REFLECTED DIFFUSION PROCESSES WITH JUMPS

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A stochastic differential equation of Wiener-Poisson type is considered in a d-dimensional bounded region. By using a penalization argument on the domain, we are able to prove the existence and uniqueness of solutions in the strong sense. The main assumptions are Lipschitzian coefficients, either convex or smooth domains and a regular outward reflecting direction. As a direct consequence, it is verified that the reflected diffusion process with jumps depends on the initial date in a Lipschitz fashion.

Introduction. In this paper we consider a stochastic differential equation of Wiener-Poisson type in a d-dimensional bounded region with reflecting conditions (cf. [7]).

This kind of problem has been studied in Anulova [1, 2] and Chaleyat-Maurel et al. [4]. Herein, we extend the results of [6] to more general situations, e.g. general domains and jumps processes.

By using a penalization argument on the domain we are able to prove the existence and uniqueness of a solution in the strong sense. Also, it is verified that the unique solution, i.e. the reflected diffusion process with jumps, depends on the initial data in a Lipschitz manner.

Let us mention that a similar penalization argument has been used in Bensoussan and Lions [3], Lions et al. [5] and Shalaumov [8].

In the first section we treat the case of convex domains not necessarily smooth. Next, via a diffeomorphism, we extend these results to smooth domains which are simply connected. Finally, in the section three, we adapt the technique to general smooth domains.

1. Convex domains. Let $(\Omega, F, P, F_t, w_t, \mu_t, t \geq 0)$ be a complete Wiener-Poisson space in $\mathbb{R}^n \times \mathbb{R}_*^m$, $\mathbb{R}_*^m = \mathbb{R}^m - \{0\}$, with the Levy measure π, i.e., (Ω, F, P) is a complete probability space, $(F_t, t \geq 0)$ is a right continuous increasing family of complete sub σ-algebras of F, $(w_t, t \geq 0)$ is a standard Wiener process in $\mathbb{R}^n$ with respect to $(F_t, t \geq 0)$, $(\mu_t, t \geq 0)$ is a martingale measure in $\mathbb{R}_*^m$, w.r.t. F_t, independent of $(w_t, t \geq 0)$, corresponding to a standard Poisson random measure $p(t, A)$, namely, for any Borel measurable subset of $\mathbb{R}_*^m$,

$$\mu_t(A) = p(t, A) - t\pi(A)$$

where

$$E(p(t, A)) = t\pi(A).$$

One can refer to Bensoussan and Lions [3], for a detailed study of diffusion processes with jumps.

Suppose that $\mathbb{O}$ is an open subset of $\mathbb{R}^d$ such that

(1.1) $\mathbb{O}$ is convex and bounded

and that we are given measurable functions

$$g : \overline{\mathbb{O}} \to \mathbb{R}^d, \quad g = (g_i(x)),$$

$$\sigma : \overline{\mathbb{O}} \to \mathbb{R}^d \times \mathbb{R}^n, \quad \sigma = (\sigma_{ik}(x)),$$

$$\gamma : \overline{\mathbb{O}} \times \mathbb{R}^m_* \to \mathbb{R}^d, \quad \gamma = (\gamma_i(x, z)),$$

satisfying

(1.2)
$$\begin{cases} |g(x)|^p + |\sigma(x)|^p + \int_{\mathbb{R}^m_*} |\gamma(x, z)|^p \pi(dz) \le C_p, \\ |g(x) - g(x')|^p + |\sigma(x) - \sigma(x')|^p \\ \qquad + \int_{\mathbb{R}^m_*} |\gamma(x, z) - \gamma(x', z)|^p \pi(dz) \le C_p |x - x'|^p, \end{cases}$$

for every $p \ge 2$, x, x' in $\overline{\mathbb{O}}$ and some constant C_p depending only on p. Note that $\overline{\mathbb{O}}$ is the closure of $\mathbb{O}$ and $|\cdot|$ denotes the appropriate Euclidian norm. Also assume that

(1.3) $x + \gamma(x, z) \in \overline{\mathbb{O}}, \quad \forall x \in \overline{\mathbb{O}}, \quad \forall z \in \mathbb{R}^m_*.$

This means that all jumps from $\overline{\mathbb{O}}$ are inside $\overline{\mathbb{O}}$.

A reflected diffusion process with jumps $(y(t), t \ge 0)$ and its associated reflecting process $(\eta(t), t \ge 0)$ is a pair of progressively measurable stochastic processes which are right continuous having left-hand limits such that

(i) $y(t)$ takes values into the closure $\overline{\mathbb{O}}$ and $\eta(t)$ is continuous, has locally bounded variation and $\eta(0) = 0$;

(1.4) (ii) $dy(t) = g(y(t)) \, dt + \sigma(y(t)) \, dw_t + \int_{\mathbb{R}^m_*} \gamma(y(t), z) \, d\mu_t(z) - d\eta(t),$ $t \ge 0$, $y(0) = x$;

(iii)   for every $z(t)$ progressively measurable process which is right continuous having left-hand limits and takes values into the closure $\overline{\mathbb{O}}$, we have

$$\int_0^T (y(t) - z(t)) \, d\eta(t) \ge 0, \quad \forall T \ge 0.$$

Since $(\eta(t), t \geq 0)$ is a continuous process, the last inequality is equivalent to

$$\int_0^T (y(t-) - z(t-)) \, d\eta(t) \geq 0, \quad \forall T \geq 0$$

where $y(t-)$, $z(t-)$ denote the left-hand limits at t.

The problem (1.4) is referred to as a stochastic variational inequality (SVI) for reflected diffusion processes with jumps in convex domains.

We approximate this SVI by means of a classical penalty argument on a diffusion process with jumps in the whole space $\mathbb{R}^d$.

Without loss of generality, we may assume that the coefficients g, σ, γ are extended to $\mathbb{R}^d$ preserving the assumption (1.2), in particular, one can take

(1.5) $$\gamma(x, z) = \gamma(\mathrm{pr}(x), z), \quad \forall x \in \mathbb{R}^d, \quad \forall z \in \mathbb{R}_*^m,$$

where $\mathrm{pr}(\cdot)$ denotes the orthogonal projection on the closure $\overline{\mathbb{O}}$.

Consider the stochastic differential equation

(1.6)
$$\begin{cases} dy^\varepsilon(t) = g(y^\varepsilon(t)) \, dt + \sigma(y^\varepsilon(t)) \, dw_t \\ \quad + \int_{\mathbb{R}_*^m} \gamma(y^\varepsilon(t), z) \, d\mu_t(z) - \frac{1}{\varepsilon} \beta(y^\varepsilon(t)) \, dt, \quad t \geq 0, \quad y^\varepsilon(0) = x, \end{cases}$$

with $\varepsilon > 0$, x in $\overline{\mathbb{O}}$ and $\beta = (\mathrm{id} - \mathrm{pr})^*$, i.e.

(1.7) $$\beta(x) = \tfrac{1}{2}\mathrm{grad}(\min\{|x - y|^2 : y \text{ in } \overline{\mathbb{O}}\}).$$

THEOREM 1. *Let the assumptions* (1.1), (1.2), (1.3) *and* (1.5) *hold. Then the SVI* (1.4) *has one and only one solution* $(y(t), \eta(t), t \geq 0)$. *Moreover, for each* $p \geq 1$, $T > 0$, *we have*

(1.8) $$E\{\sup_{0 \leq t \leq T} |y^\varepsilon(t) - y(t)|^p\} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

(1.9) $$E\left\{\sup_{0 \leq t \leq T} \left|\frac{1}{\varepsilon} \int_0^t \beta^*(y^\varepsilon(s)) \, ds - \eta(t)\right|^p\right\} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

where β^* *denotes the transpose of* β *and the limits are uniform in* x *belonging to* $\overline{\mathbb{O}}$.

PROOF. The method is very similar to [6]. For the sake of completeness we include the whole adaptation of the proof.

