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ELLIPTICAL SYMMETRY AND CHARACTERIZATION OF OPERATOR-STABLE AND OPERATOR SEMI-STABLE MEASURES

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The problem of the elliptical symmetry of an operator-stable measure on a finite dimensional vector space was studied by J. P. Holmes, W. N. Hudson and J. D. Mason (1982). The aim of this paper is to consider an analogous question for operator semi-stable measures. We prove a theorem characterizing an elliptically symmetric full operator semi-stable measure. After such a generalization some results for operator-stable measures are obtained as corollaries. At the same time, the methods of proofs seem to be simpler than those in [1], in particular, the theory of Lie algebras is not involved.

We also give a description of a full operator-stable and operator semi-stable measure μ in terms of its quasi-decomposability group, namely the group

$$\mathbb{G}(\mu) = \{t > 0\colon \exists\, A \in \operatorname{Aut} V, h \in V \quad \text{such that} \quad \mu^t = A\mu * \delta(h)\}.$$

1. Introduction. Let V be a finite dimensional real vector space with an inner product $(\cdot, \cdot)$ and let μ be a probability measure over V. For an arbitrary linear operator A acting in V, $A\mu$ is a probability measure defined by $A\mu(E) = \mu(A^{-1}E)$ for each Borel subset E of V.

We define the symmetry group $\mathbb{S}(\mu)$ of μ as

$$\mathbb{S}(\mu) = \{A \in \operatorname{Aut} V\colon \exists\, h \in V \quad \text{such that} \quad \mu = A\mu * \delta(h)\}.$$

We say that μ is elliptically symmetric if

$$\mathbb{S}(\mu) = W^{-1}\mathscr{O}W$$

for some positive linear operator W in V; here $\mathscr{O}$ denotes the orthogonal group.

Now, let μ be an infinitely divisible measure on V, let a be a positive number, and let A be a nonsingular linear operator on V. We say that μ is quasi-decomposable by the pair (a, A) if

$$\mu^a = A\mu * \delta(h) \quad \text{for some} \quad h \in V.$$

We define the quasi-decomposability group $\mathbb{G}(\mu)$ of μ as

$$\mathbb{G}(\mu) = \{t > 0\colon \exists\, A \in \operatorname{Aut} V \quad \text{such that} \quad \mu \text{ is quasi-decomposable by } (t, A)\}.$$

A probability measure μ on V is called operator semi-stable if

$$\mu = \lim_{n\to\infty}(A_n\nu^{k_n} * \delta(h_n))$$

where ν is a probability measure on V, the A_n's are linear operators on V,

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$h_n \in V$ and the sequence of positive integers $k_1 < k_2 < \cdots$ satisfies $k_{n+1}/k_n \to r$, $1 \leq r < \infty$.

For a more detailed description of operator semi-stable measures, the reader is referred to [2] and [5]. Here we only mention that if a measure is quasi-decomposable, then it is operator semi-stable and the converse is true if the measure considered is full.

2. Elliptical symmetry of an operator semi-stable measure.

LEMMA 2.1. *Let μ be an infinitely divisible measure on V such that $\mathbb{S}(\mu) = \mathscr{O}$. Then there are $h \in V$ and a probability measure ν satisfying $\mu = \nu * \delta(h)$ and $U\nu = \nu$ for $U \in \mathscr{O}$.*

PROOF. Let $\mathscr{J}$ be the operator from $\mathscr{O}$ defined by $\mathscr{J}v = -v$. There is an h' in V such that $\mu = \mathscr{J}\mu * \delta(h')$.

For the characteristic functions, we thus obtain the equality $\hat{\mu}(v) = \overline{\hat{\mu}(v)}e^{i(h',v)}$ and, consequently, $\hat{\mu}^2(v) = |\hat{\mu}(v)|^2 e^{i(h',v)}$.

