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SYMMETRIC EXCLUSION PROCESSES: A COMPARISON INEQUALITY AND A LARGE DEVIATION RESULT[1]

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We consider an infinite particle system, the simple exclusion process, which was introduced in the 1970 paper "Interaction of Markov Processes," by Spitzer. In this system, particles attempt to move independently according to a Markov kernel on a countable set of sites, but any jump which would take a particle to an already occupied site is suppressed. In the case that the Markov kernel is symmetric, an inequality by Liggett gives a comparison, for expectations of positive definite functions, between the exclusion process and a system of independent particles. We apply a special case of this inequality to an auxiliary process, to prove another comparison inequality, and to derive a large deviation result for the symmetric exclusion system. In the special case of simple random walks on Z, this result can be transformed into a large deviation result for an infinite network of queues.

1. Introduction. Let $p(x, y)$ be the transition probabilities for a Markov process on a countable set of sites S. For example, p may be simple random walk on the lattice of integers: $S = Z^d$ and $p(x, y) = 1/(2d)$ if x and y are nearest neighbors. We consider the *simple exclusion process* based on p, which was introduced in Spitzer (1970). This is a Markov process, with state space $\{0, 1\}^S \equiv \{\eta\colon \eta \subset S\}$, where η represents the set of occupied sites in a system of identical particles, with at most one particle per site. A particle at site $x \in S$ waits for an exponentially distributed time with mean 1, then chooses a site y with probability $p(x, y)$. If y is vacant at that time, the particle at x jumps to y; otherwise it stays at x. All the holding times and choices according to p are independent. Thus, a system with only one particle gives a copy of the original Markov process p, but if the system starts with all sites occupied, then there is no motion ever. Surveys of results for this system are given in Liggett (1977) and (1985).

Henceforth we assume that p is symmetric: $p(x, y) = p(y, x)$. For each $\rho \in [0, 1]$, product measure ν_ρ, with marginals

$$\nu_\rho(\{\eta\colon x \in \eta\}) = \rho \quad \forall x \in S,$$

is invariant for the symmetric exclusion process. There are other extremal invariant measures iff p has nonconstant bounded harmonic functions; see Liggett (1974). David Griffeath conjectured that for the exclusion system of

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simple random walks on Z, running in equilibrium ν_ρ for $\rho \in (0, 1)$,

$$(1) \qquad -\log P(0 \notin \eta_s \ \forall s \le t) \sim ct^{1/2} \quad \text{as} \quad t \to \infty,$$

for some constant c depending on ρ. In this paper we verify that for the exclusion system of simple random walks on Z in nontrivial equilibrium, $t^{-1/2}[-\log P(0 \notin \eta_s \ \forall s \le t)]$ remains bounded away from 0 and ∞, but the conjecture (1) remains open.

Theorem 1 deals with the exclusion system corresponding to a symmetric Markov process p, running in its equilibrium ν_ρ. Using methods which ignore the geometry of p, we prove that the log of the probability that a fixed site remains empty throughout the time interval $[0, t]$, divided by the expected range of a single trajectory run for time t, has all of its limit points as $t \to \infty$ in the interval $[\log(1-\rho), -\rho]$. Thus in the context of (1), with p a simple random walk on Z, we have proved that for all $\varepsilon > 0$, for sufficiently large t,

$$(2) \quad (1-\varepsilon)\rho(8t/\pi)^{1/2} < -\log P(0 \notin \eta_s \ \forall s \le t) < (1+\varepsilon)[-\log(1-\rho)](8t/\pi)^{1/2}.$$

