Annals of Probability
1984, Vol. 12, No. 4, 1224–1227

A CHARACTERIZATION OF ORTHOGONAL TRANSITION KERNELS

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A transition kernel $\mu = (\mu_y)_{y\in Y}$ between Polish spaces X and Y is completely orthogonal if there is a perfect statistic $\varphi: X \to Y$ for μ, i.e. the fibers of the Borel map φ separate the μ_y. Equivalent properties are: a) orthogonal, finitely additive measures p, q on Y induce orthogonal mixtures μ^p, μ^q on X; b) the Markov operator defined by μ is surjective on a certain class of Borel functions.

In [4] R. D. Mauldin, D. Preiss and H. v. Weizsäcker give a systematic study of various notions of orthogonality for transition kernels, their classification and interrelationships. For a transition kernel $(\mu_y)_{y\in Y}$ between standard measure spaces $(X, \mathscr{A})$, $(Y, \mathscr{B})$ the following two concepts may be the most important among them

(I) $(\mu_y)_{y\in Y}$ is *completely orthogonal*, if there is a Borel map $\varphi: X \to Y$ such that $\mu_y(\varphi^{-1}(y)) = 1$ for all $y \in Y$.

(II) $(\mu_y)_{y\in Y}$ is *orthogonality preserving* if for any pair (p, q) of orthogonal probability measures on $(Y, \mathscr{B})$ the corresponding mixtures $\mu^p = \int_Y \mu_y p(dy)$ and $\mu^q = \int_Y \mu_y q(dy)$ are orthogonal too.

(for more details on origin and applications of these notions, see [5]). In [4] it is shown that completely orthogonal kernels are orthogonality preserving and that the converse holds if the set $\{\mu_y, y \in Y\}$ is narrowly σ-compact but not in general. However—as I will show below—there is a stronger orthogonality preserving property which always characterizes completely orthogonal kernels. This differs from II in that not only σ-additive measures, but also finitely additive measures are involved.

THEOREM. *For a probability transition kernel $(\mu_y)_{y\in Y}$ from the standard measure space $(Y, \mathscr{B})$ to $(X, \mathscr{A})$ the following are equivalent*

a) *$(\mu_y)_{y\in Y}$ is completely orthogonal*

b) *For any pair (p, q) of orthogonal, finitely additive and positive measures on $(Y, \mathscr{B})$ the corresponding mixtures*

$$\mu^p = \int_Y \mu_y p(dy), \quad \mu^q = \int_Y \mu_y q(dy)$$

are also orthogonal.

Received August 1983.
AMS 1970 *subject classification.* Primary 60A10.
Key words and phrases. Orthogonal measures, perfect statistics, Markov operators which are Riesz homomorphisms or have the Maharam property.

c) *For every* $\mathscr{B}$*-measurable function* g, $0 \le g \le 1$, *there is a* $\mathscr{A}$*-measurable function* f, $0 \le f \le 1$, *with*

$$g(y) = \int f(x)\mu_y(dx) \quad \textit{for all} \quad y \in Y.$$

Consider the Markov operators associated with (μ_y):

$$T\colon ba(Y, \mathscr{B}) \to ba(X, \mathscr{A}), \quad Tp = \int \mu_y p(dy) \tag{1}$$

$$S\colon B(X, \mathscr{A}) \to B(Y, \mathscr{B}), \quad Tf(y) = \int f(x)\mu_y(dx) \tag{2}$$

where $ba(X, \mathscr{A})$ denotes the Banach space of finitely additive measures on $(X, \mathscr{A})$ with the variation-norm and $B(X, \mathscr{A})$ the Banach space of $\mathscr{A}$-measurable functions on X with the sup-norm.

REMARKS. 1) If (μ_y) is completely orthogonal then the operator $f \to (Sf) \circ \varphi$ is a H-sufficient statistic as defined by Dynkin ([1], page 711) for the set $\{\mu_y,\ y \in Y\}$. Hence d) gives a criterion for H-sufficient statistics in terms of surjectivity. In particular, if $(\mu_{\varphi(x)})_{x \in X}$ defines a H-sufficient statistic, then this kernel cannot have strong smoothing properties.

2) In the language of operator theory, condition b) says that T is a Riesz homomorphism and c) means that S is interval-preserving or has the Maharam property. In many situations these concepts are dual to each other (see [2], [3]). Since $ba(X, \mathscr{A})$ is the Banach space dual of $B(X, \mathscr{A})$ and T is the adjoint of S, the following proof is a variation of this theme.