But $|\hat{\mu}|$ is the characteristic function of the measure $\mathring{\mu}^{1/2}$ where "$\circ$" stands for the symmetrization, and we obtain that $\mu * \mu = [\mathring{\mu}^{1/2} * \delta(h'/2)] * [\mathring{\mu}^{1/2} * \delta(h'/2)]$, which implies that $\mu = \mathring{\mu}^{1/2} * \delta(h'/2)$ because μ and $\mathring{\mu}^{1/2}$ are infinitely divisible. Putting $h = h'/2$, $\nu = \mathring{\mu}^{1/2}$, we get that $U\nu = U\mathring{\mu}^{1/2} = (U\mathring{\mu})^{1/2} = \mathring{\mu}^{1/2} = \nu$ for each $U \in \mathscr{O}$, which ends the proof. □

We shall now discuss the question of the quasi-decomposability of an elliptically symmetric measure.

PROPOSITION 2.2. *Let μ be an elliptically symmetric measure on V; assume that μ is quasi-decomposable by a pair (a, A). Then $A = cT$ for some $c > 0$ and $T \in \mathbb{S}(\mu)$.*

PROOF. Let us first assume that $\mathbb{S}(\mu) = \mathscr{O}$. For any U from $\mathscr{O}$, we have

$$AUA^{-1}\mu = (AU)(\mu^{1/a} * \delta(h_1)) = A(U\mu)^{1/a} * \delta(h_2)$$
$$= A\mu^{1/a} * \delta(h_3) = \mu * \delta(h_4),$$

thus $AUA^{-1} \in \mathbb{S}(\mu)$. AUA^{-1} is orthogonal, hence its converse and adjoint are equal, which implies the equality

$$(AUA^{-1})^* = (A^{-1})^*U^*A^* = (AUA^{-1})^{-1} = AU^{-1}A^{-1} = AU^*A^{-1}$$

and, consequently,

$$(A^{-1})^*U^*A^* = AU^*A^{-1}.$$

Putting $|A|^2 = A^*A$, we get

$$U^*|A|^2 = |A|^2U^*$$

for any $U \in \mathscr{O}$, thus $|A|^2 = c^2I$ where I is the identity mapping, because it commutes with the whole orthogonal group $\mathcal{O}$, and

$$|A| = cI \quad \text{for some } c > 0. \tag{2.1}$$

From the polar decomposition $A = T|A|$ where $T \in \mathcal{O}$ we get that $A = cT$ with $c > 0$ and $T \in \mathbb{S}(\mu)$.

Let now $\mathbb{S}(\mu) = W^{-1}\mathcal{O}W$. It is easily seen that then

$$\mathbb{S}(W\mu) = \mathcal{O}. \tag{2.2}$$

The measure $W\mu$ is quasi-decomposable by the pair (a, WAW^{-1}); thus, from the preceding considerations we have that $WAW^{-1} = cU$ for some $c > 0$ and an orthogonal U. Putting

$$T = W^{-1}UW,$$

we infer that $T \in \mathbb{S}(\mu)$ and $A = cT$, which concludes the proof. □

COROLLARY 2.3. *Let μ be an ellipitcally symmetric measure on V. If μ is quasi-decomposable by a pair (a, A), then it is quasi-decomposable by the pair (a, cI) for some $c > 0$.*

PROOF. From Proposition 2.2 we have

$$\mu^a = A\mu * \delta(h) = (cT)\mu * \delta(h) = (cI)\mu * \delta(h')$$

because $T \in \mathbb{S}(\mu)$. □

Now, we are in a position to prove the theorem characterizing full elliptically symmetric operator semi-stable measures.

THEOREM 2.4. *Let μ be a full operator semi-stable measure on V. If μ is elliptically symmetric, then μ is multivariate semi-stable and its characteristic function has the form*:

$$\text{(i)} \quad \hat{\mu}(v) = \exp\left\{i(z, v) + \sum_{n=-\infty}^{\infty} a^{-n} \int_{c<\|u\|\le 1} [\cos c^n(W^{-1}u, v) - 1]m(du)\right\}$$

or

$$\text{(ii)} \quad \hat{\mu}(v) = \exp\{i(z, v) - b\,\|W^{-1}v\|^2\},$$

where W is a nonnegative operator on V such that $\mathbb{S}(\mu) = W^{-1}\mathcal{O}W$, $0 < a$, $c < 1$, $b > 0$ and m is an arbitrary $\mathcal{O}$-invariant finite Borel measure on $\{u\colon c < \|u\| \le 1\}$. Conversely, if $\hat{\mu}$ is of the form (i) *or* (ii) *for some nonnegative operator W on V and a, b, c, m are as above, then μ is a multivariate semi-stable measure on V such that $\mathbb{S}(\mu) = W^{-1}\mathcal{O}W$.*

