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INVARIANCE PROPERTIES OF THE CONDITIONAL INDEPENDENCE RELATION

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The conditional independence relation for a triple of σ-algebras is investigated. For certain operations on this relation, necessary and sufficient conditions are derived such that these operations leave the relation invariant. Examples of such operations are the enlargement or reduction of the σ-algebras, and an absolute continuous change of measure. A projection operator for σ-algebras is defined and some of its properties are stated. The σ-algebraic realization problem is briefly discussed.

1. Introduction. The purpose of this paper is to present certain invariance properties of the conditional independence relation, properties of a projection operator for σ-algebras, and to discuss briefly the σ-algebraic realization problem.

The conditional independence relation for a triple of σ-algebras F_1, F_2, G of a probability space is defined by the condition that for any two positive random variables x_1, x_2 that are respectively F_1, F_2 measurable, one has

$$E[x_1x_2 \mid G] = E[x_1 \mid G]E[x_2 \mid G].$$

This relation plays a key role in a large number of areas of probability theory and stochastic processes. In the area of sufficient statistics the conditional independence relation enters in a natural way [1, 2, 8, 17]. The role of the relation in sufficient statistics has recently been stressed in [3, 4, 5, 12, 14]. In stochastic processes, the conditional independence relation appears in the theory of Markov processes, in particular in the concept of germ field [9, 13]. In stochastic system theory the relation is essential for the definition of a stochastic dynamic system and the stochastic realization problem [10, 18, 19, 20]. Other areas in which the conditional independence relation arises are information theory and random fields. In all these areas the relation enters in the question of how to reduce available information.

The main problem to be posed and solved in this paper is to give necessary and sufficient conditions for the invariance of the relation under certain operations. Examples of such operations are to make F_1 smaller or larger, G smaller or larger, and to perform absolute continuous changes of measures. A second problem to be investigated is to derive properties of a projection operator for σ-algebras. Finally the σ-algebraic realization problem will briefly be mentioned.

The invariance properties of the conditional independence relation have been discovered in an investigation of the σ-algebraic realization problem [19]. These

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properties seem sufficiently interesting to other areas of probability theory and stochastic processes to receive proper attention.

The motivation of the investigation of the conditional independence relation is the stochastic realization problem. In this problem one is given a stochastic process and asked to construct a stochastic system in a specified class such that the output of this system equals the given process. The practical motivation of this problem comes from communication and control, econometrics, time series analysis, and other areas where model building is important. The stochastic realization problem for Gaussian processes has been extensively investigated [10]. For non-Gaussian processes there are still many open problems. In a static context the strong version of the stochastic realization problem reduces to the σ-algebraic realization problem.

The σ-algebraic realization problem is given two σ-algebras F_1, F_2 to classify and to construct all σ-algebras G that make F_1, F_2 conditional independent and that are minimal in a to be specified sense. This problem is unsolved [19]. For the case where the σ-algebras F_1, F_2 are generated by Gaussian random variables a rather complete solution is available [18]. A generalization of the latter case to a Hilbert space framework has been investigated [10]. However, for the σ-algebraic case the analogy of σ-algebras with Hilbert spaces is not useful because the set of σ-algebras on a probability space is a lattice on which no orthogonal complement exists. The questions that the σ-algebraic realization problem poses are rather different in nature than those posed in the statistics literature. The σ-algebraic realization problem therefore requires new tools, and the structure of its solution is likely to be rather different from the Hilbert space case. The invariance properties of the conditional independence relation are basic techniques for the investigation of this problem.

A brief outline of the paper follows. In the next section the problem is formulated and elementary properties of the conditional independence relation are mentioned. The invariance properties are derived in Section 3, while in Section 4 several properties of a projection operator for σ-algebras are investigated. The σ-algebraic realization problem is briefly discussed in Section 5.

2. The problem formulation. In this section the conditional independence relation is defined and the invariance problem posed.

Throughout the paper $\{\Omega, F, P\}$ denotes a complete probability space consisting of a set Ω, a σ-algebra F, and a probability measure P. Let

$$\mathbf{F} = \left\{ \begin{matrix} G \subset F \mid G \text{ is a } \sigma\text{-algebra that contains} \\ \text{all the null sets of } F \end{matrix} \right\}.$$

If H, $G \in \mathbf{F}$, then $H \vee G$ is the smallest σ-algebra in $\mathbf{F}$ that contains H and G. For any set $A \subset \Omega$, I_A is the indicator function of A. For $G \in \mathbf{F}$ let

$$L^+(G) = \{x\colon \Omega \to R_+ \mid x \text{ is } G \text{ measurable}\}.$$

If $x\colon \Omega \to R^n$ is a random variable, then $F^x \in \mathbf{F}$ denotes the σ-algebra generated by x and the null sets of F. All equalities are supposed to hold almost surely, unless mentioned otherwise.

