The Annals of Probability
1985, Vol. 13, No. 1, 126–144

SECOND-ORDER APPROXIMATIONS TO THE DENSITY, MEAN AND VARIANCE OF BROWNIAN FIRST-EXIT TIMES[1]

BY CHRISTEL JENNEN

Universität Heidelberg

This paper presents correction terms to the tangent approximation for the first-exit density of Brownian motion at distant boundaries. These lead to second-order approximations to the first-exit distribution. Asymptotic formulas for the mean and variance of the first-exit time are derived. Numerical comparisons show the accuracy of the approximations.

1. Introduction. Let W be a Brownian motion with drift θ and let us consider the stopping time

$$T = \inf\{t > 0 : W(t) \geq \psi(t)\}, \quad \text{with} \quad T = \infty \quad \text{if the set is empty.} \tag{1}$$

We want to study the distribution of T under drift $\theta = 0$ and $\theta > 0$. The values $P_\theta(T < t)$ and $E_\theta T$ are of interest for sequential tests. No explicit formulas for these quantities are known, except in a few special cases. In this paper we give asymptotic approximations applicable when the boundaries are "remote" that is, crossed with only small probability if the drift is 0. More precisely, we will consider families $\{\psi_a : a > 0\}$ of continuously differentiable functions on intervals $(0, t_a)$ where $0 < t_a \leq \infty$, such that $P_0(T_a < t_a) \to 0$ for the corresponding first-exit times T_a. Examples which are discussed in the literature are:

(a) $\psi_a(t) = at^p, \quad 0 < p < 1/2$,

(b) $\psi_a(t) = \sqrt{tc \log(a/t)}, \quad c > 0$,

(c) $\psi_a(t) = \sqrt{2(at + c)}, \quad c > 0$,

(d) $\psi_a(t) = \sqrt{(t + 1)(a + \log(t + 1))}$.

For these families t_a may grow like a power of a or even exponentially. For the examples above Chow, Chao and Lai (1979), Pollak and Siegmund (1975), Robbins and Siegmund (1973), Siegmund (1977) and Woodroofe (1976) present an asymptotic analysis (partly for random walks) using renewal and martingale arguments. Using completely different methods Cuzick (1981) and Jennen and Lerche (1981) derive a very general formula for $P_0(T_a < t)$. The latter study the density f_a of T_a and show that under drift $\theta = 0$, $f_a(t)$ is asymptotically equivalent to the density of the first-exit time over the tangent to ψ_a at t

$$f_a(t) = \frac{\Lambda_a(t)}{t^{3/2}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\psi_a(t)^2}{2t}\right)(1 + o(1)), \tag{2}$$

where $\Lambda_a(t) = \psi_a(t) - t\psi_a'(t)$ is the intercept on the vertical axis of the tangent

Received February 1983; revised January 1984.

[1] This work was supported by the Deutsche Forschungsgemeinschaft.

AMS 1980 *subject classifications.* Primary 62L10; secondary 60G40, 60J65.

Key words and phrases. First-exit time, Brownian motion, sequential tests.

to the curve at t. (2) holds uniformly on $0 < t < t_a$ and therefore it may be integrated to provide approximations to $P_0(T_a < t)$ for $t < t_a$.

Evaluating certain integral equations for f_a, a method first employed by Daniels (1974), we derive a second-order term to the tangent approximation (2)

$$f_a(t) = \left[\frac{\Lambda_a(t)}{t^{3/2}} + \frac{t^{3/2}\psi_a''(t)}{2\,\Lambda_a(t)^2}\,(1 + o(1)) - \frac{\Lambda_a(t)}{t^{3/2}}\,P_0(T_a < t)\,(1 + o(1))\right] \cdot \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\psi_a(t)^2}{2t}\right). \tag{3}$$

Since (3) holds uniformly for $t < t_a$ one gets second-order terms for $P_0(T_a < t)$. In the case of example c this is equal to the second-order approximation given by Siegmund (1977). The first term in the brackets at the right-hand side of (3) tends to infinity while the second one tends to zero. In most cases the third term is dominated by the second and may be neglected. The first two terms of (3) are asymptotically equivalent to the first two terms in an asymptotic expansion of the first-exit density given by Ferebee (1983). However, since his evaluation includes no uniformity in t, it does not lead to approximations for the crossing probabilities.

Multiplying (2) and (3) by the Radon-Nikodym derivative $\exp(\theta\psi_a(t) - \theta^2 t/2)$ one gets asymptotic formulas for the first-exit density $f_a^\theta(t)$ of the Brownian motion with drift $\theta \neq 0$. The tangent approximation is

$$f_a^\theta(t) = \frac{\Lambda_a(t)}{t^{3/2}}\,\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(\psi_a(t) - \theta t)^2}{2t}\right)(1 + o(1)). \tag{4}$$

This formula holds uniformly in θ and t for $t < t_a$. Thus one gets approximations of $P_\theta(T_a < t)$.

For the families we consider, the ray $x = \theta t$ with $\theta > 0$ crosses the curve $x = \psi_a(t)$ in a unique point b_a, $\psi_a(b_a) = \theta b_a$, with $b_a \to \infty$ as $a \to \infty$. We show that the distribution of T_a is asymptotically normally distributed about b_a. Using the tangent approximation (4) we derive the following asymptotic expressions for the mean and variance of T_a

$$E_\theta T_a = b_a(1 + o(1)), \tag{5}$$

$$\mathrm{Var}_\theta T_a = \frac{b_a}{(\theta - \psi_a'(b_a))^2}\,(1 + o(1)). \tag{6}$$

Using the second-order approximation for the density we get a correction term to (5)

$$E_\theta T_a = b_a + \frac{b_a\psi_a''(b_a)}{2(\theta - \psi_a'(b_a))^3}\,(1 + o(1)). \tag{7}$$

This formula can also be derived more directly using Wald's identity. For the examples above, the second term in (7) is asymptotically constant in a. Numerical studies show that the second-order approximation (7) is much better than (5), just as the second-order density approximation (3) is superior to the tangent approximation (2).

Our program is as follows. In Section 2 we derive the asymptotic formula (3) for the first-exit density at remote boundaries. The asymptotic formulas for the mean and variance of T_a in the case of drift $\theta > 0$ are given in Section 3. In Section 4 we study delayed first-exit times

$$T = \inf\{t > \tau : W(t) \geq \psi(t)\}$$

where $\tau > 0$ is fixed. In Section 5 we apply the approximations to some examples and compare them with numerical results. Section 6 contains some extensions.

2. Asymptotic evaluation of the first-exit density. Let ψ be continuously differentiable on $(0, t_0)$ and let T be given by (1). We assume $P(T = 0) = 0$. Then the distribution of T has a continuous density f on $(0, t_0)$ and under drift $\theta = 0$

$$\frac{1}{\sqrt{t}}\varphi\left(\frac{\psi(t)}{\sqrt{t}}\right) = \int_0^t \frac{1}{\sqrt{t-u}}\varphi\left(\frac{\psi(t)-\psi(u)}{\sqrt{t-u}}\right) f(u)\, du, \tag{8}$$

$$\begin{aligned} f(t) = {} & \frac{\Lambda(t)}{t^{3/2}}\varphi\left(\frac{\psi(t)}{\sqrt{t}}\right) \\ & - \int_0^t \frac{\psi(t)-\psi(u)-(t-u)\psi'(t)}{(t-u)^{3/2}}\varphi\left(\frac{\psi(t)-\psi(u)}{\sqrt{t-u}}\right) f(u)\, du, \end{aligned} \tag{9}$$

where $\Lambda(t) = \psi(t) - t\psi'(t)$. As usual φ and Φ are the density and distribution function of the standard normal distribution. Equation (8) is given by Durbin (1971), (9) by Durbin (1981) and Ferebee (1982). The integral equation (9) is our main tool for deriving an approximation to the first-exit density at remote boundaries.