PROOF. Assume that μ is elliptically symmetric with $\mathbb{S}(\mu) = W^{-1}\mathcal{O}W$. According to [2; Theorem], μ is quasi-decomposable by some pair (a, A) with $0 < a < 1$. On account of Corollary 2.3 μ is quasi-decomposable by the pair

(a, cI) for some $c > 0$ and the equalities

$$\mu^a = A\mu * \delta(h) = (cI)\mu * \delta(h)$$

show that $c^{-1}A$ belongs to $\mathbb{S}(\mu)$. Thus if λ is an eigenvalue of A we have $c = |\lambda|$ and, by virtue of the Theorem of [2], $c^2 \le a$, so that $o < c < 1$.

Put $r = 1/a$. By iterating the equality

$$\mu = (cI)\mu^r * \delta(h'),$$

we obtain

$$\mu = (c^nI)\mu^{r^n} * \delta(h_n) \quad \text{for every} \quad n.$$

Now, let $k_n = [r^n]$ and $\nu_n = (c^nI)\mu^{k_n} * \delta(h_n)$. We have that $k_{n+1}/k_n \to r$ and

$$\frac{\hat{\mu}(v)}{\hat{\nu}_n(v)} = [\hat{\mu}(c^n v)]^{r^n - [r^n]} \to 1 \quad \text{as} \quad n \to \infty;$$

thus $\nu_n \Rightarrow \mu$, which means that μ is multivariate semi-stable. In particular, on account of [3; Theorem] (cf. also [4; Theorem 3.2]), μ is either Gaussian or purely Poissonian.

Consider now the measure $W\mu$. If the pair (a, A) quasi-decomposes μ, then the pair (a, WAW^{-1}) quasi-decomposes $W\mu$, which yields that $W\mu$ is quasi-decomposable by the pair (a, cI).

Having $\mathbb{S}(W\mu) = \mathscr{O}$, let us choose, according to Lemma 2.1, an element $z' \in V$ and a measure ν, $\nu = W\mu * \delta(-z')$ such that $U\nu = \nu$ holds for each $U \in \mathscr{O}$, thus the Lévy–Khintchine representation of ν has the form

$$\hat{\nu}(v) = \exp\left\{-\frac{1}{2}(Dv, v) + \int_{V-\{0\}} [\cos(u, v) - 1]M(du)\right\}$$

with $D = UDU^*$ and $UM = M$ for each $U \in \mathscr{O}$. But the equality $D = UDU^*$ implies that $U^*D = DU^*$, $U \in \mathscr{O}$, and consequently, $D = b'I$ for some $b' > 0$. Taking $b = b'/2$, we get that

$$\hat{\nu}(v) = \exp\left\{-b\|v\|^2 + \int_{V-\{0\}} [\cos(u, v) - 1]M(du)\right\}.$$

The measure ν is (a, cI)-quasi-decomposable (since $W\mu$ is such), and from the preceding considerations and [5; Theorem 1.3] we obtain that either

$$\hat{\nu}(v) = \exp\left\{\textstyle\sum_{n=-\infty}^{\infty} a^{-n} \displaystyle\int_{c<\|u\|\le 1} [\cos c^n(u, v) - 1]m(du)\right\}$$

or

$$\hat{\nu}(v) = \exp\{-b\|v\|^2\}$$

with $Um = m$ for each $U \in \mathscr{O}$ because $m = M | \{c < \|u\| \le 1\}$.