Several steps are needed. First of all, note the following key properties of the penalization term $\beta(x)$ defined by (1.7),

(1.10) $$\begin{cases} \text{there exists } a \in \mathbb{R}^d, \quad c > 0 \quad \text{such that} \\ (x - a)\beta(x) \geq c |\beta(x)|, \quad \forall x \in \mathbb{R}^d, \end{cases}$$

(1.11) $$(x' - x)\beta(x) \leq \beta^*(x')\beta(x), \quad \forall x, x' \in \mathbb{R}^d.$$

Indeed, since $\mathbb{O}$ is convex, open and bounded, for any a in $\mathbb{O}$ we can find a

suitable positive constant c, proportional to the distance from the point a to the boundary $\partial \mathbb{O}$, such that (1.10) is satisfied. On the other hand, because $\mathbb{O}$ is convex, the function β is monotone, i.e.

$$(x' - x)(\beta(x') - \beta(x)) \geq 0, \quad \forall x, x' \in \mathbb{R}^d.$$

In particular

$$(x' - x)\beta(x) \leq 0, \quad \forall x \in \mathbb{R}^d, \quad \forall x' \in \overline{\mathbb{O}}.$$

This last inequality and the relation

$$x' - \beta^*(x') = \mathrm{pr}(x'), \quad \forall x' \in \mathbb{R}^d$$

imply the condition (1.11).

Now, we will show that for every $p \geq 1$, $T > 0$ there exists a constant C, depending only on p, T and the constants appearing in (1.2), (1.10), such that

$$(1.12) \qquad E\left\{\left(\frac{1}{\varepsilon} \int_0^T |\beta(y^\varepsilon(t))| \, dt\right)^p\right\} \geq C, \quad \forall \varepsilon > 0.$$

Indeed, we set

$$
\begin{aligned}
\alpha(p) = \sup\{ & pb_1(x)(1 + |x - a|^2)^{-1} \\
(1.13) \qquad & + p(p - 2)b_2(x)(1 + |x - a|^2)^{-2} \\
& + b_3(x, p)(1 + |x - a|^2)^{-p/2} : x \text{ in } \mathbb{R}^d\},
\end{aligned}
$$

with

$$b_1(x) = \sum_{i=1}^d (x_i - a_i)g_i(x) + \tfrac{1}{2} \sum_{i=1}^d \sum_{k=1}^n [\sigma_{ik}(x)]^2,$$

$$b_2(x) = \tfrac{1}{2} \sum_{i,j=1}^d (x_i - a_i)(x_j - a_j) \sum_{k=1}^n \sigma_{ik}(x)\sigma_{jk}(x),$$

$$
\begin{aligned}
b_3(x, p) = \int_{\mathbb{R}_*^m} [&(1 + |x - a + \gamma(x, z)|^2)^{p/2} - (1 + |x - a|^2)^{p/2} \\
& - p(1 + |x - a|^2)^{p/2-1} \sum_{i=1}^d (x_i - a_i)\gamma_i(x, z)]\pi(dz).
\end{aligned}
$$

Since

$$
\begin{aligned}
b_3(x, p) = \int_0^1 (1 - t) \, dt \int_{\mathbb{R}_*^m} \{ & p(1 + |x - a + t\gamma(x, z)|^2)^{p/2-1}(\sum_{i=1}^d \gamma_i^2(x, z)) \\
& + p(p - 2)(1 + |x - a + t\gamma(x, z)|^2)^{p/2-2} \\
& \cdot [\sum_{i=1}^d (x_i - a_i + t\gamma_i(x, z))\gamma_i(x, z)]^2\}\pi(dz),
\end{aligned}
$$

it is clear that the constant $\alpha(p)$ is finite and can be estimated by means of the constant C_p in the assumption (1.2) on the coefficients g, σ, γ. Thus, we apply Itô's formula to the function

$$(x, t) \to e^{-\alpha t}(1 + |x - a|^2)^{p/2}, \quad \alpha \geq \alpha(p),$$

with the process $y^\varepsilon(t)$, to get

$$e^{-\alpha t}(1 + |y^\varepsilon(t) - a|^2)^{p/2}$$
$$\leq (1 + |x - a|^2)^{p/2} + I(t)$$
$$- \frac{p}{\varepsilon} \int_0^t e^{-\alpha t}(y^\varepsilon(s) - a)\beta(y^\varepsilon(s))(1 + |y^\varepsilon(s) - a|^2)^{p/2-1} \, ds,$$

where $I(t)$ is a stochastic integral. This implies

(1.14) $$E\{|y^\varepsilon(t)|^p\} \leq C, \quad \forall t \in [0, T], \quad \forall \varepsilon > 0.$$

Therefore, taking $p = 2$ and using the condition (1.10) we obtain

$$\frac{2c}{\varepsilon} \int_0^t e^{-\alpha s} |\beta(y^\varepsilon(s))| \, ds$$
$$\leq (1 + |x - a|^2) + \int_0^t e^{-\alpha s}(y^\varepsilon(s) - a)\sigma(y^\varepsilon(s)) \, dw_s$$
$$+ \int_0^t \int_{\mathbb{R}_*^m} e^{-\alpha s}(y^\varepsilon(s) - a)\gamma(y^\varepsilon(s), z) \, d\mu_s(z).$$

Hence, by using standard martingale estimates on the above stochastic integrals and (1.14) we deduce (1.12). Note that we also have

(1.15) $$E\{\sup_{0 \leq t \leq T} |y^\varepsilon(t)|^p\} \leq C, \quad \forall \varepsilon > 0,$$

for some constant C independent of ε and x.

Next, we will prove that for every $p > 2$ there exists a constant C, depending only on p, T and the constant C_p appearing on (1.2), such that

(1.16) $$E\{\sup_{0 \leq t \leq T} |\beta(y^\varepsilon(t))|^p\} \leq C\varepsilon^{p/2-1}, \quad \forall \varepsilon > 0.$$

Indeed, let L be the integro-differential operator

(1.17) $$L\varphi = L_0\varphi + L_\gamma\varphi,$$

with

$$L_0\varphi = \frac{1}{2} \sum_{i,j=1}^d \left(\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right) \sum_{k=1}^n \sigma_{ik}\sigma_{jk} + \sum_{i=1}^d \left(\frac{\partial\varphi}{\partial x_i}\right)g_i$$

$$L_\gamma\varphi(x) = \int_{\mathbb{R}_*^m} \left[\varphi(x + \gamma(x, z)) - \varphi(x) - \sum_{i=1}^d \gamma_i(x, z)\frac{\partial\varphi}{\partial x_i}(x)\right]\pi(dz).$$

Consider the function

$$\varphi(x) = |\beta(x)|^p, \quad p > 2,$$

which is continuously differentiable and whose first derivatives are locally Lipschitzian. Moreover, for some constant C_0, depending only on p, the bounds of g, σ and the domain $\mathbb{O}$, and for almost every x in $\mathbb{R}^d$ we have

(1.18) $$|L_0\varphi(x)| \leq C_0(|\beta(x)|^{p-2} + |\beta(x)|^{p-1}).$$

The crucial point difference from [6], is the fact that

$$(1.19) \qquad |L_\gamma \varphi(x)| \le C_1 |\beta(x)|^{p-2}$$

holds for almost every x and some constant C_1, depending only on p, the constant C_p of (1.2) and the domain $\mathbb{O}$. In order to establish this last inequality, we notice that

$$L_\gamma \varphi(x) = \int_0^1 (1 - t) \, dt \int_{\mathbb{R}_*^m} \sum_{i,j=1}^d \gamma_i(x, z) \gamma_j(x, z) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} (x + t\gamma(x, z)) \pi(dz),$$

$$\left| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} (x + t\gamma(x, z)) \right| \le C |\beta(x + t\gamma(x, z))|^{p-2},$$

and

$$|\beta(x + t\gamma(x, z))| \le |x + t\gamma(x, z) - y|, \quad \forall y \in \overline{\mathbb{O}}.$$