Let $\{\psi_a : a > 0\}$ be a family of continuously differentiable functions on R^+ and let T_a be the first-exit time of the standard Brownian motion W through ψ_a

$$T_a = \inf\{t > 0 : W(t) \geq \psi_a(t)\}, \quad \text{with} \quad T_a = \infty \quad \text{if the set is empty.} \tag{10}$$

Let $0 < t_a \leq \infty$ and $I_a = (0, t_a)$. We make the following assumptions:

- A1. $P(T_a = 0) = 0$,
- A2. $\int_{I_a} (\psi_a(u)/u^{3/2})\, \varphi(\psi_a(u)/\sqrt{u})\, du \to 0$ as $a \to \infty$,
- A3. there exist α and β with $1/2 < \alpha < 1$ and $\beta > 2\alpha - 1$ such that $\psi_a(t)/t^\alpha$ is decreasing and $\psi_a(t)/t^\beta$ is increasing in t for all a,
- A4. ψ_a is twice continuously differentiable on I_a and for every $\rho_1 > 0$ there exist $\rho_2 > 0$ and a_0, such that if $a \geq a_0$ and $s, t \in I_a$ with $|s/t - 1| < \rho_2$ then $|\psi_a'(s)/\psi_a'(t) - 1| < \rho_1$ and $|\psi_a''(s)/\psi_a''(t) - 1| < \rho_1$,
- A5. there exist $\varepsilon < 1$ and $B < \infty$ such that $|t^{3/2}\psi_a''(t)| < B(\psi_a(t)/\sqrt{t})^{1+\varepsilon}$ for all $t \in I_a$ and all a.

These assumptions imply that the boundaries are remote.

LEMMA 1. *Let conditions A1–A3 hold. Then*

$$P(T_a < t_a) \to 0 \tag{11}$$

and

$$\psi_a(t)/\sqrt{t} \to \infty \quad \text{uniformly on} \quad I_a \quad \text{as} \quad a \to \infty. \tag{12}$$

PROOF. From (8) and (9) one can derive another integral equation for the first-exit density f_a

$$f_a(t) = \frac{\psi_a(t)}{t^{3/2}}\varphi\left(\frac{\psi_a(t)}{\sqrt{t}}\right) - \int_0^t \frac{\psi_a(t) - \psi_a(u)}{(t-u)^{3/2}}\varphi\left(\frac{\psi_a(t) - \psi_a(u)}{\sqrt{t-u}}\right)f_a(u)\,du.$$

Since ψ_a is increasing by A3 the integral is positive and it follows that

$$f_a(t) \le (\psi_a(t)/t^{3/2})\varphi(\psi_a(t)/\sqrt{t}). \tag{13}$$

This inequality together with assumption A2 gives (11).

(12) follows from (11) since for $t < t_a$

$$1 - \Phi(\psi_a(t)/\sqrt{t}) = P(W(t) \ge \psi_a(t)) \le P(T_a < t_a) \to 0.$$

Under assumptions A1–A4, Theorem 1 of Jennen and Lerche holds, i.e.

$$f_a(t) = (\Lambda_a(t)/t^{3/2})\varphi(\psi_a(t)/\sqrt{t})(1 + o(1)) \text{ uniformly on } I_a \text{ as } a \to \infty. \tag{14}$$

THEOREM 1. *Let conditions A1–A5 hold. Then*

$$\begin{aligned} f_a(t) = {} & [\Lambda_a(t)/\sqrt{t} + \psi_a''(t)t^{5/2}/2\Lambda_a(t)^2(1 + o(1)) \\ & - (\Lambda_a(t)/\sqrt{t})P(T_a < t)(1 + o(1)) + o(R_a(t))]t^{-1}\varphi(\psi_a(t)/\sqrt{t}) \end{aligned} \tag{15}$$

holds uniformly on I_a. The remainder R_a satisfies $R_a(t) = \exp(-(\psi_a(t)/\sqrt{t})^\kappa)$ for some $\kappa > 0$ depending on α, β and ε.

COROLLARY. *Let condition A1–A5 hold. Then*

$$\begin{aligned} P(T_a < t) = {} & \int_0^t (\Lambda_a(u)/u^{3/2})\varphi(\psi_a(u)/\sqrt{u})\,du \\ & + \int_0^t (\psi_a''(u)u^{3/2}/2\Lambda_a(u)^2)\varphi(\psi_a(u)/\sqrt{u})\,du(1 + o(1)) \\ & - \left(\frac{1}{2}\right)\left(\int_0^t (\Lambda_a(u)/u^{3/2})\varphi(\psi_a(u)/\sqrt{u})\,du\right)^2(1 + o(1)) \\ & + o\left(\int_0^t (R_a(u)/u)\varphi(\psi_a(u)/\sqrt{u})\,du\right) \end{aligned} \tag{16}$$

holds uniformly for $t \in I_a$.

Note that $\int_0^t f_a(u)P(T_a < u)\,du = P(T_a < t)^2/2$. Thus the corollary follows by integrating (15).

Let us consider the terms in the brackets at the right-hand side of (15). The first term $\Lambda_a(t)/\sqrt{t}$ corresponds to the tangent approximation. This term is positive since by A3

$$\beta\psi_a(t)/t \le \psi_a'(t) \le \alpha\psi_a(t)/t \tag{17}$$

and therefore

$$(1-\alpha)\psi_a(t) \le \Lambda_a(t) \le (1-\beta)\psi_a(t). \tag{18}$$

From (12) and (18) it follows that $\Lambda_a(t)/\sqrt{t} \to \infty$ uniformly on I_a. The second term $\psi_a''(t)t^{5/2}/2\Lambda_a(t)^2$ is a "local" correction term to the tangent approximation, i.e. it only depends on the behaviour of the curve at t. By the assumptions it is of the order $O((\psi_a(t)/\sqrt{t})^{\varepsilon-1})$ and since $\varepsilon < 1$ it tends to zero. For the examples given in the introduction $\varepsilon = 0$ and the term behaves like $\sqrt{t}/\Lambda_a(t)$. $P(T_a < t)\Lambda_a(t)/\sqrt{t}$ is a "global" correction term, i.e. it depends on the whole curve $\psi_a(s)$, $s \in (0, t)$. The unknown probability $P(T_a < t)$ is asymptotically equal to $\int_0^t (\Lambda_a(u)/u^{3/2})\varphi(\psi_a(u)/\sqrt{u})\,du$. If t_a is not too large $P(T_a < t_a)$ tends to zero very rapidly and the global term is of lower order than the local one. In general the remainder $R_a(t)$ tends to zero very rapidly and may be neglected.

