$ is obtained if ε is replaced by a real-valued predictable sequence v uniformly bounded in absolute value by 1. Furthermore, $\beta_p(B)$ is finite if and only if there is a biconvex function $\zeta\colon B \times B \to \mathbb{R}$ satisfying $\zeta(0, 0) > 0$ and

$$\zeta(x, y) \le |x + y| \quad \text{if} \quad |x| = |y| = 1. \tag{17.1}$$

Here $|x|$ denotes the norm of x. See [5], where the condition on ζ is slightly different.

By Theorem 1.1, $\beta_p(\mathbb{R}) = p^* - 1$. An elementary calculation gives

$$\cot(\pi/2p^*) \le p^* - 1$$

so $\alpha_p(\mathbb{R}) \le \beta_p(\mathbb{R})$. But this is no mere happenstance. McConnell and the author [7] proved that $\alpha_p(B)$ is finite if $\beta_p(B)$ is finite; in response, Bourgain (personal communication) showed the converse. Some of the ideas of the present paper can be used to prove that, for all Banach spaces B,

$$\alpha_p(B) \le \beta_p(B). \tag{17.2}$$

The details will appear elsewhere.

REFERENCES

[1] ANDO, T. (1966). Contractive projections in L_p spaces. *Pacific J. Math.* **17** 391–405.

[2] BICHTELER, K. (1981). Stochastic integration and L^p-theory of semimartingales. *Ann. Probab.* **9** 49–89.

[3] BURKHOLDER, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504.

[4] BURKHOLDER, D. L. (1979). A sharp inequality for martingale transforms. *Ann. Probab.* **7** 858–863.

[5] BURKHOLDER, D. L. (1981). A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional. *Ann. Probab.* **9** 997–1011.

[6] BURKHOLDER, D. L. (1982). A nonlinear partial differential equation and the unconditional constant of the Haar system in L^p. *Bull. Amer. Math. Soc.* (N.S.) **7** 591–595.

[7] BURKHOLDER, D. L. (1983). A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions. *Conference on Harmonic Analysis in Honor of Antoni Zygmund* **1** 270–286. Edited by William Beckner, Alberto P. Calderón, Robert Fefferman, and Peter W. Jones. Wadsworth, Belmont, California.

[8] CLARKSON, J. A. (1936). Uniformly convex spaces. *Trans. Amer. Math. Soc.* **40** 396–414.

[9] DELLACHERIE, C. AND MEYER, P.-A. (1980). *Probabilités et potentiel: Théorie des martingales.* Hermann, Paris.

[10] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[11] DOOB, J. L. (1954). Semimartingales and subharmonic functions. *Trans. Amer. Math. Soc.* **77** 86–121.

[12] DOR, L. E. and ODELL, E. (1975). Monotone bases in L_p. *Pacific J. Math.* **60** 51–61.

[13] GAMELIN, T. W. (1978). *Uniform Algebras and Jensen Measures.* Cambridge University Press, London.

[14] LINDENSTRAUSS, J. and PEŁCZYŃSKI, A. (1971). Contributions to the theory of the classical Banach spaces. *J. Funct. Anal.* **8** 225–249.

[15] LINDENSTRAUSS, J. AND TZAFRIRI, L. (1979). *Classical Banach Spaces* II: *Function Spaces.* Springer, New York.

[16] MARCINKIEWICZ, J. (1937). Quelques théorèmes sur les séries orthogonales. *Ann. Soc. Polon. Math.* **16** 84–96.

[17] MAUREY, B. (1975). Système de Haar. *Séminaire Maurey-Schwartz* (1974–1975). École Polytechnique, Paris.

[18] OLEVSKIĬ, A. M. (1967). Fourier series and Lebesgue functions. *Uspehi Mat. Nauk* **22** 237–239. (Russian)

[19] OLEVSKIĬ, A. M. (1975). *Fourier Series with Respect to General Orthogonal Systems.* Springer, New York.

[20] PALEY, R. E. A. C. (1932). A remarkable series of orthogonal functions I. *Proc. London Math. Soc.* **34** 241–264.

[21] PEŁCZYŃSKI, A. and ROSENTHAL, H. P. (1975). Localization techniques in L^p spaces. *Studia Math.* **52** 263–289.

[22] PICHORIDES, S. K. (1972). On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov. *Studia Math.* **44** 165–179.

[23] RIESZ, M. (1927). Sur les fonctions conjuguées. *Math. Z.* **27** 218–244.

[24] SCHAUDER, J. (1928). Eine Eigenschaft des Haarschen Orthogonalsystems. *Math. Z.* **28** 317–320.

[25] STARR, N. (1965). On an operator limit theorem of Rota. *Ann. Math. Statist.* **36** 1864–1866.

[26] TZAFRIRI, L. (1969). Remarks on contractive projections in L_p-spaces. *Israel J. Math.* **7** 9–15.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS 61801