Theorem 1 may be recast as a statement about the motion of a tagged particle. Consider the exclusion system starting from equilibrium ν_ρ, conditioned to have a particle at the origin at time 0. Let $y(t)$ denote the position of this particle at time t. In the case of simple random walk on Z, the tagged particle is "trapped" between its neighbors; Arratia (1983) proved that $\mathrm{var}(y(t)) \sim (2t/\pi)^{1/2}(1-\rho)/\rho$. In the case of simple random walk on Z^d, $d \ge 2$, Kipnis and Varadhan (1984) proved that $\mathrm{var}(y(t))$ grows linearly with t. Now the set of empty sites, $S \setminus \eta_t$, evolves according to exactly the same dynamics as the symmetric exclusion system η_t itself. Thus the event $\{0 \notin \eta_s \ \forall s \le t\}$ for the exclusion system starting in equilibrium ν_ρ is equivalent to the event $\{0 \in \eta_s \ \forall s \le t\}$, for the exclusion system in equilibrium $\nu_{1-\rho}$. Theorem 1 may be restated as: for the exclusion system corresponding to a symmetric Markov process p, running in its equilibrium ν_ρ, the log of the probability that a tagged particle remains motionless at the origin throughout the time interval $[0, t]$, divided by the expected range of a single trajectory run for time t, has all of its limit points as $t \to \infty$ in the interval $[\log \rho, \rho - 1]$.

The key to Theorem 1 is a remarkable inequality from Liggett (1977, pages 226–228) which states that, for any symmetric positive definite function on S^n, evaluated at the positions of an n-particle system at some fixed time, the expectation does not decrease if the exclusion interaction is replaced by independent particle motions. (This result for $n = 2$ particles appears in Liggett, 1974. A bounded symmetric function $g(x, y)$ on S^2 is called positive definite if $\sum_{x,y} \beta(x)\beta(y)g(x, y) \ge 0$ whenever $\sum_x |\beta(x)| < \infty$ and $\sum_x \beta(x) = 0$. For $n > 2$, a bounded symmetric function on S^n will be called positive definite if it is a positive definite function of each pair of variables. In the special case where the symmetric positive definite function is

$$f(x_1, \cdots, x_n) = \textstyle\prod_i 1(x_i \in A),$$

this inequality says: for one fixed time s, for any initial configuration $B \subset S$ of size $|B| = n$, for any $A \subset S$, the set η_s^B of sites occupied by the exclusion process satisfies

$$P(\eta_s^B \subset A) \le \prod_{x\in B} P(\xi(x, s) \in A). \tag{3}$$

Here, $\xi(x, \cdot)$ denotes the one particle Markov process governed by p, starting from x. Now to get the large deviation bounds of Theorem 1, all that is needed is an inequality like (3), applied to $A = S\backslash\{0\}$, but with the quantification "$\forall s \le t$" inserted into every event:

$$P(\eta_s^B \subset A \ \forall s \le t) \le \prod_{x\in B} P(\xi(x, s) \in A \ \forall s \le t). \tag{4}$$

The idea for deriving (4) from (3) is to introduce an auxiliary process with lots of additional sites connected to each site of $S\backslash A$, so that a particle moves in A as before, but if it leaves A before time t, then with high probability it wanders off among the additional sites and is still "lost" at time t.

2. Comparison inequalities. We note that the proof in Liggett (1977) of the comparison inequality for symmetric positive definite functions remains valid whenever the one particle motion is a Markov process on a countable state space with bounded, not necessarily time independent, symmetric transition rates.

Couple the family $(\eta^B; B \subset S)$ of exclusion processes with various initial configurations B, using a system $(\xi(x, t); x \in S, t \ge 0)$ of random stirrings of pairs of particles, by setting:

$$\text{for} \quad B \subset S, \quad \eta^B(t) \equiv \{\xi(x, t): x \in B\}. \tag{5}$$

For the stirring system, $\xi(x, t)$ represents the position at time t of a particle which starts at site x. At rate $p(x, y)$, independently for all $x, y \in S$, the particles sitting at x and at y are interchanged. Thus for each x, $\xi(x, \cdot)$ is a realization of the original one-particle process starting from x, while for fixed t, $\xi(\cdot, t)$ is a permutation on S. Notice that, via the coupling (5), Liggett's inequality for the exclusion system can be restated as an inequality for the stirring system. The stirring system was introduced in Harris (1972); a recent exposition is given in Griffeath (1979). Random stirrings of a finite set S are studied in Diaconis and Shashahani (1981), and in Flatto, Odlyzko, and Wales (1984).