PROOF OF THEOREM 1. In what follows we omit the index a. The remainder estimates in o- and O-notations always refer to the limit $a \to \infty$ and are uniform on I_a. Define

$$g(t) = \frac{tf(t)}{\varphi(\psi(t)/\sqrt{t})}. \tag{19}$$

From (9) we get the following integral equation for g

$$g(t) = \Lambda(t)/\sqrt{t} - \int_0^t K(t, u)g(u)\,du, \tag{20}$$

where

$$K(t, u) = \frac{t(\psi(t) - \psi(u) - (t-u)\psi'(t))}{u(t-u)^{3/2}\sqrt{2\pi}} \exp\left(-\frac{t(\psi(u) - \psi(t)u/t)^2}{2u(t-u)}\right). \tag{21}$$

We split the integral into three parts $\int_0^r$, $\int_r^s$, $\int_s^t$, where

$$r = t(t/\psi(t)^2)^\gamma \quad\text{and}\quad s = t(1 - (t/\psi(t)^2)^\delta) \tag{22}$$

with $1/\beta < \gamma < 1/(2\alpha - 1)$ and $1/2 + \varepsilon/4 < \delta < 1$. It will be shown that there is a $\kappa > 0$ such that

$$\begin{aligned}\int_0^r K(t, u)g(u)\,du \\ = (\Lambda(t)/\sqrt{t})P(T < t)(1 + o(1)) + o(\exp(-(\psi(t)/\sqrt{t})^\kappa)),\end{aligned} \tag{23}$$

$$\int_r^s K(t, u)g(u)\,du = o(\exp(-(\psi(t)/\sqrt{t})^\kappa)), \tag{24}$$

$$(25)\qquad \int_s^t K(t, u)g(u)\,du = -\psi''(t)t^{5/2}/2\Lambda(t)^2(1 + o(1)).$$

The theorem will then follow from (19) and (20). First we prove (23). Let $u \le r$. From (17), (18) and A3 it follows that $\psi(u) = O(\Lambda(t)(r/t)^\beta)$ and $u\psi'(t) = O(\Lambda(t)r/t)$. Since $r/t \to o$, we get

$$(26)\qquad \frac{t(\psi(t) - \psi(u) - (t - u)\psi'(t))}{u(t - u)^{3/2}} = \frac{\Lambda(t)}{u\sqrt{t}}(1 + o(1)).$$

Now we show that

$$(27)\qquad \varphi\left(\frac{\psi(t) - \psi(u)}{\sqrt{t - u}}\right) \Big/ \varphi\left(\frac{\psi(t)}{\sqrt{t}}\right) = 1 + o(1).$$

Since $r/t \to 0$ we have for a large enough

$$\begin{aligned}\left|\frac{(\psi(t) - \psi(u))^2}{t - u} - \frac{\psi(t)^2}{t}\right| &= \left|\frac{\psi(t)^2u/t + \psi(u)^2 - 2\psi(t)\psi(u)}{t - u}\right| \\ &\le \frac{2\psi(t)^2}{t}\left(\frac{r}{t}\right) + \frac{2\psi(t)^2}{t}\left(\frac{r}{t}\right)^{2\beta} + \frac{4\psi(t)^2}{t}\left(\frac{r}{t}\right)^\beta \\ &\le \frac{8\psi(t)^2}{t}\left(\frac{r}{t}\right)^\beta = 8\left(\frac{\psi(t)^2}{t}\right)^{1-\gamma\beta}.\end{aligned}$$

Since $\gamma\beta > 1$ this expression tends to 0 and (27) follows. (26), (27) and the definition (19) of g yield

$$\begin{aligned}\int_0^r K(t, u)g(u)\,du &= \frac{\Lambda(t)}{\sqrt{t}}(1 + o(1))\int_0^r \varphi\left(\frac{\psi(u)}{\sqrt{u}}\right)\frac{g(u)}{u}\,du \\ &= (\Lambda(t)/\sqrt{t})(1 + o(1))P(T < r).\end{aligned}$$

From this we get (23) if we show that

$$(28)\qquad P(r \le T < t) \le \exp(-(\psi(t)^2/t)^\eta) \quad \text{for some} \quad \eta > 0 \quad \text{and large } a.$$

Since by (13)

$$P(r \le T < t) \le \int_r^t (\psi(u)/u^{3/2})\varphi(\psi(u)/\sqrt{u})\,du$$

and since by A3

$$\psi(u)^2/u \ge (\psi(t)^2/t)(u/t)^{2\alpha-1} \ge (\psi(t)^2/t)^{1-\gamma(2\alpha-1)},$$

there exist $0 < \eta < \nu < 1 - \gamma(2\alpha - 1)$ such that for a large enough

$$\begin{aligned}P(r \le T < t) &\le \exp(-(\psi(t)^2/t)^\nu)\int_r^t u^{-1}\,du \\ &= \exp(-(\psi(t)^2/t)^\nu)\log((\psi(t)^2/t)^\gamma) \le \exp(-(\psi(t)^2/t)^\eta).\end{aligned}$$

To prove (24) we first estimate the exponent in K for $u \in [r, s]$. By A3 we have $\psi(u) \geq \psi(t)(u/t)^{\alpha}$ and therefore

$$\frac{t(\psi(u) - \psi(t)u/t)^2}{u(t-u)} \geq \frac{\psi(t)^2}{t}\left(\left(\frac{t}{u}\right)^{1-\alpha} - 1\right)^2\left(\frac{t}{u} - 1\right)^{-1}.$$

The right-hand side tends to infinity. To see this, note that the function $((t/u)^{1-\alpha} - 1)^2/(t/u - 1)$ is 0 for $u = 0$ and $u = t$ and strictly positive for $0 < u < t$. Since $r/t \to 0$ and $s/t \to 1$ the function takes its infimum on $[r, s]$ in $u = r$ or $u = s$ if a is large enough. We have for large a

$$((t/r)^{1-\alpha} - 1)^2/(t/r - 1) \geq ½(r/t)^{2\alpha-1} = ½(t/\psi(t)^2)^{\gamma(2\alpha-1)},$$

$$((t/s)^{1-\alpha} - 1)^2/(t/s - 1) \geq ½(1 - \alpha)^2(t-s)/t = ½(1-\alpha)^2(t/\psi(t)^2)^{\delta}.$$

Since $\gamma(2\alpha - 1) < 1$ and $\delta < 1$ it follows that there is some $\nu > 0$ such that for large a

$$\exp\left(-\frac{t(\psi(u) - \psi(t)u/t)^2}{2u(t-u)}\right) \leq \exp\left(-\left(\frac{\psi(t)}{\sqrt{t}}\right)^{\nu}\right).$$

Using (13), (19) and A3 we get

$$|t(\psi(t) - \psi(u) - (t-u)\psi'(t))g(u)/(t-u)^{3/2}| \leq (\psi(t)/\sqrt{t})^{\rho}$$

for some $\rho < \infty$. Hence

$$\left|\int_r^s K(t, u)g(u)\,du\right| \leq (\psi(t)/\sqrt{t})^{\rho}\exp(-(\psi(t)/\sqrt{t})^{\nu})\int_r^s u^{-1}\,du$$
$$\leq \exp((\psi(t)/\sqrt{t})^{\eta})$$

for some $0 < \eta < \nu$ and large a which proves (24).