Thus, choosing

$$y = \mathrm{pr}(x) + t\gamma(\mathrm{pr}(x), z)$$

and using (1.5), we verify (1.19). Then, approximating φ by smooth functions and applying Itô's formula for the process $y^\varepsilon(t)$, we may justify the following inequality

$$\varphi(y^\varepsilon(t)) + \frac{p}{\varepsilon} \int_0^t \varphi(y^\varepsilon(s)) \, dx$$

$$\le I(t) + \int_0^t [(C_0 + C_1) |\beta(y^\varepsilon(s))|^{p-2} + C_0 |\beta(y^\varepsilon(s))|^{p-1}] \, ds,$$

where $I(t)$ is a stochastic integral, precisely

$$I(t) = p \int_0^t \beta^*(y^\varepsilon(s)) \sigma(y^\varepsilon(s)) |\beta(y^\varepsilon(s))|^{p-2} \, dw_s$$

$$+ p \int_0^t \int_{\mathbb{R}_*^m} \beta^*(y^\varepsilon(s)) \gamma(y^\varepsilon(s), z) |\beta(y^\varepsilon(s))|^{p-2} \, d\mu_s(z),$$

and C_0, C_1 are the constants in (1.18), (1.19). Hence, using the elementary inequality

$$0 \le AB \le A^q/q + B^{q'}/q', \quad 1/q + 1/q' = 1$$

for appropriate factors A and B, we deduce

$$(1.20) \qquad \varphi(y^\varepsilon(t)) + \frac{p}{2\varepsilon} \int_0^t \varphi(y^\varepsilon(s)) \, ds \le I(t) + C(\varepsilon^{p-1} + \varepsilon^{p/2-1}),$$

for some constant C independent of ε. Therefore, taking the mathematical

expectation in (1.20), we obtain

$$(1.21) \qquad E\left\{\int_0^t \varphi(y^\varepsilon(s))\,ds\right\} \le C\varepsilon^{p/2}, \quad \text{if} \quad 0 < \varepsilon < 1,$$

for a suitable constant C. Similarly, taking the supremum over t belonging to $[0, T]$ in (1.20) and, then, using the stochastic integral estimate

$$E\left\{\sup_{0 \le t \le T}\left(\left|\int_0^t h_1(s)\,dw_s\right| + \left|\int_0^t \int_{\mathbb{R}_*^m} h_2(s, z)\,d\mu_s(z)\right|\right)\right\}$$

$$\le 3E\left\{\left(\int_0^T |h_1(s)|^2\,ds\right)^{1/2} + \left(\int_0^T ds \int_{\mathbb{R}_*^m} |h_2(s, z)|^2 \pi(dz)\right)^{1/2}\right\},$$

we have

$$E\{\sup_{0 \le t \le T} |\beta(y^\varepsilon(t))|^p\} \le C\varepsilon^{p/2-1} + CE\left\{\left(\int_0^T |\beta(y^\varepsilon(t))|^{2p-2}\,dt\right)^{1/2}\right\}.$$

Since the last term is dominated by

$$C'E\left\{\int_0^T |\beta(y^\varepsilon(t))|^{p-2}\,dt\right\} + \tfrac{1}{2}E\{\sup_{0 \le t \le T}|\beta(y^\varepsilon(t))|^p\},$$

we deduce for a new constant C that

$$E\{\sup_{0 \le t \le T}|\beta(y^\varepsilon(t))|^p\} \le C\varepsilon^{p/2-1} + CE\left\{\left(\int_0^T |\beta(y^\varepsilon(t))|^p\,dt\right)^{1-2/p}\right\},$$

by means of Hölder's inequality. Hence, the estimate (1.16) follows after using (1.21).

Let ε, ε' be two positive numbers. We will show that for every $p \ge 1$, $0 < 2q < p$, there exists a constant C, depending only on p, q, T, the constant C_p in (1.2) and the domain $\mathbb{O}$, such that

$$(1.22) \qquad E\left\{\left(\frac{1}{\varepsilon'}\int_0^T |\beta^*(y^\varepsilon(t))\beta(y^{\varepsilon'}(t))|\,dt\right)^p\right\} \le C\varepsilon^q.$$

Indeed, we have

$$\frac{1}{\varepsilon'}\int_0^T |\beta^*(y^\varepsilon(t))\beta(y^{\varepsilon'}(t))|\,dt \le AB$$

$$\equiv (\sup_{0 \le t \le T}|\beta(y^\varepsilon(t))|)\left(\frac{1}{\varepsilon'}\int_0^T |\beta(y^{\varepsilon'}(t))|\,dt\right).$$

Since, for $r > 2$

$$E\{(AB)^p\} \le (E\{A^{pr}\})^{1/r}(E\{B^{pr'}\})^{1/r'}, \quad r' = r/(r-1),$$

from (1.12) and (1.16) we get

$$E\{(AB)^p\} \le C\varepsilon^{p/2-1/r},$$

which implies (1.22), if r is chosen sufficiently large.

Now we will prove that if $2 < 2q < p$, there exists a constant C, depending only on p, q, T, the constant C_p in (1.2) and the domain $\textcircled{\scriptsize O}$, such that for every ε, $\varepsilon' > 0$

$$(1.23) \qquad E\{\sup_{0 \leq t \leq T} | y^{\varepsilon}(t) - y^{\varepsilon'}(t) |^p\} \leq C(\varepsilon + \varepsilon')^q,$$

$$(1.24) \qquad E\left\{ \sup_{0 \leq t \leq T} \left| \frac{1}{\varepsilon} \int_0^t \beta^*(y^{\varepsilon}(s))\, ds - \frac{1}{\varepsilon'} \int_0^t \beta^*(y^{\varepsilon'}(s))\, ds \right|^p \right\}$$
$$\leq C(\varepsilon + \varepsilon')^q.$$

Indeed, it is clear that by proving (1.23) we may deduce (1.24) after using the equation (1.6). In order to show the estimate (1.23), we apply Itô's formula to the function

$$\xi \rightarrow | \xi |^2$$

with the process

$$\xi_t = y^{\varepsilon}(t) - y^{\varepsilon'}(t)$$

to get

$$| y^{\varepsilon}(t) - y^{\varepsilon'}(t) |^2 \leq I(t) + C \int_0^t | y^{\varepsilon}(s) - y^{\varepsilon'}(s) |^2\, ds$$
$$+ \frac{2}{\varepsilon} \int_0^t (y^{\varepsilon'}(s) - y^{\varepsilon}(s)) \beta(y^{\varepsilon}(s))\, ds$$
$$+ \frac{2}{\varepsilon'} \int_0^t (y^{\varepsilon}(s) - y^{\varepsilon'}(s)) \beta(y^{\varepsilon'}(s))\, ds,$$

where $I(t)$ is the stochastic integral

$$I(t) = 2 \int_0^t (y^{\varepsilon}(s) - y^{\varepsilon'}(s))[\sigma(y^{\varepsilon}(s)) - \sigma(y^{\varepsilon'}(s))]\, dw_s$$
$$+ 2 \int_0^t \int_{\mathbb{R}_*^m} (y^{\varepsilon}(s) - y^{\varepsilon'}(s))[\gamma(y^{\varepsilon}(s), z) - \gamma(y^{\varepsilon'}(s), z)]\, d\mu_s(z),$$

and C is a constant depending only on the constant C_p in (1.2). Notice that by means of standard estimates on stochastic integrals we have, for $r > 1$,

$$E\{\sup_{0 \leq t \leq T} | I(t) |\}^r$$

$$\leq CE\left\{ \left(\int_0^T | y^{\varepsilon}(t) - y^{\varepsilon'}(t) |^4\, dt \right)^{r/2} \right\}$$

$$\leq \tfrac{1}{2} E\{\sup_{0 \leq t \leq T} | y^{\varepsilon}(t) - y^{\varepsilon'}(t) |^{2r}\} + \tfrac{1}{2} C^2 E\left\{ \left(\int_0^T | y^{\varepsilon}(t) - y^{\varepsilon'}(t) |^2\, dt \right)^r \right\}.$$

Therefore, by using the property (1.11) and with a new constant C, $2r = p$,

$0 \leq t \leq T$, we deduce the following inequality

$$E\{\sup_{0 \leq s \leq t} | y^{\varepsilon}(s) - y^{\varepsilon'}(s)|^p\} \leq C \int_0^t E\{| y^{\varepsilon}(s) - y^{\varepsilon'}(s)|^p\} \, ds$$

$$+ CE\left\{\left(\frac{1}{\varepsilon} \int_0^T | \beta^*(y^{\varepsilon'}(s))\beta(y^{\varepsilon}(s))| \, ds\right)^p\right\}$$

$$+ CE\left\{\left(\frac{1}{\varepsilon'} \int_0^T | \beta^*(y^{\varepsilon}(s))\beta(y^{\varepsilon'}(s))| \, ds\right)^p\right\},$$

which implies (1.23) after using (1.22) and Gronwall's inequality.