Consequently, we have

$$\widehat{W\mu}(v) = \exp\left\{i(z', v) + \textstyle\sum_{n=-\infty}^{\infty} a^{-n} \displaystyle\int_{c<\|u\|\le 1} \cos[c^n(u, v) - 1]m(du)\right\}$$

or

$$\widehat{W\mu}(v) = \exp\{i(z', v) - b \| v \|^2\}$$

and, taking into account the relation $\widehat{W\mu}(v) = \hat{\mu}(W^*v)$, we get that $\hat{\mu}(v) = \widehat{W\mu}(W^{-1}v)$, which gives conditions (i), (ii).

Let us now assume that $\hat{\mu}$ is of the form (i) or (ii). Then, the measure $W\mu$ satisfies either

$$(*) \qquad \widehat{W\mu}(v) = \exp\left\{i(z, v) + \textstyle\sum_{n=-\infty}^{\infty} a^{-n} \displaystyle\int_{c<\|u\|\leq 1} [\cos c^n(u, v) - 1]m(du)\right\}$$

or

$$(**) \qquad \widehat{W\mu}(v) = \exp\{i(z, v) - b \| v \|^2\}.$$

Consider $(*)$ first. Using again Theorem 1.3 from [5], we conclude that $W\mu$ is quasi-decomposable by the pair (a, cI); thus it is multivariate semi-stable. From equality (1.5) in [5] and the fact that m is $\mathscr{O}$-invariant it follows that the Lévy spectral measure of $W\mu$ is $\mathscr{O}$-invariant; thus $\mathbb{S}(W\mu) = \mathscr{O}$, which implies $\mathbb{S}(\mu) = W^{-1}\mathscr{O}W$.

It is clear that, for $W\mu$ satisfying $(**)$, $\mathbb{S}(W\mu) = \mathscr{O}$ so that $\mathbb{S}(\mu) = W^{-1}\mathscr{O}W$ and μ, being Gaussian, is multivariate semi-stable. The proof of the theorem is completed. □

Applying the above results to operator-stable measures, we obtain the following corollary (cf. [1; Theorem 3]):

COROLLARY 2.5. *Let μ be a full operator-stable measure on V. If μ is elliptically symmetric, then it is multivariate stable with the characteristic function of the form*

$$(2.3) \qquad \hat{\mu}(v) = \exp\{i(z, v) - b \| W^{-1}v \|^{\gamma}\}$$

where W is a nonnegative operator on V such that

$$\mathbb{S}(\mu) = W^{-1}\mathscr{O}W, \quad z \in V, \ \gamma \in (0, 2], \quad b > 0.$$

Conversely, if $\hat{\mu}$ is of the form (2.3) *with some nonnegative operator W on V and z, γ, b as above, then μ is multivariate stable and $\mathbb{S}(\mu) = W^{-1}\mathscr{O}W$.*

PROOF. μ is quasi-decomposable by the pair (t, t^B) for each $t > 0$; consequently, on account of Corollary 2.3, it is quasi-decomposable by the pair $(t, c(t)I)$. In particular, taking $t = n$, we get

$$\mu^n = (c(n)I)\mu * \delta(h_n),$$

and hence

$$\mu = (c^{-1}(n)I)\mu^n * \delta(h_n) = \lim_{n\to\infty}(c^{-1}(n)I)\mu^n * \delta(h_n),$$

which proves the multivariate stability of μ.

Now, if $\mathbb{S}(\mu) = W^{-1}\mathscr{O}W$, then, considering the measure $W\mu$, from Lemma 2.1 we obtain that there exist $z' \in V$ and a measure ν, such that $\nu = W\mu * \delta(-z')$

and $U\nu = \nu$ for $U \in \mathscr{O}$. ν is multivariate stable since W is multivariate stable and symmetric, thus

$$\hat{\nu}(v) = \exp\{-b \| v \|^{\gamma}\} \quad \text{for some} \quad b > 0, \quad \gamma \in (0, 2].$$

The equality $\hat{\mu}(v) = \hat{\nu}(W^{-1}v)\exp(i(W^{-1}z', v))$ yields the claim.