For $u \in [s, t]$ we expand $\psi(u)$ about t. Since $s/t \to 1$ we have by A4 for $v \in [s, t]\psi''(v) = \psi''(t)(1 + o(1))$ and therefore

(29) $\quad (\psi(t) - \psi(u) - (t-u)\psi'(t))/(t-u)^{3/2} = -\sqrt{t-u}\,\psi''(t)/2(1 + o(1)).$

For the exponent in K we get

$$\frac{t(\psi(u) - \psi(t)u/t)^2}{u(t-u)} = \frac{t((t-u)\Lambda(t)/t + (t-u)^2\psi''(t)/2(1 + o(1)))^2}{u(t-u)}$$
$$= (t-u)(\Lambda(t)/t)^2 + o(1),$$

for by assumption A5 and the definition (22) of s the remainder is of the order

$$O((\psi(t)/\sqrt{t})^{2+\varepsilon}((t-s)/t)^2) = O((\psi(t)/\sqrt{t})^{2+\varepsilon-4\delta}) = o(1)$$

since $\delta > ½ + \varepsilon/4$. Hence

(30) $$\exp\left(-\frac{t(\psi(u) - \psi(t)u/t)^2}{2u(t-u)}\right) = \exp\left(-\frac{t-u}{2}\left(\frac{\Lambda(t)}{t}\right)^2\right)(1 + o(1)).$$

By (14) $g(u) = \Lambda(u)/\sqrt{u}\,(1 + o(1))$. One can derive from A3 and A4 that $\Lambda(u) = \Lambda(t)(1 + o(1))$ for $u \in [s, t]$. This together with (29) and (30) gives

$$\int_s^t K(t, u)g(u)\,du = -(\psi''(t)\Lambda(t)/2\sqrt{t})(1 + o(1))J$$

where

$$J = \int_s^t \sqrt{t-u}\,\varphi(\sqrt{t-u}\,\Lambda(t)/t)\,du = 2(t/\Lambda(t))^3 \int_0^{\sqrt{t-s}\Lambda(t)/t} x^2\varphi(x)\,dx.$$

Since $\sqrt{t-s}\,\Lambda(t)/t \geq (1-\alpha)(\psi(t)/\sqrt{t})^{1-\delta} \to \infty$ it follows that

$$J = (t/\Lambda(t))^3(1 + o(1))$$

which completes the proof of (25).

The functions $\psi_a(t) = \sqrt{2(at + c)}$, $c > 0$, of Siegmund's repeated significance tests (Siegmund, 1977) do not fulfill the assumption A3 since $\psi_a(t)/t^\beta$ is not increasing in t for small t if $\beta > 0$. Therefore one cannot apply Theorem 1 directly. In the proof the assumption in question is only used to evaluate the integral $\int_0^r K(t, u)g(u)\,du$ (see (23)). If one makes the weaker assumption ψ_a increasing on I_a and if one requires in addition to the other assumptions that

$$\exp(\psi_a(t)\psi_a(r)/t - \psi_a(r)^2/2t) \int_0^r (\psi_a(u)/u^{3/2})\varphi(\psi_a(u)/\sqrt{u})\,du = o((\sqrt{t}/\psi_a(t))^\eta)$$

for some $\eta > 0$, where $r = t(t/\psi_a(t)^2)^\gamma$ and $1 < \gamma < 1/(2\alpha - 1)$, then one can show that

$$f_a(t) = [\Lambda_a(t)/\sqrt{t} + \psi_\alpha''(t)t^{5/2}/2\Lambda_\alpha(t)^2(1 + o(1)) + o((\psi_\alpha(t)/\sqrt{t})^{1-\eta})]t^{-1}\varphi(\psi_a(t)/\sqrt{t}) \tag{31}$$

holds uniformly on I_a as $a \to \infty$.

3. Asymptotic mean and variance of the first-exit time. Now we study the first-exit times (10) for a Brownian motion with positive drift θ. For the family $\{\psi_a : a > 0\}$ of continuously differentiable functions on R^+ we require that A3 holds for all t.

Under this assumption the ray $x = \theta t$ crosses the curve $x = \psi_a(t)$ in a unique point b_a

$$\theta b_a = \psi_a(b_a). \tag{32}$$

We further assume that the other assumptions of Section 2 hold with $t_a = cb_a$ for some $c > 1$. Then b_a must tend to infinity since $\theta^2 b_a = \psi_a(b_a)^2/b_a \to \infty$ by (12).

A3 implies that for every a there is a straight line h_a with slope less than θ such that $\psi_a(t) < h_a(t)$ for all $t \geq 0$. Since T_a is smaller than the first-exit time through h_a, all of those moments are finite, all moments of T_a are also finite.

Since the Brownian motion remains within a band of width $t^{1/2+\rho}$, $\rho > 0$, about the ray $x = \theta t$, it is plausible that the distribution of T_a is asymptotically concentrated around the point b_a where this ray intersects the boundary. It turns out that the mean and variance of T_a are asymptotically equal to the mean and variance of the first-exit time through the tangent to ψ_a at b_a.

THEOREM 2. *Let conditions A1–A5 hold. Then*

$$E_\theta T_a = b_a(1 + o(1)) \quad \text{as} \quad a \to \infty. \tag{33}$$

If, in addition

$$\int_0^{b_a} (\psi_a(t)/t^{3/2})\varphi(\psi_a(t)/\sqrt{t})\, dt = o(b_a^{-1}), \tag{34}$$

then

$$\operatorname{Var}_\theta T_a = b_a/\theta_a^2(1 + o(1)), \tag{35}$$

$$E_\theta T_a = b_a + b_a\psi_a''(b_a)/(2\theta_a^3) + o(b_a^{\varepsilon/2}), \tag{36}$$

where $\theta_a = \theta - \psi_a'(b_a)$ *and* ε *is determined by A5.*

Note that by A3 $(1 - \alpha)\theta \le \theta_a \le (1 - \beta)\theta$. Therefore the mean and variance of T_a are of the same order. The second-order term in (36) is $O(b_a^{\varepsilon/2})$. For the examples given in the introduction this term is asymptotically constant in a and the remainder is $o(1)$ (see Section 5).

As in Section 2 one can dispense with the assumption $\psi_a(t)/t^\beta$ increasing in t on I_a. If the ψ_a are increasing and if in addition to the other assumptions

$$\exp(\theta\psi_a(r_a)) \int_0^{r_a} (\psi_a(t)/t^{3/2})\varphi(\psi_a(t)/\sqrt{t})\, dt = o(b_a^{-1})$$

where $r_a = b_a^{-\nu}$, $0 < \nu < (2 - 2\alpha)/(2\alpha - 1)$ then Theorem 2 also holds.

We prove Theorem 2 using the tangent approximation to the first-exit density. Since

$$f_a^\theta(t) = \exp(\theta\psi_a(t) - \theta^2 t/2)f_a(t) \tag{37}$$

where f_a^θ is the density of T_a under drift θ and f_a is the density under drift 0, it follows from (14) that

$$f_a^\theta(t) = (\Lambda_a(t)/t^{3/2})\varphi((\psi_a(t) - \theta t)/\sqrt{t})(1 + o(1)) \tag{38}$$

holds uniformly on I_a.

The density is asymptotically normal in an interval about b_a and it vanishes very rapidly outside this interval. For small t the density under drift θ behaves like the density under drift 0.

LEMMA 2. *Let*

$$r_a = b_a^{-\nu}, \quad s_a = b_a - b_a^\mu \quad \text{and} \quad S_a = b_a + b_a^\mu \tag{39}$$

where $(1 - \beta)/\beta < \nu < (2\alpha - 2)/(2\alpha - 1)$ *and* $1/2 < \mu < 2/3 - \varepsilon/6$. *Then for* $n \ge 0$

there is some $\kappa > 0$ *such that*

$$E_\theta(T_a^n 1_{\{T_a > S_a\}}) = o(\exp(-b_a^\kappa)), \tag{40}$$

$$P_\theta(r_a < T_a < s_a) = o(\exp(-b_a^\kappa)), \tag{41}$$

$$f_a^\theta(t) = f_a(t)(1 + o(1)) \tag{42}$$

uniformly on $(0, r_a)$,

$$f_a^\theta(t) = (\theta_a/\sqrt{b_a})\varphi(\theta_a(t - b_a)/\sqrt{b_a})(1 + o(1)) \tag{43}$$

uniformly on (s_a, S_a).

Lemma 2 together with inequality (13) and Assumption A2 implies

$$P_\theta(T_a < r_a) = o(1). \tag{44}$$

Under the stronger condition (34) one has

$$P_\theta(T_a < r_a) = (b_a^{-1}). \tag{45}$$

From (44) and Lemma 2 we get

PROPOSITION. *Let conditions A1–A5 hold. Then*

$$P_\theta(\theta_a(T_a - b_a)/\sqrt{b_a} \le y) = \Phi(y) + o(1) \tag{46}$$

uniformly in y.