The next step is to define the processes

(1.25) $$y(t) = \lim_{\varepsilon \to 0} y^{\varepsilon}(t),$$

(1.26) $$\eta(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \beta^*(y^{\varepsilon}(s)) \, ds,$$

where both limits exist in the sense of (1.8), (1.9). Clearly, the estimate (1.12) shows that $\eta(t)$ has locally bounded variation, precisely for any $p \geq 1$, $T > 0$,

$$E\{(\sum_{i=1}^n | \eta(t_i) - \eta(t_{i-1})|)^p\} \leq C(p, T) < +\infty$$

$$\text{for every} \quad 0 \leq t_0 < \cdots < t_n \leq T, \quad n = 1, 2, \cdots.$$

On the other hand, the condition (1.16) implies that the process $y(t)$ takes values into the closure $\overline{\mathbb{O}}$, almost surely. If we define

$$\eta^{\varepsilon}(t) = \frac{1}{\varepsilon} \int_0^t \beta^*(y^{\varepsilon}(s)) \, ds,$$

then

$$d\eta^{\varepsilon}(t) = (1/\varepsilon)\beta(y^{\varepsilon}(t)) \, dt.$$

Hence, the property (1.11) proves that

$$\int_0^T (y^{\varepsilon}(t) - z(t)) \, d\eta^{\varepsilon}(t) \geq 0, \quad \forall T \geq 0,$$

for every right continuous process having left-hand limits $z(t)$, which is progressively measurable and takes values into the closure $\overline{\mathbb{O}}$. Thus, passing to the limit as ε tends to zero, we obtain the SVI (1.4).

To conclude the proof, let us suppose that $(y_i(t), \eta_i(t), t \geq 0)$, $i = 1, 2$ are two solutions of the SVI (1.4). By applying a combined version of the Itô's formula and an integration by part to the function

$$\xi \to | \xi |^2$$

with the process

$$\xi_t = y_1(t) - y_2(t)$$

we get

$$E\{|y_1(t) - y_2(t)|^2\} \le CE\left\{\int_0^t |y_1(s) - y_2(s)|^2\, ds\right\}$$

$$+ 2E\left\{\int_0^t (y_1(s) - y_2(s))(d\eta_2(s) - d\eta_1(s))\right\}.$$

Since the last term is not positive, we deduce

$$y_1(t) = y_2(t), \quad \forall t \ge 0.$$

Hence, the SVI (1.4) poses a unique solution given by the limits (1.25) and (1.26). $\square$

REMARK 1. We also have for any numbers $p > 2q > 2$, the estimates

(1.27) $E\{\sup_{0 \le t \le T} |y^\varepsilon(t) - y(t)|^p\} \le C\varepsilon^q, \quad \forall \varepsilon > 0,$

(1.28) $E\left\{\sup_{0 \le t \le T}\left|\dfrac{1}{\varepsilon}\int_0^t \beta^*(y^\varepsilon(s))\, ds - \eta(t)\right|\right\} \le C\varepsilon^q, \quad \forall \varepsilon > 0,$

where the constant C depends only on p, q, the domain $\mathbb{O}$ and the constant C_p in (1.2). $\square$

REMARK 2. The reflected diffusion process with jumps $y(t) = y_x(t)$ is Lipschitz continuous with respect to the initial data x in $\overline{\mathbb{O}}$. Precisely, if $p \ge 2$ and

(1.29)
$$\alpha_p = \sup\{pC_1(x, x')|x - x'|^{-2} + p(p - 2)C_2(x, x')|x - x'|^{-4}$$
$$+ C_3(x, x', p)|x - x'|^{-p} : x, x' \text{ in } \mathbb{R}^d\},$$

with

$C_1(x, x')$
$$= \sum_{i=1}^d (x_i - x_i')(g_i(x) - g_i(x')) + \tfrac{1}{2}\sum_{i=1}^d \sum_{k=1}^n [\sigma_{ik}(x) - \sigma_{ik}(x')]^2,$$

$C_2(x, x')$
$$= \tfrac{1}{2}\sum_{i,j=1}^d (x_i - x_i')(x_j - x_j') \sum_{k=1}^n [\sigma_{ik}(x) - \sigma_{ik}(x')][\sigma_{jk}(x) - \sigma_{jk}(x')],$$

$C_3(x, x', p)$
$$= \int_{\mathbb{R}_*^m} [|x - x' + \gamma(x, z) - \gamma(x', z)|^p - |x - x'|^p$$
$$- p|x - x'|^{p-2}\sum_{i=1}^d (x_i - x_i')(\gamma_i(x, z) - \gamma_i(x', z))]\pi(dz),$$

then for $\alpha \ge \alpha_p$, $T \ge 0$ and x, x' in $\overline{\mathbb{O}}$, we have

$$E\left\{|y_x(T) - y_{x'}(T)|^p e^{-\alpha T} + (\alpha - \alpha_p)\int_0^T |y_x(t) - y_{x'}(t)|^p e^{-\alpha t}\, dt\right\}$$

(1.30)
$$\le |x - x'|^p.$$

Note that the constant α_p is finite and can be estimated by means of the constant C_p in (1.2). $\square$

REMARK 3. Assume that the domain $\mathbb{O}$ is bounded, convex and smooth, i.e. there exists a function $\rho(x)$ from $\mathbb{R}^d$ into $\mathbb{R}$, which is twice continuously differentiable and satisfies

$$\mathbb{O} = \{x \in \mathbb{R}^d : \rho(x) < 0\},$$

(1.31)
$$\partial\mathbb{O} = \{x \in \mathbb{R}^d : \rho(x) = 0\},$$

$$|\nabla\rho(x)| \geq 1, \quad \forall x \in \partial\mathbb{O}.$$

Moreover, without loss of generality, we may suppose also that

$$\rho(x) = |\beta(x)|, \quad \forall x \in \mathbb{R}^d \setminus \mathbb{O}.$$

Then, by applying Itô's formula to the function $\rho(x)$ with the process $(y^\varepsilon(t), t \geq 0)$, we have

$$\frac{1}{\varepsilon} \int_0^t |\beta(y^\varepsilon(s))|\, ds = \rho(x) - \rho(y^\varepsilon(t)) + I_\varepsilon(t) + \int_0^t L\rho(y^\varepsilon(s))\, ds,$$

where L denotes the integro-differential operator (1.17) and $I_\varepsilon(t)$ is a stochastic integral. This equality and the estimate (1.23) allow us to establish the convergence

$$(1.32) \qquad E\left\{\sup_{0 \leq t \leq T}\left|\frac{1}{\varepsilon}\int_0^t |\beta(y^\varepsilon(s))|\, ds - \zeta(t)\right|^p\right\} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

with the limit being uniform for x belonging to $\overline{\mathbb{O}}$. The process $(\zeta(t), t \geq 0)$ is nonnegative, increasing, continuous, progressively measurable and satisfies

$$(1.33) \qquad \eta(t) = \int_0^t n(y(s))\, d\zeta(s), \quad \forall t \geq 0,$$

$$(1.34) \qquad \zeta(t) = \int_0^t \chi(y(s) \in \partial\mathbb{O})\, d\zeta(s), \quad \forall t \geq 0,$$

where $(y(t), \eta(t), t \geq 0)$ is the solution of the SVI (1.4), $\chi(\cdot)$ denotes the characteristic function and $n = \nabla\rho$ stands for the outward unit normal to $\mathbb{O}$. $\square$

REMARK 4. In order to obtain the estimates (1.12), (1.14), (1.15), (1.16) and (1.21), we did not make use of the Lipschitzian character of the coefficients g, σ, γ, i.e. the second part of (1.2). This fact allows us to extend the technique of Theorem 1 to the so-called weak formulation. In that case, the convergences (1.8) and (1.9) become weak convergence in probability. On the other hand, if we consider unbounded domains, then similar results can be deduced. $\square$

2. Simply connected domains. Let $(\Omega, F, P, F_t, w_t, \mu_t, t \geq 0)$ be complete Wiener-Poisson space in $\mathbb{R}^n \times \mathbb{R}_*^m$ with Levy measure π.

Suppose that $\mathbb{O}$ is an open subset of $\mathbb{R}^d$ such that

(2.1)   there exists a diffeomorphism of class C^3
between the closure $\overline{\mathbb{O}}$ and the closed unit ball.

In other words there is a one-to-one map ψ from a neighbourhood of $\overline{\mathbb{O}}$ into a neighbourhood of the closed unit ball $\overline{B}$ in $\mathbb{R}^d$, such that ψ and its inverse ψ^{-1} are three times continuously differentiable satisfying

$$\psi(\mathbb{O}) = B, \quad \psi(\partial\mathbb{O}) = \partial B.$$

For instance, if $\mathbb{O}$ is a bounded simply connected set in $\mathbb{R}^d$ with a smooth, connected and orientable boundary $\partial\mathbb{O}$, then the condition (2.1) is verified.