In the following the concept of a projection of one σ-algebra on another is needed. This definition is essentially due to H. P. McKean [13, page 343]; see also [14, page II.14; 19].

2.1 DEFINITION. For $H, G \in \mathbf{F}$ let the *projection of H on G* be the σ-algebra

$$\sigma(H \mid G) = \sigma(\{E[h \mid G] \mid \forall h \in L^+(H)\}) \in \mathbf{F}$$

with the understanding that all null sets of F are adjoined to it. The operator $\sigma(\cdot \mid \cdot)$: $\mathbf{F} \times \mathbf{F} \to \mathbf{F}$ will be called the *projection operator for σ-algebras.*

Recall that $F_1, F_2 \in \mathbf{F}$ are *independent σ-algebras* if for any $A_1 \in F_1$, and $A_2 \in F_2$

$$P(A_1 \cap A_2) = P(A_1)P(A_2);$$

equivalently, if for any $x_1 \in L^+(F_1)$, $x_2 \in L^+(F_2)$

$$E[x_1 x_2] = E[x_1]E[x_2].$$

[15, IV.4]. The notation $(F_1, F_2) \in I$ will be used to indicate that F_1, F_2 are independent σ-algebras, and $I \subset (\mathbf{F} \times \mathbf{F})$ will be called the *independence relation.*

2.2 DEFINITION. The *conditional independence relation* CI is a relation for a triple of σ-algebras $F_1, F_2, G \in \mathbf{F}$ defined by the condition that for all $x_1 \in L^+(F_1)$, $x_2 \in L^+(F_2)$

$$E[x_1 x_2 \mid G] = E[x_1 \mid G]E[x_2 \mid G] \quad \text{a.s.}$$

Then one calls F_1, F_2 *conditional independent* given G, or one says that G *splits* F_1, F_2. Notation: $(F_1, G, F_2) \in \mathrm{CI}$.

Note the analogy between the conditional independence relation and the independence relation.

In this paper attention will be concentrated on the following problem.

2.3 PROBLEM. *The invariance problem for the conditional independence relation is, given certain operations to determine necessary and sufficient conditions, such that these operations leave the relation invariant.*

Below the above defined problem will be solved for several operations.

In the following some elementary properties of the conditional independence relation are derived that will be used in the sequel.

2.4 PROPOSITION. *Let $F_1, F_2, G \in \mathbf{F}$. The following statements are equivalent:*

a. $(F_1, G, F_2) \in \mathrm{CI}$;

b. $(F_2, G, F_1) \in \mathrm{CI}$;

c. *for all* $x_1 \in L^+(F_1)$

$$E[x_1 \mid F_2 \vee G] = E[x_1 \mid G];$$

d. *for all* $x_1 \in L^+(F_1)$, $E[x_1 \mid F_2 \vee G]$ *is G-measurable*;

e. $\sigma(F_1 \mid F_2 \vee G) \subset G$;
f. $(F_1 \vee G, G, G \vee F_2) \in \mathrm{CI}$;
g. *for all* $z \in L^+(F_1 \vee G)$

$$E[E[z \mid G] \mid F_2] = E[z \mid F_2].$$

Condition 2.4.g. is due to Mouchart and Rolin [14, Theorem 2.1], and to Döhler [7, Lemma 4]. Below the proof is given for the sake of completeness.