Now we will sketch the proofs of Lemma 2 and Theorem 2.

PROOF OF LEMMA 2. From (37) and (13) we get

$$f_a^\theta(t) \le (\psi_a(t)/t^{3/2})\exp(-(\psi_a(t) - \theta t)^2/2t). \tag{47}$$

Estimating the right-hand side of (47) and integrating yields (40) and (41). (42) follows from (37).

To prove (43) one has to expand the right-hand side of (38) about b_a.

PROOF OF THEOREM 2. From Lemma 2 it follows that

$$E_\theta(T_a - b_a)^n = (\sqrt{b_a}/\theta_a)^n \int_{-\infty}^{+\infty} x^n\varphi(x)\,dx(1 + o(1)) + P_\theta(T_a < r)O(b_a^n). \tag{48}$$

Putting $n = 1$ we get (33) by (44). If (34) holds then (48) and (45) imply that $E_\theta(T_a - b_a) = o(\sqrt{b_a})$ and $E_\theta(T_a - b_a)^2 = b_a/\theta_a^2 + o(b_a)$. This yields (35).

To prove (36) we use the identity

$$\theta E_\theta T_a = E_\theta \psi_a(T_a). \tag{49}$$

Expanding ψ_a about b_a and using A3, A5, Lemma 2 and (45) we get (36).

We derived the second-order term for $E_\theta T_a$ in (36) using only the first-order approximation for the density. We can also prove (36) expanding the second-order approximation for the density about b_a. But the method above is shorter. Furthermore it can be used to derive higher-order terms for the mean and variance (see Section 6).

4. Delayed stopping times. Now we consider stopping times for the standard Brownian motion of the form

$$T = \inf\{t > \tau : W(t) \geq \psi(t)\}, \quad \text{with} \quad T = \infty \text{ if the set is empty,} \tag{50}$$

where $\tau > 0$. Delayed stopping times of this kind are of interest especially for curves like $\psi(t) = \sqrt{t}$ which are crossed immediately after zero. For the density of T one can derive similar equations as in the case $\tau = 0$.

LEMMA 5. *Let $0 < \tau < t_0$ and let ψ be continuously differentiable on (τ, t_0). Assume $\lim_{t \to \tau} \psi(t) > -\infty$. Then the stopping time* (50) *has a continuous density f on (τ, t_0) and the following integral equations hold*

$$\frac{1}{\sqrt{t}} \varphi\left(\frac{\psi(t)}{\sqrt{t}}\right) \Phi\left(\frac{\psi(\tau) - \psi(t)\tau/t}{\sqrt{(t-\tau)\tau/t}}\right) = \int_\tau^t \frac{1}{\sqrt{t-u}} \varphi\left(\frac{\psi(t) - \psi(u)}{\sqrt{t-u}}\right) f(u)\, du, \tag{51}$$

$$\begin{aligned} f(t) = &\left[\frac{\Lambda(t)}{t^{3/2}} \Phi\left(\frac{\psi(\tau) - \psi(t)\tau/t}{\sqrt{(t-\tau)\tau/t}}\right)\right. \\ &\left. + \left(\frac{\tau}{t-\tau}\right)^{1/2} \frac{1}{t} \varphi\left(\frac{\psi(\tau) - \psi(t)\tau/t}{\sqrt{(t-\tau)\tau/t}}\right)\right] \varphi\left(\frac{\psi(t)}{\sqrt{t}}\right) \\ &- \int_\tau^t \frac{\psi(t) - \psi(u) - (t-u)\psi'(t)}{(t-u)^{3/2}} \varphi\left(\frac{\psi(t) - \psi(u)}{\sqrt{t-u}}\right) f(u)\, du. \end{aligned} \tag{52}$$

When the boundary is a straight line, $\psi(t) = b + ct$, the integral in (52) vanishes and we get the formula

$$f(t) = \left[\frac{b}{t^{3/2}} \Phi\left(b\left(\frac{t-\tau}{\tau t}\right)^{1/2}\right) + \left(\frac{\tau}{t-\tau}\right)^{1/2} \frac{1}{t} \varphi\left(b\left(\frac{t-\tau}{\tau t}\right)^{1/2}\right)\right] \varphi\left(\frac{\psi(t)}{\sqrt{t}}\right). \tag{53}$$

PROOF OF LEMMA 5. Let $F(t) = P(T < t) = P(\tau < T < t) + P(T = \tau)$. Since $\lim_{t \to \tau} \psi(t) > -\infty$ it follows that $P(\tau < T < t) > 0$.

We have to show that $P(\tau < T < t)$ is continuously differentiable with respect to t and that the derivative $f(t)$ fulfills the equations (51) and (52). We have

$$P(\tau < T < t) = \int_{-\infty}^{\psi(\tau)} P(\tau < T < t \mid W(\tau) = x)(1/\sqrt{\tau})\varphi(x/\sqrt{\tau})\, dx. \tag{54}$$

By the remarks in Section 2 the conditional probability $P(\tau < T < t \mid W(\tau) = x)$ is continuously differentiable with respect to t for $x < \psi(\tau)$ and the conditional density $f(t \mid W(\tau) = x)$ fulfills the integral equation

$$f(t \mid W(\tau) = x) = \frac{\psi(t) - (t-\tau)\psi'(t) - x}{(t-\tau)^{3/2}} \varphi\left(\frac{\psi(t) - x}{\sqrt{t-\tau}}\right) - \int_\tau^t \frac{\psi(t) - \psi(u) - (t-u)\psi'(t)}{(t-u)^{3/2}} \cdot \varphi\left(\frac{\psi(t) - \psi(u)}{\sqrt{t-u}}\right) f(u \mid W(\tau) = x)\, du. \tag{55}$$

With arguments similar to those employed by Ferebee (1981) one can show that the right-hand side of (54) has a continuous derivative and that differentiation and integration can be exchanged, so

$$f(t) = \frac{d}{dt} P(\tau < T < t) = \int_{-\infty}^{\psi(\tau)} f(t \mid W(\tau) = x) \frac{1}{\sqrt{\tau}} \varphi\left(\frac{x}{\sqrt{\tau}}\right) dx.$$

Now we insert the expression (55) for $f(t \mid W(\tau) = x)$. After integrating over x we get for the first term

$$\left[\frac{\Lambda(t)}{t^{3/2}} \Phi\left(\frac{\psi(\tau) - \psi(t)\tau/t}{\sqrt{(t-\tau)\tau/t}}\right) + \left(\frac{\tau}{t-\tau}\right)^{1/2} \frac{1}{t} \varphi\left(\frac{\psi(\tau) - \psi(t)\tau/t}{\sqrt{(t-\tau)\tau/t}}\right)\right] \varphi\left(\frac{\psi(t)}{\sqrt{t}}\right).$$

In the second part we change the order of integration and get the integral in (52) since

$$\int_{-\infty}^{\psi(\tau)} f(u \mid W(\tau) = x)(1/\sqrt{\tau})\varphi(x/\sqrt{\tau})\, dx = f(u).$$

In the same way one can derive (51) using (8) for the conditional density.

Let us now consider a family of stopping times

$$T_a = \inf\{t > \tau : W(t) \geq \psi_a(t)\}, \quad \text{with} \quad T_a = \infty \quad \text{if the set is empty.}$$

We assume that the functions ψ_a are continuously differentiable on (τ, ∞) and that conditions A2–A5 of Section 2 hold, where now $I_a = (\tau, t_a)$ and $\tau < t_a \leq \infty$. Instead of A1 we require

A1′. $\psi_a(\tau) \to \infty$ as $a \to \infty$.