PROOF. a) $\Rightarrow$ c) If $\varphi\colon X \to Y$ is the separating map and g is given, choose $f = \varphi \circ g$.

c) $\Rightarrow$ b) Since T in (1) is a quotient map, it follows from the Hahn-Banach theorem that $T = S'$ is an isometric embedding. It is well known that $p, q \in ba(Y, \mathscr{B})$ are orthogonal if and only if $\| p + q \| = \| p - q \|$. Hence the isometry T satisfies b).

b) $\Rightarrow$ a) First we show for $U_X = \{f \in B(X, \mathscr{A})\colon 0 \le f \le 1\}$:

$$S(U_X) \quad \text{is} \quad \| \cdot \|_\infty\text{-dense in} \quad U_Y. \tag{3}$$

Otherwise, by the Hahn-Banach theorem, there are $h \in U_Y$, $\mu \in ba(Y, \mathscr{B}) = B(Y, \mathscr{B})^*$ and $\alpha \in \mathbb{R}$ with

$$\mu(h) > \alpha \ge \mu(Sf) \quad \text{for all} \quad f \in U_x. \tag{4}$$

But from $T = S'$, b) and (4) a contradiction results:

$$\begin{aligned} \mu^+(1_Y) &= \mu^+(S1_X) = (T\mu^+)(1) = (T\mu)^+(1) \\ &= \sup\{(T\mu)(f)\colon f \in U_X\} \le \alpha < \mu(h) \le \mu^+(1_Y). \end{aligned}$$

From (3) we obtain

(5) For any $B \in \mathscr{B}$ there is some $A \in \mathscr{A}$ with $S\chi_A = \chi_B$.

Indeed, choose a sequence $f_n \in U_X$ such that

$$\sum_{n=1}^{\infty} \| \chi_B - Sf_n \|_\infty < \infty.$$

Put $g_m = \sup_{n \geq m} f_n$ and observe that

$$\chi_B \geq \chi_B \cdot S(g_m) \geq \sup_{n \geq m} \chi_B \cdot S(f_n) \geq \chi_B$$

and

$$\| \chi_{B^c} \cdot S(g_m) \|_\infty \leq \| \chi_{B^c}(\sum_{n=m}^{\infty} Sf_n) \|_\infty \leq \sum_{n=m}^{\infty} \| \chi_{B^c}(Sf_n - \chi_B) \|_\infty \to_{m \to \infty} 0.$$

Together with the monotone convergence theorem it follows that $Sg = \lim_n Sg_n = \chi_B$ where $g = \inf_m g_m$. For $A = \{g \neq 0\}$ we have $1 \geq S\chi_A \geq \chi_B$. On the other hand, we have

$$n \cdot g \wedge 1 \nearrow \chi_A \quad \text{for} \quad n \to \infty, \quad S(ng \wedge 1) \leq \chi_B$$

and it follows again from the monotone convergence theorem that $S\chi_A = \chi_B$.

Finally, in order to construct the separating map φ, we choose a generating sequence $B_n \in \mathscr{B}$ and then $A_n \in \mathscr{A}$ with $S\chi_{A_n} = \chi_{B_n}$ by (5). The measurable map

$$b\colon Y \to \{0, 1\}^{\mathbb{N}}, \quad b(y) = (\chi_{B_n}(y))_{n \in \mathbb{N}}$$

is injective and therefore—since $(Y, \mathscr{B})$ is a standard measure space—the inverse $b^{-1}\colon b(Y) \to Y$ is measurable too. Let a be the measurable map

$$a\colon X \to \{0, 1\}^{\mathbb{N}}, \quad a(x) = (\chi_{A_n}(x))_{n \in \mathbb{N}}$$

and define

$$\varphi(x) = \begin{cases} b^{-1}(a(x)) & \text{for} \quad x \in a^{-1}(b(Y)) \\ y_0 & \text{otherwise} \end{cases}$$

where $y_0 \in Y$ is arbitrary and fixed. Then φ is measurable and for all $y \in Y$ we have

$$\varphi^{-1}(\{y\}) \supset \{x\colon b(y) = a(x)\} = \cap\{A_n\colon y \in B_n\} \cap \cap\{A_n^c\colon y \notin B_n\}.$$

Since $\chi_{B_n}(y) = S\chi_{A_n}(y) = \mu_y(A_n)$ we only intersect sets with μ_y-measure equal to 1. Therefore $\mu_y(\varphi^{-1}(y)) = 1$. □

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