LEMMA 1. *For any $B \subset S$, for any $A \subset S$, and fixed time t,*

$$P(\eta_s^B \subset A \ \forall s \le t) \le \prod_{x\in B} P(\xi(x, s) \in A \ \forall s \le t).$$

PROOF. Assume that B is finite; the case of B infinite will follow by taking limits. (That the left side is monotone decreasing in B can be seen from the coupling (5).) In order to keep the notation simple we fix a distinguished site $0 \in S$ and give the proof for the case $A = S\backslash\{0\}$. The idea is to attach lots of additional sites to the site 0. For the convenience of arguing with transience, we let the additional sites be a copy of Z^3. The distinguished site 0 in S and the origin 0 in Z^3 are identified. A particle performs simple random walk on the copy of Z^3 at rate c, so that in the limit as $c \to \infty$, a particle "disappears from S" immediately after it hits 0. More precisely, we let the state space for the auxiliary process be $S^* \equiv \{(x, a) \in S \times Z^3: a = 0 \text{ or } x = 0\}$; the site $(x, 0)$ in S^* corresponds to the site x in S, while the site $(0, a)$ in S^* corresponds to the site a in Z^3. Let the transition rates for the process on S be denoted by $q(x, y)$. The transition intensities for the auxiliary process are

$$\begin{aligned} q^*((x, 0), (y, 0)) &= q(x, y) && \text{unless} \quad x = y = 0 \\ q^*((0, a), (0, b)) &= c/6 && \text{if} \quad |a - b| = 1 \\ &= 0 && \text{if} \quad |a - b| > 1, \end{aligned}$$

with the appropriate negative values for q^* on the diagonal of $S^* \times S^*$. This specifies a symmetric Markov process on a countable state space, so that the special case (3) of Liggett's inequality may be applied, in particular with A in (3) taken to be $A^* \equiv \{(x, a) \in S^*: a = 0 \text{ and } x \neq 0\}$. This yields, for the auxiliary exclusion process η^* starting from $B \times \{0\}$, that

$$P(\eta_t^* \subset A^*) \leq \textstyle\prod_{x \in B} P(\xi^*((x, 0), t) \in A^*). \tag{6}$$

In the above formula, ξ^* denotes the one-particle process governed by q^*. The processes $\xi^*((x, 0), \cdot)$ and $\xi(x, \cdot)$ can be coupled so that they agree until the first time that $\xi(x, \cdot)$ hits 0. It is clear that as $c \to \infty$,

$$\begin{aligned} P(\xi^*((x, 0), t) \in A^*) &\to P(\xi^*((x, 0), s) \in A^* \ \forall s \leq t) \\ &= P(\xi(x, s) \in S \backslash \{0\} \ \forall s \leq t). \end{aligned}$$

Combining this with the coupling specified in (5), it follows that

$$P(\eta_t^* \subset A^*) \to P(\eta_s^* \subset A^* \ \forall s \leq t) = P(\eta_s \subset S \backslash \{0\} \ \forall s \leq t).$$

Taking the limit of (6) as $c \to \infty$ yields the statement of this lemma. □

The following lemma, which is a slight generalization of Lemma 1, should be useful for giving bounds on the moments of occupation times for the symmetric exclusion process. See Cox and Griffeath (1984) for the computation of moments, and large deviation results, for occupation times of systems of independent random walks in equilibrium.