Then $P(T_a = \tau) \to 0$. As in Section 2 one can show that $P(T_a < t_a) \to 0$ and therefore $\psi_a(t)/\sqrt{t} \to \infty$ uniformly on I_a (cf. (11) and (12)). Examples of boundaries which meet our requirements are

(e) $\psi_a(t) = a\sqrt{t}$,

(f) $\psi_a(t) = \sqrt{t(a^2 + \log t)}$.

They are discussed in Section 5.

First we will derive asymptotic expressions for the density f_a of T_a using equation (52).

THEOREM 3. *Let assumptions A1′ and A2–A5 hold. Let*

$$\tau_a = \tau(1 + (\tau/\psi_a(\tau)^2)^\rho) \tag{56}$$

where $\frac{1}{2} + \varepsilon/4 < \rho < 1$. *Then*

$$\begin{aligned} f_a(t) = {} & [\Lambda_a(t)/\sqrt{t} + \psi_a''(t)t^{5/2}/2\Lambda_a(t)^2(1 + o(1)) \\ & -(\Lambda_a(t)/\sqrt{t})P(T_a < t)(1 + o(1)) + o(R_a(t))]t^{-1}\varphi(\psi_a(t)/\sqrt{t}) \end{aligned} \tag{57}$$

uniformly on $[\tau_a, t_a]$ *and*

$$\begin{aligned} f_a(t) = {} & \{[\Lambda_a(t)/\sqrt{t} + \psi_a''(\tau)\tau^{5/2}/2\Lambda_a(\tau)^2(1 + o(1))]\Phi(\sqrt{t-\tau}\,\Lambda_a(\tau)/\tau) \\ & + [\sqrt{\tau/(t-\tau)} - \sqrt{\pi/2}\,\psi_a''(\tau)\tau^{5/2}/2\Lambda_a(\tau)^2 \\ & \quad - \sqrt{t-\tau}\,\psi_a''(\tau)\tau^{3/2}/2\Lambda_a(\tau) + O(1/\psi_a(\tau)^3)] \\ & \cdot \varphi(\sqrt{t-\tau}\,\Lambda_a(\tau)/\tau)\}t^{-1}\varphi(\psi_a(t)/\sqrt{t}) \end{aligned} \tag{58}$$

uniformly on (τ, τ_a), *where* $R_a(t) = \exp(-(\psi_a(t)/\sqrt{t})^\kappa)$ *for some* $\kappa > 0$.

Note that $\tau_a \to \tau$. For $t > \tau_a$, $f_a(t)$ has asymptotically the same form as in the case $\tau = 0$. Integrating (57) and (58) one gets approximation for $P(\tau < T_a < t)$. For the examples above the integral from τ to τ_a only plays a role in a second-order approximation.

We only sketch the proof since the methods are similar to those employed in Section 2. Instead of (9) we use the integral equation (52). We omit the index a. We put

$$g(t) = tf(t)/\varphi(\psi(t)/\sqrt{t}) \tag{59}$$

and get

$$g(t) = (\Lambda(t)/\sqrt{t})\Phi(h(t)) + \sqrt{\tau/(t-\tau)}\varphi(h(t)) - \int_\tau^t K(t,u)g(u)\,du, \tag{60}$$

where K is given in (21) and

$$h(t) = (\psi(\tau) - \psi(t)\tau/t)/\sqrt{(t-\tau)\tau/t}. \tag{61}$$

One can show that the integral in (60) is dominated by the other terms so that

$$g(t) = (\Lambda(t)/\sqrt{t})\Phi(h(t))(1 + o(1)) + \sqrt{\tau/(t-\tau)}\varphi(h(t)) \tag{62}$$

holds uniformly on I_a. Proceeding as in the proof of Theorem 1 one can derive (57). Instead of (23) one gets

$$\int_\tau^r K(t,u)g(u)\,du = (\Lambda(t)/\sqrt{t})P(\tau < T < t)(1 + o(1)) + o(R(t)).$$

Furthermore one has

$$(\Lambda(t)/\sqrt{t})\Phi(h(t)) = (\Lambda(t)/\sqrt{t}) - (\Lambda(t)/\sqrt{t})P(T = \tau)(1 + o(1)) + o(R(t)).$$

The second term in (60) yields only o-terms.

To prove (58) one has to estimate the integral in (60). For $g(u)$ one can insert the expression from (62). After expanding about τ one gets

$$\begin{aligned}\int_\tau^t K(t, u)g(u)\,du &= -(\sqrt{\tau}\,\psi''(\tau)/2)(1 + o(1)) \\ &\quad\cdot \int_\tau^t \sqrt{(t-u)/(u-\tau)}\,\varphi(\sqrt{t-u}\,\Lambda(\tau)/\tau)\varphi(\sqrt{u-\tau}\Lambda(\tau)/\tau)\,du \\ &\quad - (\psi''(\tau)\Lambda(\tau)/2\sqrt{\tau})(1 + o(1)) \\ &\quad\cdot \int_\tau^t \sqrt{t-u}\varphi(\sqrt{t-u}\Lambda(\tau)/\tau)\Phi(\sqrt{u-\tau}\Lambda(\tau)/\tau)\,du. \end{aligned} \tag{63}$$

After integration by parts the last integral may be rewritten as

$$\begin{aligned}&\int_\tau^t \sqrt{t-u}\varphi(\sqrt{t-u}\Lambda(\tau)/\tau)\Phi(\sqrt{u-\tau}\Lambda(\tau)/\tau)\,du \\ &\quad= -(\tau/\Lambda(\tau))^2\sqrt{t-\tau}\varphi(\sqrt{t-\tau}\Lambda(\tau)/\tau) \\ &\qquad + (\tau/\Lambda(\tau))^2 \int_\tau^t (1/\sqrt{t-u})\varphi(\sqrt{t-u}\Lambda(\tau)/\tau)\Phi(\sqrt{u-\tau}\Lambda(\tau)/\tau)\,du \\ &\qquad - (\tau/\Lambda(\tau)) \int_\tau^t \sqrt{(t-u)/(u-\tau)}\varphi(\sqrt{t-u}\Lambda(\tau)/\tau) \\ &\qquad \cdot \varphi(\sqrt{u-\tau}\Lambda(\tau)/\tau)\,du. \end{aligned} \tag{64}$$

It can be shown that

$$\begin{aligned}&\int_\tau^t (1/\sqrt{t-u})\varphi(\sqrt{t-u}\Lambda(\tau)/\tau)\Phi(\sqrt{u-\tau}\Lambda(\tau)/\tau)\,du \\ &\qquad = (\tau/\Lambda(\tau))[\Phi(\sqrt{t-\tau}\Lambda(\tau)/\tau) - \sqrt{\pi/2}\varphi(\sqrt{t-\tau}\Lambda(\tau)/\tau)]. \end{aligned} \tag{65}$$

(63), (64) and (65) together with an evaluation of the first two terms in (60) lead to (58).

One can proceed as in Section 3 to derive approximations for the mean and variance of T_a under drift $\theta > 0$. We assume that condition A3 of Section 2 holds on (τ, ∞) and that A1′ and A2–A5 hold with $I_a = (\tau, cb_a)$ where $c > 1$ and b_a is given by $\psi_a(b_a) = \theta b_a$. Then the expected value and variance of T_a are finite and they are asymptotically of the same form as in the case $\tau = 0$ (see Theorem 2).

$$E_\theta T_a = b_a + b_a\psi_a''(b_a)/2\theta_a^3 + o(b_a^{\epsilon/2}), \tag{66}$$

$$\operatorname{Var}_\theta T_a = b_a/\theta_a^2(1 + o(1)). \tag{67}$$

5. Examples. As before let

$$T_a = \inf\{t > \tau : W(t) \geq \psi_a(t)\}$$

where now $\tau \geq 0$. We use formulas (35) and (36) for the asymptotic mean and variance of T_a under drift $\theta > 0$. To give approximations to the distribution function of T_a under drift 0 we integrate the asymptotic density (see (16), (31), (57) and (58)) and evaluate the integrals. We omit the calculations.