On the other hand, assume we are given coefficients $g(x)$, $\sigma(x)$, $\gamma(x)$ satisfying (1.2) and (1.3), and also a vector field ν defined in a neighbourhood of the closure $\overline{\mathbb{O}}$, which is twice continuously differentiable such that

(2.2) $\nu(x) \cdot n(x) \geq \varepsilon > 0, \quad \forall x \in \partial\mathbb{O},$

where $n(x)$ is the outward unit normal to $\mathbb{O}$ at the point x.

A diffusion process with jumps $(y(t), t \geq 0)$ and with instantaneous reflection at the boundary $\partial\mathbb{O}$ in the direction ν is a right continuous process having left-hand limits, which is progressively measurable and satisfies

(i) $y(t)$ takes values in the closure $\overline{\mathbb{O}}$ and there exists an increasing continuous process $\zeta(t)$, with $\zeta(0) = 0$, such that

(ii) $dy(t) = g(y(t))\, dt + \sigma(y(t))\, dw_t$

(2.3) $+ \int_{\mathbb{R}^m_*} \gamma(y(t), z)\, d\mu_t(z) - \nu(y(t))\, d\zeta(t), \quad t \geq 0,$

$y(0) = x;$

(iii) $d\zeta(t) = \chi(y(t) \in \partial\mathbb{O})\, d\zeta(t), \quad t \geq 0,$

where $\chi(y \in \partial\mathbb{O})$ denotes the characteristic function of the boundary $\partial\mathbb{O}$. The stochastic process $(\zeta(t), t \geq 0)$ is called the associate increasing process to the reflected diffusion process with jumps $(y(t), t \geq 0)$.

THEOREM 2.   *Under the assumptions* (1.2), (1.3), (2.1) *and* (2.2) *there exists a unique solution of the stochastic equation* (2.3) *for each x belonging to* $\overline{\mathbb{O}}$.

PROOF.   The point is to build a diffeomorphism between $\overline{\mathbb{O}}$ and the closed unit ball $\overline{B}$ such that the direction ν is transformed into an outward normal direction. Hence, by using Itô's formula, we determine a SVI (1.4) on the ball $\overline{B}$, which can be solved by means of the Theorem 1. Thus, going back to the initial domain $\overline{\mathbb{O}}$ via the diffeomorphism, we establish the existence and uniqueness of solutions for the problem (2.3). The construction of such a diffeomorphism is based on an idea of Williams [9].

Because of the assumption (2.1), there is a diffeomorphism between $\overline{\mathbb{O}}$ and $\overline{B}$

i.e. for some δ, $\delta_1 > 0$,

$$\psi_1 \colon \overline{\mathbb{O}} + \delta B \to \overline{B} + \delta_1 B$$

together with its inverse are three times continuously differentiable such that

$$\psi_1(\mathbb{O}) = B, \quad \psi_1(\partial \mathbb{O}) = \partial B.$$

Notice that if C_1, C_2 are two curves in $\overline{\mathbb{O}} + \delta B$ with nonzero angle, then $\psi_1(C_1)$, $\psi_1(C_2)$ are also two curves of $\overline{B} + \delta_1 B$ with nonzero angle. This fact, which is valid for any smooth diffeomorphism, allows us to verify the condition (2.2) is preserved by ψ_1, i.e. for some $0 < \varepsilon_1 < 1$

$$\nu_1(x) \cdot n_1(x) \geq \varepsilon_1, \quad \forall x \in \partial B,$$

where $n_1(x)$ stands for the outward unit normal to B at the point x in ∂B. Note that $\nu_1(x)$ is indeed a tangent vector of a curve C at the point x, provided the curve $\psi_1^{-1}(C)$ possesses $\nu(y)$ as a tangent vector at the point $y = \psi_1^{-1}(x)$, i.e.

$$\nu_1 = \nu \nabla \psi_1,$$

with $\nabla \psi_1$ being the matrix of the first derivatives of ψ_1.

Now the problem is reduced to the case $\mathbb{O} = B$, in which we only need to transform ν_1 into the normal. To this end, we consider a vector field $f(x)$ which is twice continuously differentiable on the open set

$$0 < |x| < 1 + \delta_1$$

and satisfies

$$f(x) = \nu_1(x), \qquad \text{if} \quad |x| = 1,$$

$$f(x) \cdot x \geq \tfrac{1}{2}\varepsilon_1 |x|, \quad \text{for every} \quad x,$$

$$f(x) = |x|^{-1} x, \quad \text{if} \quad |x| \leq \tfrac{2}{3}.$$

Notice that the assumptions made on the direction ν and the diffeomorphism ψ_1 ensure that ν_1 is twice continuously differentiable and, therefore, the vector field f can be constructed. Thus, we define for some $\delta_2 > 0$

$$\psi_2 \colon \overline{B} + \delta_1 B \to \overline{B} + \delta_2 B$$

by means of an ordinary differential equation. Let $x(t, a)$ be the unique solution of the initial value problem

$$dx/dt = f(x), \quad x(\tfrac{1}{2}) = \tfrac{1}{2}a,$$

where $|a| = 1$. We set

$$\psi_2^{-1}(y) = \begin{cases} y, & \text{if } |y| \leq \tfrac{1}{2}, \\ x(|y|, y/|y|), & \text{if } |y| \geq \tfrac{1}{2}. \end{cases}$$

This means that the curve

$$\{ta \colon 0 \leq t < 1 + \delta_2\} \quad \text{in} \quad \overline{B} + \delta_2 B$$

becomes the curve

$$\{ta : 0 \leq t \leq \tfrac{1}{2}\} \cup \{x(t, a) : \tfrac{1}{2} \leq t < 1 + \delta_1\} \quad \text{in} \quad \bar{B} + \delta_1 B.$$

Clearly, the classical theory of ordinary differential equation guarantees the smoothness of the diffeomorphism Ψ_2 and by construction

$$\nu_2 = \nu_1 \nabla \Psi_2 = n_2,$$

where n_2 is the outward unit normal, i.e. $n_2(y) = y$. By the way, note that if

$$\nu(x) = \lambda(x) \nu'(x), \quad \forall x,$$

with λ being smooth and satisfying

$$\lambda(x) \geq \varepsilon' > 0, \quad \forall x \in \partial \mathbb{O},$$

then the reflected diffusion processes with jumps $(y(t), t \geq 0)$ and $(y'(t), t \geq 0)$ associated with the direction ν and ν', respectively possess the property

$$y(t) = y'(t), \qquad\qquad \forall t \geq 0$$

$$d\zeta(t) = \lambda(y(t)) \, d\zeta'(t), \quad \forall t \geq 0$$

where $(\zeta(t), t \geq 0)$ and $(\zeta'(t), t \geq 0)$ are the increasing processes corresponding to the directions ν and ν' respectively. This completes the proof. $\square$

REMARK 5. The hypothesis (2.1) is not really needed. It suffices to know that the domain $\mathbb{O}$ is such that it can be transformed into a convex set $\mathbb{O}'$, via a diffeomorphism of class C^2 with the property of mapping the direction ν into the normal n' of $\mathbb{O}$. Note that the problem of characterizing domains $\mathbb{O}$ having the property just mentioned is an open question for us. However, it is clear that $\mathbb{O}$ could have only a piecewise smooth boundary and still satisfy the above property. $\square$

3. Regular domains. Let $(\Omega, F, P, F_t, w_t, \mu_t, t \geq 0)$ be a complete Wiener-Poisson space in $\mathbb{R}^n \times \mathbb{R}_*^m$ with Levy measure π, and $g(x)$, $\sigma(x)$, $\gamma(x)$ be coefficients satisfying (1.2).

In order to be able to treat domains which are not simple connected, we assume that $\mathbb{O}$ admits the following representation

(3.1)
$$\begin{cases} \text{there exists a function } \rho(x) \text{ from } \mathbb{R}^d \text{ into } \mathbb{R} \text{ which is twice continuously} \\ \text{differentiable and such that} \\ \\ \mathbb{O} = \{x \in \mathbb{R}^d : \rho(x) < 0\}, \\ \\ \partial\mathbb{O} = \{x \in \mathbb{R}^d : \rho(x) = 0\}, \\ \\ |\nabla\rho| \geq 1 \quad \text{on} \quad \partial\mathbb{O}, \end{cases}$$

and also

(3.2) $\mathbb{O}$ is bounded.