LEMMA 2. *For any* $B \subset S$, $\forall t_1 < \cdots < t_k$, $\forall A_1, \cdots, A_k \subset S$,

$$P(\eta^B(t_i) \subset A_i \text{ for } i = 1 \text{ to } k) \leq \textstyle\prod_{x \in B} P(\xi(x, t_i) \in A_i \text{ for } i = 1 \text{ to } k). \tag{7}$$

PROOF. Begin with the case that B is finite; the general case then follows by approximation. We give the proof for the case with each $A_i = S \backslash \{0\}$ in order to keep the notation simple. Let S^* be as in the proof of Lemma 1; let $A^* \equiv \{(x, a) \in S^*: a = 0\}$. The idea now is to choose time inhomogeneous rates for the auxiliary process so that away from the times t_i, we have a copy of the original process, with no chance to move off among the additional states; while immediately after the times t_i, the random walk on the additional sites runs so fast that, if a particle is at $(0, 0)$ at t_i, then with high probability it gets lost quickly among the additional states. Formally, for a fixed value of the parameter

c we set

$$\begin{aligned} q_t^*((x,0),(y,0)) &= q(x,y) && \text{unless} \quad x = y = 0 \\ q_t^*((0,a),(0,b)) &= c^2/6 && \text{if} \quad |a-b| = 1 \text{ and } t \in [t_i, t_i + 1/c] \text{ for some } i \\ &= 0 && \text{if} \quad |a-b| > 1 \\ &= 0 && \text{if} \quad t - t_i > 1/c \quad \text{for all } i, \quad \text{and not} \quad a = b = 0, \end{aligned}$$

with the appropriate negative rates on the diagonal of $S^* \times S^*$. Fix any $t > t_k$. Now the auxiliary process ξ^* and the original process ξ can be coupled so that for each $x \in S$, with probability tending to 1 as $c \to \infty$, on the event $\{\xi(x, t_i) \in A_i$ for $i = 1$ to $k\}$, the trajectories $\xi^*((x, 0), \cdot)$ and $\xi(x, \cdot)$ agree for all time, while on the complementary event, we have that $\xi^*((x, 0), t) \notin A^*$. Combining this with (5), the finite exclusion systems η and η^* can be coupled to show that, as $c \to \infty$, $P(\eta_t^* \subset A^*) \to P(\eta^B(t_i) \subset A_i$ for $i = 1$ to $k)$. Taking the limit of (6) for the time-inhomogeneous process η^* yields (7). □

3. A large deviation result. We will express our large deviation probabilities in terms of ER_t, the expected range of a single random trajectory starting from 0. Formally, define the random variable R_t which is the number of distinct sites visited up to time t, by a particle starting at the origin: $R_t \equiv \sum_{y \in S} 1(\xi(0, s) = y$ for some $s \le t)$. Let $h_t(x, y) \equiv P(\xi(x, s) = y$ for some $s \le t)$, so that

(8) $$ER_t = \sum_{x \in S} h_t(0, x) = \sum_{x \in S} h_t(x, 0).$$

In case p is simple random walk on Z^d for $d = 1, 2, 3, \cdots$, the expected range of a single particle satisfies, asymptotically as $t \to \infty$,

(9) $$\begin{aligned} ER_t &\sim 4(t/2\pi)^{1/2} && d = 1 \\ &\sim \pi t/\log t && d = 2 \\ &\sim \gamma_d t && d \ge 3 \end{aligned}$$

where γ_d is the probability that a d-dimensional simple random walk never returns to its origin. (See Dvoretzky and Erdös, 1951, for the cases $d \ge 2$ above.)

THEOREM 1. *Consider the simple exclusion process governed by a symmetric Markov kernel on a countable set S, starting in its equilibrium state $\nu_\rho \equiv$ product measure with density $\rho \in (0, 1)$. Let $p_{t,\rho} = P(0 \notin \eta_s \; \forall s \le t)$ be the probability that site 0 is empty throughout the time interval $[0, t]$. Then for any $t \ge 0$,*

(10) $$(1-\rho)^{ER_t} \le p_{t,\rho} \le \prod_{x \in S}(1 - \rho h_t(x, 0)) \le \exp(-\rho ER_t),$$

and hence $(\log p_{t,\rho})/ER_t$ has all of its limit points, as $t \to \infty$, in the interval $[\log(1-\rho), -\rho]$.