EXAMPLE A. $\psi_a(t) = at^p$, $0 < p < 1/2$, $\tau = 0$. The assumptions of Theorem 1 hold if $t_a = o(a^{2/(1-2p)})$. On $(0, t_a)$ the local second-order term dominates the global term. We get:

$$P_0(T_a < t) = (1 - \Phi(at^{p-1/2}))\left[\frac{2-2p}{1-2p} - \frac{pt^{1-2p}}{(1-p)(1-2p)a^2}(1 + o(1))\right] \tag{68}$$

$$E_\theta T_a = \frac{a^q}{\theta^q} - \frac{pq^2}{2\theta^2} + o(1), \tag{69}$$

$$\mathrm{Var}_\theta T_a = \frac{a^q q^2}{\theta^{q+2}} + o(a^q), \tag{70}$$

where $q = 1/(1-p)$. (69) and (70) also hold for $1/2 \le p < 1$ if $\tau > 0$ (cf. Woodroofe, 1976, Chow, Chao, Lai, 1979).

EXAMPLE B. $\psi_a(t) = \sqrt{ct\log(a/t)}$, $c > 0$, $\tau = 0$. Theorem 1 holds if $\log t_a = o(\log a)$. We get

$$P_0(T_a < t) = (t/a)^{c/2}[\sqrt{\log(a/t)} + 1/\sqrt{\log(a/t)}(1 + o(1))]/\sqrt{2\pi c}, \tag{71}$$

$$E_\theta T_a = (1/\theta^2)[c\log a - c\log_2 a - c\log(c/\theta^2) - 1 + o(1)], \tag{72}$$

$$\mathrm{Var}_\theta T_a = (4c/\theta^4)\log a + o(\log a), \tag{73}$$

where $\log_2 a = \log(\log a)$.

EXAMPLE C. $\psi_a(t) = \sqrt{2(at + c)}$, $c > 0$, $\tau = 0$. Since assumption A3 of Theorem 1 does not hold we apply (31) and get

$$P_0(T_a < t) = \varphi(\sqrt{2a})\left[(\sqrt{2a} - 1/\sqrt{2a}(1 + o(1)))\int_{\sqrt{2/t}}^{\infty} x^{-1}\exp(-cx^2/2)\,dx + (3/\sqrt{8a})\exp(-c/t)(1 + o(1))\right] \tag{74}$$

Formula (74) is valid for every fixed t. It holds uniformly for $1 \le t \le \exp(a^{-p}e^a)$, where $p > 3/2$. Approximations for the corresponding crossing probabilities of a normal random walk are given by Siegmund (1977) and Woodroofe, Takahashi (1982).

$$E_\theta T_a = 2a/\theta^2 - 1/\theta^2 + o(1), \tag{75}$$

$$\mathrm{Var}_\theta T_a = 8a/\theta^4 + o(a). \tag{76}$$

EXAMPLE D. $\psi_a(t) = \sqrt{(t + 1)(a + \log(t + 1))}$, $\tau = 0$. Although assumption A3 does not hold it can be shown that formula (15) of Theorem 1 remains true for large t. For $t \geq \exp(e^{a/2})$ the global term in (15) dominates the local second-order term.

$$P_0(T_a < t) = e^{-a/2}\left[\left(1 - \Phi\left(\left(\frac{a + \log(t+1)}{t}\right)^{1/2}\right)\right)(1 + o(1/a)) + \left(\frac{a + \log(t+1)}{t}\right)^{1/2} \frac{1}{a}\varphi\left(\left(\frac{a + \log(t+1)}{t}\right)^{1/2}\right)(1 + o(1))\right] \tag{77}$$

uniformly for $a^p \leq t \leq e^{ca}$, where $c < 1$ and $p > 0$,

$$P_0(T_a < \infty) = \tfrac{1}{2} e^{-a/2}(1 + o(1/a)) \tag{78}$$

(cf. Robbins, Siegmund, 1973; and Lai, Siegmund, 1977).

$$E_\theta T_a = (1/\theta^2)[a + \log(a/\theta^2) - 1 + \theta^2] + o(1), \tag{79}$$

(cf. Pollak, Siegmund, 1975).

$$\mathrm{Var}_\theta T_a = 4a/\theta^4 + o(a). \tag{80}$$

EXAMPLE E. $\psi_a(t) = a\sqrt{t}$, $\tau = 1$. Theorem 3 holds with $t_a = \exp(a^{-q}e^{a^2/2})$, $q > 1$, and implies that uniformly for $1 + 1/a \leq t \leq t_a$

$$P_0(T_a < t) = \varphi(a)[(a/2)\log t + a^{-1}(2 - \tfrac{1}{2}\log t)(1 + o(1))]. \tag{81}$$

$$E_\theta T_a = a^2/\theta^2 - 1/\theta^2 + o(1), \tag{82}$$

$$\mathrm{Var}_\theta T_a = 4a^2/\theta^4 + o(a^2). \tag{83}$$

EXAMPLE F. $\psi_a(t) = \sqrt{(a^2 + \log t)t}$, $\tau = 1$. Here Theorem 3 holds with $t_a = \infty$.

$$P_0(T_a < t) = \varphi(a)[a(1 - 1/\sqrt{t}) + a^{-1}(1 + 1/\sqrt{t} - \log t/2\sqrt{t})(1 + o(1))], \tag{84}$$

$$P_0(T_a < \infty) = \varphi(a)[a + 1/a + o(1/a)], \tag{85}$$

(cf. Robbins, Siegmund, 1973).

$$E_\theta T_a = (1/\theta^2)[a^2 + \log(a^2/\theta^2) - 1] + o(1), \tag{86}$$

$$\mathrm{Var}_\theta T_a = 4a^2/\theta^4 + o(a^2). \tag{87}$$

By solving the integral equation (9) numerically one gets approximations to the first-exit density. This was done for Example D,

$$\psi_a(t) = \sqrt{(t+1)(a + \log(t+1))}$$

for several values of a (5, 10, 15 and 20). We compare these numerical values for the density $f_a(t)$ with the tangent approximation $f_a^*(t)$

$$f_a^*(t) = (\Lambda_a(t)/t^{3/2})\varphi(\psi_a(t)/\sqrt{t})$$

and with the second-order approximation f_a^{**}

$$f_a^{**}(t) = [\Lambda_a(t)/t^{3/2} + t^{3/2}\psi_a''(t)/2\Lambda_a(t)^2]\varphi(\psi_a(t)/\sqrt{t}).$$

TABLE 1
Relative errors for the density approximations

$r(f_a^*) = \sup\{|f_a^*(t) - f_a(t)|/f_a(t): 0 < t \le 100\}$

$r(f_a^{**}) = \sup\{|f_a^{**}(t) - f_a(t)|/f_a(t): 0 < t \le 100\}$

a	5	10	15	20
$r(f_a^*)$	0.22	0.10	0.07	0.05
$r(f_a^{**})$	0.05	0.02	0.01	<0.01

TABLE 2
Numerically calculated values and approximations for the mean of T_a.

a	5	10	15	20
E_a	7.4	12.8	18.0	23.2
E_a^*	8.1	13.6	18.9	24.2
E_a^{**}	7.3	12.7	18.0	23.3

TABLE 3
Numerically calculated values and approximations for the variance of T_a.

a	5	10	15	20
V_a	27	50	71	93
V_a^*	33	55	76	97

Table 1 shows the maximal relative errors of these approximations for $t \le 100$. These errors are independent of the drift. In Table 2 we give the numerically calculated values E_a for the mean $E_\theta T_a$ and the approximations E_a^* and E_a^{**} of Theorem 2

$$E_a^* = b_a, \quad E_a^{**} = b_a + b_a \psi_a''(b_a)/2\theta_a^3.$$

The drift θ of the Brownian motion is 1. Table 3 compares the numerical values V_a for the variance with the approximation $V_a^* = b_a/\theta_a^2$.