Note that without loss of generality, the function $\rho(x)$ may satisfy

(3.3) $\begin{cases} \rho(x) = |x - a|, \text{ if the distance between } x \text{ and } \overline{\mathbb{O}} \text{ is greater than some} \\ \delta > 0 \text{ for a suitable } a \text{ in } \mathbb{O}. \end{cases}$

On the other hand, it is clear that under the hypothesis (3.1), we may construct a Lipschitzian function $\mathrm{pr}(x)$ from $\mathbb{R}^d$ in itself such that

(3.4) $\begin{cases} |x - \mathrm{pr}(x)| \leq \mathrm{dist}(x, \overline{\mathbb{O}}) \text{ and } \mathrm{pr}(x) \in \overline{\mathbb{O}}, \text{ for every } x \text{ in } \mathbb{R}^d \\ \text{satisfying } \mathrm{dist}(x, \overline{\mathbb{O}}) < \delta_0, \end{cases}$

where δ_0 is a positive constant and $\mathrm{dist}(x, \overline{\mathbb{O}})$ stands for the distance between the point x and the set $\overline{\mathbb{O}}$. Hence, we can extend the definition of the coefficients g, σ, γ to the whole space $\mathbb{R}^d$, in such a way that (1.2) is preserved. In particular, for every x in a neighborhood of $\overline{\mathbb{O}}$ we have

(3.5) $$\gamma(x, z) = \gamma(\mathrm{pr}(x), z), \quad \forall z \in \mathbb{R}^m_*.$$

Instead of (1.3), we assume

(3.6) $x + t\gamma(x, z) \in \overline{\mathbb{O}}, \quad \forall x \in \overline{\mathbb{O}}, \quad \forall z \in \mathbb{R}^m_*, \quad \forall t \in [0, 1].$

Notice that from (3.4), (3.5) and (3.6) we deduce

(3.7) $\begin{cases} \mathrm{dist}(x + t\gamma(x, z), \overline{\mathbb{O}}) \leq \mathrm{dist}(x, \overline{\mathbb{O}}), \text{ for every } x \text{ in } \mathbb{R}^d \\ \text{satisfying } \mathrm{dist}(x, \overline{\mathbb{O}}) < \delta_0 \text{ and any } z \text{ in } \mathbb{R}^m_*, 0 \leq t \leq 1. \end{cases}$

Suppose we are given a function $M(x)$ from a neighbourhood of the boundary $\partial\mathbb{O}$ into the set of symmetric matrices $d \times d$, which is twice continuously differentiable and such that for every x,

(3.8) $z^*M(x)z \geq \delta |z|^2, \quad \forall z \in \mathbb{R}^d,$

for some constant $\delta > 0$. Remark that any vector field $\nu(x)$, which is twice continuously differentiable in $\partial\mathbb{O}$ and satisfies (2.2), can be represented as

(3.9) $\nu(x) = M(x)n(x), \quad \forall x$

for some matrix satisfying (3.4). Note that

$$n(x) = |\nabla\rho(x)|^{-1}\nabla\rho(x),$$

with $\nabla\rho$ being the gradient of the function ρ.

We construct the following vector field $\beta(x)$ defined and Lipschitzian on the whole $\mathbb{R}^d$,

(3.10) $\beta(x) = \rho^+(x)\chi(x)M(x)\nabla\rho(x) + (1 - \chi(x))\rho^+(x)\nabla\rho(x),$

where $\chi(x)$ is a smooth function satisfying

$0 \leq \chi(x) \leq 1, \quad \forall x \in \mathbb{R}^d,$

$\chi(x) = 1$ in a neighbourhood of $\partial\mathbb{O}$

$\chi(x) = 0$ if either $\rho(x) = |x - a|$ or $M(x)$ is not defined,

and ρ^+ denotes the positive part of the function ρ given in (3.1). Notice that in

view of (3.1) and (3.3), we may suppose without loss of generality that

(3.11) $$|\nabla\rho| \geq 1 \quad \text{outside of } \mathbb{O}.$$

Consider the stochastic differential equation

(3.12)
$$\begin{cases} dy^\varepsilon(t) = g(y^\varepsilon(t)) \, dt + \sigma(y^\varepsilon(t)) \, dw_t \\ \quad + \int_{\mathbb{R}^m_*} \gamma(y^\varepsilon(t), z) \, d\mu_t(z) - \frac{1}{\varepsilon} \beta(y^\varepsilon(t)) \, dt, \quad t \geq 0, \quad y^\varepsilon(0) = x, \end{cases}$$

i.e. the same equation as (1.6), but with a different β, and denote by

(3.13) $$\zeta^\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t \rho^+(y^\varepsilon(s)) |\nabla\rho(y^\varepsilon(s))| \, ds, \quad t \geq 0.$$

THEOREM 3. *Assume the conditions* (1.2), (3.1), $\cdots$, (3.11) *hold. Then the problem* (2.3) *has one and only one solution* $(y(t), \zeta(t), t \geq 0)$. *Moreover, for each* $p \geq 1$, $T > 0$, *we have*

(3.14) $$E\{\sup_{0\leq t\leq T} |y^\varepsilon(t) - y(t)|^p\} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

(3.15) $$E\{\sup_{0\leq t\leq T} |\zeta^\varepsilon(t) - \zeta(t)|^p\} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

where the limits are uniform in x *belonging to* $\overline{\mathbb{O}}$.

PROOF. The methods to be used are essentially the same as those of Theorem 1. We just need to establish some key facts before applying the technique developed in [6]. First we will prove that for some constants $K_0 > 0$, $K > 0$, and any x, x' in $\mathbb{R}^d$, we have

(3.16) $$\text{dist}(x, \overline{\mathbb{O}}) \leq \rho^+(x) \leq K_0 \text{dist}(x, \overline{\mathbb{O}}),$$

(3.17) $$\rho^+(x)(x' - x)\nabla\rho(x) \leq \rho^+(x)\rho^+(x') + K\rho^+(x)|x - x'|^2,$$

where $\text{dist}(x, \overline{\mathbb{O}})$ stands for the distance from the point x to the set $\overline{\mathbb{O}}$. Indeed, the inequality (3.16) follows from

$$\rho^+(x) = \int_0^1 (x - a)\nabla\rho(a + t(x - a))\chi(a + t(x - a) \notin \mathbb{O}) \, dt,$$

for any a in $\overline{\mathbb{O}}$, and in view of (3.11), if we take

$$K_0 = \sup\{|\nabla\rho(x)| : x \text{ in } \mathbb{R}^d - \mathbb{O}\}.$$

Similarly, the identity

$$\rho(x') - \rho(x)$$
$$= (x' - x)\nabla\rho(x) + \int_0^1 (x' - x)[\nabla\rho(x + t(x' - x)) - \nabla\rho(x)] \, dt$$

yields (3.17) with

$$2K = \sup\{|\nabla\rho(x) - \nabla\rho(x')| |x - x'|^{-1} : x, x' \text{ in } \mathbb{R}^d\}.$$

A second clue is that we can find a constant $c > 0$ such that

(3.18) $$\nabla\rho(x)\cdot\beta(x) \geq c\rho^+(x)|\nabla\rho(x)|, \quad \forall x \in \mathbb{R}^d.$$

Indeed, from (3.8) and (3.10) follows, for the same $\delta > 0$ of (3.8),

$$\begin{aligned}
\nabla\rho(x)\cdot\beta(x) &= \rho^+(x)\chi(x)(\nabla\rho(x)\cdot M(x)\nabla\rho(x)) \\
&\quad + \rho^+(x)(1 - \chi(x))(\nabla\rho(x)\cdot\nabla\rho(x)) \\
&\geq [\delta\chi(x) + (1 - \chi(x))]\rho^+(x)|\nabla\rho(x)|^2,
\end{aligned}$$

which implies (3.18), after using (3.11) and choosing

$$c = \min\{1, \delta\}.$$

Now, we will show that for every constant $\alpha > 0$ there exist constants C, λ depending only on the constants C_p in (1.2), δ in (3.8), α and the domain $\mathbb{O}$ such that

(3.19) $$E\{\exp(\alpha\zeta^\varepsilon(t))\} \leq C \exp(\lambda t), \quad \forall t \geq 0, \quad \forall\varepsilon > 0,$$

where $(\zeta^\varepsilon(t), t \geq 0)$ is the stochastic process (3.13). Indeed, by means of Itô's formula applied to the function $\rho(x)$ with the process $y^\varepsilon(t)$, we get