PROOF. From now on, write $B = \eta_0$ for the initial configuration of the exclusion process, so that B has distribution ν_ρ, and B is independent of the stirring system.

The lower bound on the large deviation probability in (10) is easy to obtain. Let H_t be the set of all sites $x \in S$ for which the stirring path starting from x has hit 0 before time t:

$$H_t \equiv \{x \in S: \xi(x, s) = 0 \text{ for some } s \le t\},$$

so that H grows with t and by the symmetry of p, $E \mid H_t \mid = ER_t$. Using Jensen's inequality,

$$\begin{aligned} P(0 \notin \eta_s \ \forall s \le t) &= P(H_t \cap B = \varnothing) = E((1-\rho)^{|H_t|}) \\ &\ge (1-\rho)^{E|H_t|} = (1-\rho)^{ER_t}. \end{aligned}$$

The upper bound is an easy consequence of Lemma 1. For a deterministic initial configuration B, taking $A \equiv S\{0\}$, and using the notation $h_t(x, y)$ defined in (8), Lemma 1 says that

$$P(0 \notin \eta_s \ \forall s \le t) \le \prod_{x \in B}(1 - h_t(x, 0)).$$

Averaging this with respect to the distribution ν_ρ of B yields $p_{t,\rho} \le \prod_{x \in S} (1 - \rho h_t(x, 0))$. □

The basic upper bound in Theorem 1, $\prod_{x \in S}(1 - \rho h_t(x, 0))$, comes from and is exactly the probability that 0 is not reached by any particle before time t, in a system of independent particles, starting with the initial distribution ν_ρ—product measure with intensity ρ, and at most one particle per site. If the same independent particle system starts with another initial distribution with intensity ρ, namely the Poisson distribution, then the number of particles to have hit the origin by time t is Poisson, with expectation ρER_t. For this system, the probability that the origin has not been hit by time t is exactly $\exp(-\rho ER_t)$, which is the secondary upper bound in Theorem 1.

To further analyze the difference between our two upper bounds, use the expansion $\log(1 - y) = -y - y^2/2 - y^3/3 - \cdots$, so that our basic upper bound becomes $\log p_{t,\rho} \le \sum_{x \in S} \log(1 - \rho h_t(x, 0)) = -\sum_{k \le 1} c_k \rho^k/k$, where $c_k \equiv \sum_{x \in S} (h_t(x, 0))^k$. The first coefficient is $c_1 = ER_t$, so our two upper bounds give the same bound on $\limsup_{t\to\infty}(1/ER_t)\log p_{t,\rho}$ iff p has the property: $c_2/c_1 \to 0$ as $t \to \infty$.

In the case that p is one-dimensional simple random walk, the reflection principle yields $h_t(x, 0) = P(\xi(0, t) \notin [-x, x))$, so with X a standard normal,

$$t^{-1/2}ER_t \to \int_R P(X \notin (-a, a))\, da = \frac{4}{\sqrt{2\pi}},$$

and

$$t^{-1/2}c_2 \to \int_R (P(X \notin (-a, a)))^2\, da = \frac{4(2 - \sqrt{2})}{\sqrt{2\pi}}.$$

Thus an expansion with two terms gives the large deviations bound

$$\limsup_{t\to\infty}\left(\frac{1}{ER_t}\right)\log p_{t,\rho} \le -\rho - \frac{(2-\sqrt{2})\rho^2}{2}. \tag{11}$$

4. A network of queues: the zero-range process. A certain queueing network can be realized as a sample path transformation of the exclusion system of simple random walks, so that a server with n customers corresponds to a block of n adjacent occupied sites. As a corollary to Theorem 1 we get a large deviation result for this queueing network.