The converse is proved in the standard way by considering the measure $W\mu$. □

3. A characterization of operator-stable and operator semi-stable measures. Now, we are going to describe a full infinitely divisible measure μ on V in terms of its quasi-decomposability group $\mathbb{G}(\mu)$. To this end, we begin with

LEMMA 3.1. *Let μ be a full infinitely divisible measure on V. Then $\mathbb{G}(\mu)$ is a closed multiplicative subgroup of $\mathbb{R}_+$.*

PROOF. It is easily seen that $\mathbb{G}(\mu)$ is a multiplicative subgroup of $\mathbb{R}_+$. We shall prove that $\mathbb{G}(\mu)$ is closed in $\mathbb{R}_+$.

Let $t_n \in \mathbb{G}(\mu)$ and $t_n \to t$, $t > 0$. We have that $\mu^{t_n} = A_n\mu * \delta(h_n) \Rightarrow \mu^t$. Since μ is full, therefore, on account of the Compactness Lemma, [6; Proposition 4], $\{A_n\}$ is relatively compact and, if A is a limit point of $\{A_n\}$, then $\mu^t = A\mu * \delta(h)$ for some $h \in V$. This shows that $t \in \mathbb{G}(\mu)$ and the proof is finished.

The following observation is of some importance for the theorem below; if $\mathbb{G}(\mu) \neq \{1\}$, then there are only two possibilities: either $\mathbb{G}(\mu) = \mathbb{R}_+$ or $\mathbb{G}(\mu) = \{a^n: n = 0, \pm 1, \cdots\}$ for some $0 < a < 1$ because $\mathbb{G}(\mu)$ is a closed subgroup of $\mathbb{R}_+$. For a full operator semi-stable and thus, operator-stable measure μ, always $\mathbb{G}(\mu) \neq \{1\}$ and we have

THEOREM 3.2. *Let μ be a full infinitely divisible measure on V. Then*

(i) *μ is operator-stable if and only if $\mathbb{G}(\mu) = \mathbb{R}_+$;*
(ii) *μ is strictly operator semi-stable if and only if*

$$\mathbb{G}(\mu) = \{a^n: n = 0, \pm 1, \cdots\} \quad \textit{for some} \quad 0 < a < 1.$$

PROOF. (i) If μ is operator-stable, then $\mu^t = t^B\mu * \delta(h_t)$ for $t > 0$ so that $\mathbb{G}(\mu) = \mathbb{R}_+$.

Assume that $\mathbb{G}(\mu) = \mathbb{R}_+$. We have

$$\mu^n = A_n\mu * \delta(h_n),$$

thus

$$\mu = A_n^{-1}\mu^n * \delta(h_n') = \lim_{n\to\infty}(A_n^{-1}\mu^n * \delta(h_n')),$$

which shows that μ is operator-stable.

(ii) If μ is strictly operator semi-stable, then, by (i), $\mathbb{G}(\mu) \neq \mathbb{R}_+$; so, there must be $\mathbb{G}(\mu) = \{a^n: n = 0, \pm 1, \cdots\}$ for some $0 < a < 1$.

Conversely, if $\mathbb{G}(\mu) = \{a^n: n = 0, \pm 1, \cdots\}$, then μ is quasi-decomposable by some pair (a, A) and, by virtue of [5; Remark 1.2], μ is operator semi-stable.

The proof of the theorem has thus been completed. □

REFERENCES

[1] HOLMES, J. P., HUDSON, W. N. and MASON, J. D. (1982). Operator-stable laws: multiple exponents and elliptical symmetry. *Ann. Probab.* **10** 602–612.

[2] JAJTE, R. (1977). Semi-stable probability measures on $\mathbb{R}^N$. *Studia Math.* **61** 29–39.

[3] KRUGLOV, W. M. (1972). On a class of limit distribution in a Hilbert space (in Russian). *Litovsk. Mat. Sb.* **12** 85–88.

[4] KUMAR, A. (1976). Semi-stable probability measures on Hilbert spaces. *J. Multivariate Anal.* **6** 309–318.

[5] ŁUCZAK, A. (1981). Operator semi-stable probability measures on $\mathbb{R}^N$. *Colloq. Math.* **45** 287–300.

[6] SHARPE, M. (1969). Operator-stable probability distributions on vector groups. *Trans. Amer. Math. Soc.* **136** 51–65.

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