(3.20) $$\begin{aligned}
\rho(y^\varepsilon(t)) &= \rho(x) + I(t) + \int_0^t L\rho(y^\varepsilon(s)) \, ds \\
&\quad - \frac{1}{\varepsilon} \int_0^t \nabla\rho(y^\varepsilon(s))\cdot\beta(y^\varepsilon(s)) \, ds, \quad t \geq 0
\end{aligned}$$

where L is the integro-differential operator (1.17) and

$$I(t) = \int_0^t \nabla\rho(y^\varepsilon(s))\sigma(y^\varepsilon(s)) \, dw_s + \int_0^t \int_{\mathbb{R}_*^m} \nabla\rho(y^\varepsilon(s))\cdot\gamma(y^\varepsilon(s), z) \, d\mu_s(z).$$

Since $\rho(x)$ is bounded from below, the inequality (3.18) implies

(3.21) $$c\zeta^\varepsilon(t) \leq C_0 + C_1 t + I(t), \quad \forall t \geq 0,$$

for some constants C_0, C_1 depending on ρ, g, σ, γ, and the same c of (3.18). Again, by using Itô's formula for the function

$$\xi \to \exp(\alpha c^{-1}\xi)$$

and the process $I(t)$, we have

$$E\{\exp(\alpha c^{-1}I(t))\} \leq 1 + \alpha^2 c^{-2} C_2 \int_0^t E\{\exp(\alpha c^{-1}I(s))\} \, ds, \quad t \geq 0,$$

where C_2 is a suitable constant depending only on ρ, σ and γ. Hence, Gronwall's inequality implies

$$E\{\exp(\alpha c^{-1}I(t))\} \leq \exp(\alpha^2 c^{-2} C_2 t), \quad t \geq 0.$$

This fact together with (3.21) yield the estimate (3.19) with

$$\lambda = \alpha c^{-1}C_1 + \alpha^2 c^{-2}C_2, \quad C = \exp(\alpha c^{-1}C_0).$$

It is clear that from (3.19) follows

$$(3.22) \qquad\qquad E\{|\zeta^\varepsilon(t)|^p\} \le C, \quad \forall \varepsilon > 0$$

and going back to (3.20), we deduce

$$(3.23) \qquad\qquad E\{\sup_{0 \le t \le T}|y^\varepsilon(t)|^p\} \le C, \quad \forall \varepsilon > 0,$$

for any $p \ge 1$, $T > 0$.

Let us show that for any $p > 2$ there exists a constant C, depending only on p, T, the constant C_p appearing in (1.2) and the domain $\mathbb{O}$, such that

$$(3.24) \qquad\qquad E\{\sup_{0 \le t \le T}[\rho^+(y^\varepsilon(t))]^p\} \le C\varepsilon^{p/2-1}, \quad \forall \varepsilon > 0.$$

Indeed, similar to Theorem 1, (1.18), (1.19), and in view of inequalities (3.7) and (3.16), we verify that

$$|L\varphi(x)| \le C_1[\rho^+(x)]^{p-1} + C_2[\rho^+(x)]^{p-2}, \quad \forall x \in \mathbb{R}^d,$$

for some suitable constants C_1, C_2 and

$$\varphi(x) = [\rho^+(x)]^p, \quad p > 2,$$

with L being the integro-differential operator (1.17). Hence, by using Itô's formula for the function $\varphi(x)$ we deduce (3.24), after noting that

$$\nabla\rho(x)\cdot\beta(x) = p[\rho^+(x)]^{p-1}\nabla\rho(x)\cdot\beta(x) \ge pc\varphi(x),$$

with the same c of (3.18).

Therefore, it follows

$$E\left\{\left(\frac{1}{\varepsilon'}\int_0^T \rho^+(y^\varepsilon(t))\rho^+(y^{\varepsilon'}(t))\,dt\right)^p\right\} \le C\varepsilon^q, \quad \forall \varepsilon, \varepsilon' > 0,$$

with $p \ge 1$, $0 < 2q < p$ and some constant C.

At this point, we apply Itô's formula to the function

$$(3.25) \qquad \begin{cases} (\xi, y, y', \zeta) \to \xi^*(Q(y) + Q(y'))\xi \exp(-\alpha\zeta), \\ Q(y) = \chi(y)M^{-1}(y) + (1 - \chi(y))Id, \end{cases}$$

where Id stands for the identity matrix and χ the function of (3.10), with the process

$$\xi_t = y^\varepsilon(t) - y^{\varepsilon'}(t)$$

$$y_t = y^\varepsilon(t), \quad y'_t = y^{\varepsilon'}(t),$$

$$\zeta_t = \zeta^\varepsilon(t) + \zeta^{\varepsilon'}(t)$$

to get

$$c \, | \, y^\varepsilon(t) - y^{\varepsilon'}(t) |^2 \exp(-\alpha \zeta_t)$$

$$\leq \xi_t^*(Q(y_t) + Q(y_t'))\xi_t \exp(-\alpha \zeta_t)$$

$$\leq I(t) + C \int_0^t | y^\varepsilon(s) - y^{\varepsilon'}(s) |^2 \exp(-\alpha \zeta_s) \, ds$$

$$+ (C - c\alpha) \int_0^t | y^\varepsilon(s) - y^{\varepsilon'}(s) |^2 \exp(-\alpha \zeta_s)(d\zeta^\varepsilon(s) + d\zeta^{\varepsilon'}(s))$$

$$+ \frac{2}{\varepsilon} \int_0^t \beta(y^\varepsilon(s))[Q(y^\varepsilon(s)) + Q(y^{\varepsilon'}(s))](y^{\varepsilon'}(s) - y^\varepsilon(s))\exp(-\alpha \zeta_s) \, ds$$

$$+ \frac{2}{\varepsilon'} \int_0^t \beta(y^{\varepsilon'}(s))[Q(y^\varepsilon(s)) + Q(y^{\varepsilon'}(s))](y^\varepsilon(s) - y^{\varepsilon'}(s))\exp(-\alpha \zeta_s) \, ds,$$

where c, C are positive constants and the stochastic integral satisfies

$$E\{\sup_{0 \leq t \leq T} | I(t) |^r\} \leq CE\left\{\left(\int_0^t | y^\varepsilon(t) - y^{\varepsilon'}(t) |^4 \exp(-2\alpha \zeta_t) \, dt\right)^{r/2}\right\}.$$

Hence, by means of (3.10) and (3.17) we have

$$\beta(y)Q(y)(y' - y)$$

$$= \rho^+(y)(y' - y)\nabla\rho(y)$$

$$+ \rho^+(y)\chi(y)(1 - \chi(y))(y' - y)^*[-2Id + M(y) + M^{-1}(y)]\nabla\rho(y)$$

$$\leq \rho^+(y)\rho^+(y') + K\rho^+(y) | y - y' |^2$$

$$+ C_1[\rho^+(y)\rho^+(y') + \rho^+(y) | y - y' |^2],$$

and

$$\beta(y)[Q(y) - Q(y')](y' - y) \leq C_2 | y' - y |^2,$$

for some constants C_1, C_2. Thus, noting that

$$d\zeta^\varepsilon(t) = (1/\varepsilon)\rho^+(y^\varepsilon(s)) | \nabla\rho(y^\varepsilon(s)) | \, ds,$$

we obtain

$$c \, | \, y^\varepsilon(t) - y^{\varepsilon'}(t) |^2 \exp(-\alpha \zeta_t)$$

$$\leq I(t) + C \int_0^t | y^\varepsilon(s) - y^{\varepsilon'}(s) |^2 \exp(-\alpha \zeta_s) \, ds$$

$$+ \frac{C}{\varepsilon} \int_0^t \rho^+(y^{\varepsilon'}(s))\rho^+(y^\varepsilon(s)) \, ds + \frac{C}{\varepsilon'} \int_0^t \rho^+(y^\varepsilon(s))\rho^+(y^{\varepsilon'}(s)) \, ds,$$

provided the constants α and C are sufficiently large. Therefore, as in Theorem

1 and in view of (3.19), we get

(3.26) $E\{\sup_{0\leq t\leq T}|y^\varepsilon(t) - y^{\varepsilon'}(t)|^p\} \leq C(\varepsilon + \varepsilon')^q, \quad \forall \varepsilon, \varepsilon' > 0,$

for $2 < 2q < p$, $T > 0$ and some constant C. Hence, from the stochastic equation (3.12) follows