Consider a queueing system in which there is a server at each site $x \in Z$; let $M_t(x) \ge 0$ be the number of customers at x at time t, including the customer currently being served. The service times are all exponentially distributed with mean 1. As soon as a customer has been served at x, he chooses to move to $x+1$ or $x-1$ according to a fair coin, joining the queue at that site. Customers never enter from outside the system and never leave the system. All the service times and coin tosses are independent. This system $M_t = (M_t(x);\ x \in Z)$ is a special case of the zero-range process introduced in Spitzer (1970); see Andjel (1982) for a recent discussion. The system M_t is a Markov process with state space $\{0, 1, 2, \cdots\}^Z$. It has a one-parameter family μ_ρ of extremal equilibrium measures (Liggett, 1973). The measure μ_ρ is a product measure, with geometric marginals with parameter $1-\rho$:

$$\mu_\rho(\{M: M(x) = k\}) = \rho^k(1-\rho), \quad \forall x \in Z. \tag{12}$$

COROLLARY 1. *For the queueing network M described above, in its equilibrium μ_ρ, the probability that a given server remains idle for time t satisfies*

$$P(M_s(0) = 0\ \forall s \le t) = p_{t,\rho}, \tag{13}$$

where $p_{t,\rho}$ is the probability that simple symmetric exclusion on Z, in its equilibrium ν_ρ, has no particle at 0 throughout the time interval $[0, t]$. In particular $(\pi/8t)^{1/2}\log P(M_s(0) = 0\ \forall s \le t)$ has all of its limits, as $t \to \infty$, in the interval $[\log(1-\rho), -\rho-(1-1/\sqrt{2})\rho^2]$.

PROOF. Consider the exclusion system of simple random walks on Z, realized via the coupling (5). Let $B \subset Z$ have distribution ν_ρ, with B independent of the stirring system, and let $C = B\setminus\{0\}$. Tag the "hole" initially at the origin in η^C, and let $z(t)$ be the position of this hole at time t. Note that the hole moves only by exchanging positions with an adjacent particle. On the event $\{\forall t \ge 0,\ |Z^+\setminus\eta_t| = |Z^-\setminus\eta_t| = \infty\}$, for each fixed $t \ge 0$, label the translated configuration $\eta_t^C - z(t)$ and define M_t so that

$$\begin{gathered}\eta_t^C - z(t) = \{\cdots, a_{-2}, a_{-1}, a_0, a_1, \cdots\}\\ \text{with}\quad \cdots < a_{-2} < a_{-1} < a_0 < 0 < a_1 < \cdots;\\ M_t(x) \equiv |\{i \in Z: a_i - i = x\}|, \quad \forall x \in Z.\end{gathered} \tag{14}$$

It can be checked that M defined by (14) above evolves as the network of queues, and the distribution of M is μ_ρ as specified at (12). We note that a similar transformation is used in Rost (1981).

Our theorem will follow from two relations:

$$\begin{aligned}\{M_s(0) = 0\ \forall s \le t\} &= \{z(s) = 0 \text{ and } 1 \notin \eta_s^C,\ \forall s \le t\}\\ &= \{0, 1 \notin \eta_s^C\ \forall s \le t\}\end{aligned} \tag{15}$$

and

$$P(0, 1 \notin \eta_s^C\ \forall s \le t) = P(0 \notin \eta_s^B\ \forall s \le t). \tag{16}$$

To see (15), note that the tagged hole initially at the origin can move only by trading places with an adjacent particle. To prove (16), define a transform ζ_s of the sample paths of η_s^C,

$$\zeta_s \equiv \{x\colon x \in \eta_s^C \text{ and } x < 0, \text{ or } x + 1 \in \eta_s^C \text{ and } x > 0\}:$$

Until the first time that 0 or $1 \in \eta_s^C$, ζ_s evolves like the simple exclusion system, and the distribution of ζ_0 is ν_ρ. Thus

$$\{0, 1 \notin \eta_s^C\ \forall s \le t\} = \{0 \notin \zeta_s\ \forall s \le t\},$$

and

$$P(0 \notin \zeta_s\ \forall s \le t) = P(0 \notin \eta_s^B\ \forall s \le t),$$

which proves (16). □

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