The tables show that the improvement of the second-order approximation is quite large.

6. Concluding remarks. Higher-order terms.

An improved evaluation of the integral in (9) leads, under additional assumptions on the higher derivatives of ψ_a, to higher-order terms for the density. The next term in an expansion is

$$\left[-\frac{3t^{5/2}\psi_a''(t)}{2\Lambda_a(t)^4} - \frac{2t^{9/2}\psi_a''(t)^2}{\Lambda_a(t)^5} - \frac{t^{7/2}\psi_a'''(t)}{2\Lambda_a(t)^4}\right]\varphi\left(\frac{\psi_a(t)}{\sqrt{t}}\right). \tag{88}$$

For the examples of Section 5 all terms in the brackets are of the order of magnitude $\sqrt{t}/\psi_a(t)^3$.

Refining the techniques in the proof of Theorem 2 one can derive higher-order terms for the mean and variance of T_a. Using the identities

$$E_\theta(\psi_a(T_a) - \theta T_a)^2 = E_\theta T_a$$

$$E_\theta(\psi_a(T_a) - \theta T_a)^3 = 3E_\theta(T_a\psi_a(T_a) - \theta T_a^2)$$

one gets under assumptions on ψ_a''' and $\psi_a^{(4)}$

(89) $$E_\theta T_a = b_a + b_a\psi_a''(b_a)/2\theta_a^3 + m_a/b_a + o(b_a^{3\varepsilon/2-1}),$$

(90) $$\mathrm{Var}_\theta T_a = b_a/\theta_a^2 + n_a + o(b_a^\varepsilon),$$

where

$$m_a = b_a^2\psi_a'''(b_a)/2\theta_a^5 + b_a^3\psi_a^{(4)}(b_a)/8\theta_a^5 + 7b_a^2\psi_a''(b_a)^2/4\theta_a^6$$
$$+ 5b_a^3\psi''(b_a)\psi'''(b_a)/4\theta_a^6 + 15b_a^3\psi_a''(b_a)^3/8\theta_a^7,$$

$$n_a = 7b_a\psi_a''(b_a)/2\theta_a^5 + b_a^2\psi_a'''(b_a)/\theta_a^5 + 7b_a^2\psi_a''(b_a)^2/2\theta_a^6.$$

For the examples of Section 5, m_a and n_a are asymptotically constants in a.

Approximations to the first-exit density for small t.

Let us now consider the first-exit time (1) through a fixed curve ψ. Applying the results of Section 2, one can get second-order terms to Strassen's (1966) tangent approximation for the density f. After the space-time transformation

(91) $$t \to a^2t, \quad x \to ax, \quad a > 0,$$

W transforms to a new Brownian motion and ψ becomes ψ_a

$$\psi_a(t) = a\psi(t/a^2).$$

The first exit-density f_a at ψ_a and f are related by the formula $f(t) = a^2f_a(a^2t)$. If the family $\{\psi_a : a > 0\}$ fulfills the assumptions of Theorem 1 then as $t \to 0$

(92) $$f(t) = [\Lambda(t)/t^{3/2} + \psi''(t)t^{3/2}/2\Lambda(t)^2(1 + o(1)) - (\Lambda(t)/t^{3/2})P(T < t)(1 + o(1))]\varphi(\psi(t)/\sqrt{t}).$$

Furthermore putting $a = \theta$ in (91) and appealing to Theorem 2 one gets

(93) $$E_\theta T = b_\theta + b_\theta\psi''(b_\theta)/2\theta_*^3 + o(b_\theta^{\varepsilon/2}) \quad \text{as} \quad \theta \to \infty,$$

(94) $$Var_\theta\ T = b_\theta/\theta_*^2(1 + o(1)) \quad \text{as} \quad \theta \to \infty,$$

where b_θ is given by $\theta b_\theta = \psi(b_\theta)$ and $\theta_* = \theta - \psi'(b_\theta)$.

From (92) one can derive asymptotic formulas for the density of last-entry times (see the proof of Theorem 3.6 of Strassen, 1966).

Acknowledgement. This work is part of a doctoral dissertation which was performed under the guidance of Dr. R. Lerche. I would like to thank B. Ferebee and R. Lerche for helpful advice during the preparation of this paper.

REFERENCES

CHOW, Y. S., CHAO, A. H. and LAI, T. L. (1979). Extended renewal theory and moment convergence in Anscombe's theorem. *Ann. Probab.* **7** 304–318.

CUZICK, J. (1981). Boundary crossing probabilities for stationary Gaussian processes and Brownian motion. *Trans. Amer. Math. Soc.* **263** 469–492.

DANIELS, H. E. (1974). An approximation technique for a curved boundary problem. *Adv. in Appl. Probab.* **6** 194–196.

DURBIN, J. (1971). Boundary-crossing probabilities for the Brownian motion and Poisson processes and techniques for computing the power of the Kolmogorov-Smirnov test. *J. Appl. Probab.* **8** 431–453.

DURBIN, J. (1981). The first passage density of a Gaussian process to a general boundary. University of London, Preprint.

FEREBEE, B. (1982). The tangent approximation to one-sided Brownian exit densities. *Z. Wahrsch. verw. Gebiete* **61** 309–366.

FEREBEE, B. (1983). An asymptotic expansion for one-sided Brownian exit densities. *Z. Wahrsch. verw. Gebiete* **63** 1–15.

JENNEN, C. and LERCHE, H. R. (1981). First exit densities of Brownian motion through one-sided moving boundaries. *Z. Wahrsch. verw. Gebiete* **55** 133–148.

LAI, T. L. and SIEGMUND, D. (1977). A nonlinear renewal theory with applications to sequential analysis I. *Ann. Statist.* **5** 946–954.

POLLAK, M. and SIEGMUND, D. (1975). Approximations to the expected sample size of certain sequential tests. *Ann. Statist.* **3** 1267–1282.

ROBBINS, H. and SIEGMUND, D. (1973). Statistical tests of power one and the integral representation of solutions of certain partial differential equations. *Bull. Inst. Math. Acad. Sinica* **1** 93–120.

SIEGMUND, D. (1977). Repeated significance tests for a normal mean. *Biometrika* **64** 177–189.

STRASSEN, V. (1966). Almost sure behaviour of sums of independent random variables and martingales. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.*, Univ. Calif. Press, Vol. II, Part I, 315–343.

WOODROOFE, M. (1976). A renewal theorem for curved boundaries and moments of first passage times. *Ann. Probab.* **4** 67–80.

WOODROFFE, M. and TAKAHASHI, H. (1982). Asymptotic expansions for the error probabilities of some repeated significance tests. *Ann. Statist.* **10** 895–908.

ZENTRALINSTITUT FÜR SEELISCHE GESUNDHEIT
P.O.B. 5970
6900 MANNHEIM 1
FEDERAL REPUBLIC OF GERMANY