(3.27) $E\{\sup_{0\leq t\leq T}|\eta^\varepsilon(t) - \eta^{\varepsilon'}(t)|^p\} \leq C(\varepsilon + \varepsilon')^q, \quad \forall \varepsilon, \varepsilon' > 0$

where C is another constant and

$$\eta^\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t \beta^*(y^\varepsilon(s))\, ds, \quad t \geq 0.$$

Now, let us verify that for every $T > 0$, $p > 2$ we have

(3.28) $E\{\sup_{0\leq t\leq T}|\zeta^\varepsilon(t) - \zeta^{\varepsilon'}(t)|^p\} \to 0 \quad \text{as} \quad \varepsilon, \varepsilon' \to 0.$

Indeed, define the processes

$a_\varepsilon(t)$

$\quad = [\chi(y^\varepsilon(t))M(y^\varepsilon(t))\nabla\rho(y^\varepsilon(t)) + (1 - \chi_0(y^\varepsilon(t)))\nabla\rho(y^\varepsilon(t))]\,|\nabla\rho(y^\varepsilon(t))|^{-1},$

$$t \geq 0,$$

$b_\varepsilon(t)$

$\quad = [\chi(y^\varepsilon(t))M^{-1}(y^\varepsilon(t))\nabla\rho(y^\varepsilon(t)) + (1 - \chi_0(y^\varepsilon(t)))\nabla\rho(y^\varepsilon(t))]\,|\nabla\rho(y^\varepsilon(t))|^{-1},$

$$t \geq 0,$$

where χ is the same function of (3.10) and χ_0 is a smooth function such that

$$\chi_0(x) = \chi(x) \quad \text{in a neighborhood of } \mathbb{R}^d \setminus \mathbb{O}$$

$$\chi_0(x) = 1, \quad \forall x \in \mathbb{O}.$$

It is clear that

$$d\eta^\varepsilon(t) = a_\varepsilon(t)\, d\zeta^\varepsilon(t),$$

and since $(a \cdot b$ denoting the scalar product of vectors a and $b)$

$$a_\varepsilon(t)\cdot b_\varepsilon(t) = \chi^2(y^\varepsilon(t)) + [1 - \chi_0(y^\varepsilon(t))]^2$$
$$+ \chi(y^\varepsilon(t))[1 - \chi_0(y^\varepsilon(t))]\,\nabla\rho(y^\varepsilon(t))$$
$$\cdot[M(y^\varepsilon(t)) + M^{-1}(y^\varepsilon(t))]\nabla\rho(y^\varepsilon(t))\,|\nabla\rho(y(t))|^2$$

we have

(3.29) $|\chi_1(y^\varepsilon(t)) - a_\varepsilon(t)\cdot b_\varepsilon(t)| \leq C\rho^+(y^\varepsilon(t)), \quad \forall t \geq 0, \quad \forall \varepsilon > 0,$

with C an appropriate constant and χ_1 a smooth function equal to one in a neighborhood of $\mathbb{R}^d \setminus \mathbb{O}$, precisely

$$\chi_1 = \chi^2.$$

Also note that

(3.30) $\qquad |a_\varepsilon(t)| + |b_\varepsilon(t)| \le C, \quad \forall t \ge 0, \quad \forall \varepsilon > 0,$

for some constant C, and by means of (3.26) we get

(3.31) $\qquad E\{\sup_{0 \le t \le T} |b_\varepsilon(t) - b_{\varepsilon'}(t)|^p\} \le C(\varepsilon + \varepsilon')^q, \quad \forall \varepsilon, \varepsilon' > 0,$

where $2 < 2q < p$, $T > 0$ and a suitable constant C. Hence, the identity

$$d\zeta^\varepsilon(t) - d\zeta^{\varepsilon'}(t)$$
$$= [\chi_1(y^\varepsilon(t)) - a_\varepsilon(t) \cdot b_\varepsilon(t)]\, d\zeta^\varepsilon(t) + a_\varepsilon(t) \cdot [b_\varepsilon(t) - b_{\varepsilon'}(t)]\, d\zeta^\varepsilon(t)$$
$$+ [d\eta^\varepsilon(t) - d\eta^{\varepsilon'}(t)] \cdot b_{\varepsilon'}(t) + [a_{\varepsilon'}(t) \cdot b_{\varepsilon'}(t) - \chi_1(y^\varepsilon(t))]\, d\zeta^{\varepsilon'}(t)$$

together with (3.22), (3.24), (3.29), (3.30) and (3.31) prove that (3.28) holds provided we know that

(3.32) $\qquad \begin{cases} E\left\{\sup_{0 \le t \le T} \left| \int_0^t b_{\varepsilon'}(s) \cdot [d\eta^\varepsilon(t) - d\eta^{\varepsilon'}(t)] \right|^p \right\} \to 0 \\ \text{as} \quad \varepsilon, \varepsilon' \to 0. \end{cases}$

In order to establish (3.32), consider the process

$$b_{\varepsilon,\delta}(t) = h_\delta(y^\varepsilon(t)) \quad \delta > 0,$$

where $h_\delta(x)$ is a smooth function and satisfies

$$|b_\varepsilon(t) - b_{\varepsilon,\delta}(t)| \le C\delta, \quad \forall t \ge 0, \quad \forall \varepsilon, \delta > 0,$$

and some constant C. Thus, it follows

$$\left| \int_0^t b_{\varepsilon'}(s) \cdot [d\eta^\varepsilon(s) - d\eta^{\varepsilon'}(s)] \right|$$
$$\le C\delta(\zeta^\varepsilon(t) + \zeta^{\varepsilon'}(t)) + \left| \int_0^t b_{\varepsilon',\delta}(s) \cdot [d\eta^\varepsilon(s) - d\eta^{\varepsilon'}(s)] \right|.$$

But, by using Itô's formula to calculate the stochastic differential of $b_{\varepsilon',\delta}(s)$ and after an integration by parts, we can dominate the last term by the expression

$$C_\delta \sup_{0 \le t \le T} |\eta^\varepsilon(t) - \eta^{\varepsilon'}(t)| + |I(t)|,$$

where the stochastic integral $I(t)$ satisfies

$$E\{\sup_{0 \le t \le T} |I(t)|^p\} \le C_p E\{\sup_{0 \le t \le T} |\eta^\varepsilon(t) - \eta^{\varepsilon'}(t)|^p\},$$

for some appropriate constant C_p. Clearly, from this we deduce (3.32).

Next, it is easy to verify from the above properties that the processes

(3.33) $\qquad y(t) = \lim_{\varepsilon \to 0} y^\varepsilon(t), \quad t \ge 0,$

(3.34) $\qquad \zeta(t) = \lim_{\varepsilon \to 0} \zeta^\varepsilon(t), \quad t \ge 0,$

are well defined, solve the problem (2.3) and satisfy (3.14), (3.15).

In order to show the uniqueness of solutions, suppose we are given two solutions $y_i(t)$, $\zeta_i(t)$, $i = 1, 2$, of the problem (2.3). Then, applying Itô's formula to the function (3.25) with the processes

$$\xi_t = y_1(t) - y_2(t),$$

$$y_t = y_1(t), \quad y'_t = y_2(t),$$

$$\zeta_t = \zeta_1(t) + \zeta_1(t),$$

we deduce

$$E\{|y_1(t) - y_2(t)|^2\} = 0, \quad \forall t \ge 0,$$

after using the fact

(3.35) $(x' - x)\nabla\rho(x) \le K|x' - x|^2, \quad \forall x \in \partial\mathbb{O}, \forall x' \in \overline{\mathbb{O}}.$

Note that the inequality (3.35) follows from (3.17). □

REMARK 6.   Let us mention that Remarks 1, 2 and 4 extend to these general smooth domains. In particular (1.30) holds for a suitable constant α_p, different from (1.29). This Lipschitz continuous property can be deduced by using Itô's formula to a function similar to (3.25). Note that the clues are the inequalities (3.17), (3.19) and (3.35). Also, the constant α_p may be estimated by means of C_p in (1.2), K in (3.17) and C, λ in (3.19). □

REMARK 7.   It is clear that we may consider an evolution version of either (1.4) or (2.3). Also, it is possible to use stopping times and random variables as the initial data. All results extend to this situation with obvious changes. □

REMARK 8.   The techniques of this paper permit us to give a stochastic interpretation to several analytical results related to the Neumann problem of Hamilton-Jacobi-Bellman equation, variational inequality or quasivariational inequality for an integro-differential operator, as studied in Bensoussan and Lions [3]. □

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