PROOF. a $\Leftrightarrow$ b. This follows from the symmetry in F_1, F_2 of Definition 2.2.
a $\Leftrightarrow$ c. This is known, see [6, II.45].
c $\Rightarrow$ d. This is obvious.
d $\Rightarrow$ e. This follows from the definition of $\sigma(F_1 \mid F_2 \vee G)$.
e $\Rightarrow$ c. Let $x_1 \in L^+(F_1)$. Then

$$E[x_1 \mid G] = E[E[x_1 \mid F_2 \vee G] \mid G] = E[x_1 \mid F_2 \vee G]$$

by e.
c $\Rightarrow$ f. Let $x_1 \in L^+(F_1)$. Then c implies that

$$E[x_1 \mid (F_2 \vee G) \vee G] = E[x_1 \mid F_2 \vee G] = E[x_1 \mid G],$$

hence $(F_1, G, G \vee F_2) \in \mathrm{CI}$. Using the equivalence of a and the above one obtains $(F_2 \vee G, G, F_1) \in \mathrm{CI}$ and with the above $(F_2 \vee G, G, G \vee F_1) \in \mathrm{CI}$, and thus the result.
f $\Rightarrow$ g. From f follows by restriction that $(F_1 \vee G, G, F_2) \in \mathrm{CI}$. Let $z \in L^+(F_1 \vee G)$. Then

$$\begin{aligned} E[E[z \mid G] \mid F_2] &= E[E[z \mid F_2 \vee G] \mid F_2] \quad \text{by} \quad (F_1 \vee G, G, F_2) \in \mathrm{CI}, \\ &= E[z \mid F_2]. \end{aligned}$$

g $\Rightarrow$ a. Let $x_1 \in L^+(F_1)$, $x_2 \in L^+(F_2)$, $g \in L^+(G)$. Then

$$\begin{aligned} E[x_1 x_2 g] &= E[x_2 E[x_1 g \mid F_2]] = E[x_2 E[E[x_1 g \mid G] \mid F_2]] \quad \text{by} \quad g, \\ &= E[x_2 E[x_1 g \mid G]] = E[x_2 g E[x_1 \mid G]] \\ &= E[g E[x_1 \mid G] E[x_2 \mid G]] \end{aligned}$$

and the result follows from the definition of conditional expectation. □

There follow two sufficient conditions for a triple of σ-algebras to be conditional independent.

2.5 PROPOSITION. *Given* $F_1, F_2, G \in \mathbf{F}$.
a. *If* $F_1 \subset G$ *or* $F_2 \subset G$, *then* $(F_1, G, F_2) \in \mathrm{CI}$. *In particular* $(F_1, F_1, F_2) \in \mathrm{CI}$ *and* $(F_1, F_2, F_2) \in \mathrm{CI}$.
b. *If* $(F_1, F_2 \vee G) \in I$ *then* $(F_1, G, F_2) \in \mathrm{CI}$.

PROOF. a. This follows directly from the Definition 2.2.
b. Again for $x_1 \in L^+(F_1)$

$$E[x_1 \mid F_2 \vee G] = E[x_1] = E[x_1 \mid G]$$

by independence and [15, IV.4.2]. □

Several other elementary properties of the conditional independence relation follow.

2.6 PROPOSITION. *Let F_1, F_2, $G \in \mathbf{F}$ with $G = \{\emptyset, \Omega\}$ up to null sets of F. Then $(F_1, F_2) \in I$ iff $(F_1, G, F_2) \in \mathrm{CI}$.*

PROOF. The elementary proof is omitted. □

2.7 PROPOSITION. *Let F_1, F_2, $G \in \mathbf{F}$.*
a. *If $(F_1, G, F_2) \in \mathrm{CI}$ then $(F_1 \cap F_2) \subset G$.*
b. *Assume that $F_2 \subset F_1$. Then $(F_1, G, F_2) \in \mathrm{CI}$ iff $F_2 \subset G$. In particular, $(F_1, G, F_1) \in \mathrm{CI}$, iff $F_1 \subset G$.*

PROOF. a. Let $A \in (F_1 \cap F_2)$. Then

$$E[I_A \mid G] = E[I_A \mid F_2 \vee G] \quad \text{by} \quad A \in (F_1 \cap F_2) \subset F_1 \quad \text{and} \quad (F_1, G, F_2) \in \mathrm{CI},$$

$$= I_A \quad \text{by} \quad A \in (F_1 \cap F_2) \subset F_2.$$

Hence A is G measurable.
b. This is a direct consequence of a. □

3. The invariance problem.

In this section results for the invariance problem are derived. Some of these results have been stated without proof in [19].

The investigation of the invariance problem for the conditional independence relation as defined in 2.3 is initiated with the invariance with respect to F_2 in $(F_1, G, F_2) \in \mathrm{CI}$. Due to the symmetry of the conditional independence relation with respect to F_1 and F_2, the invariance of the relation with respect to F_1 follows.

3.1 THEOREM. *Let F_1, F_2, F_3, $G \in \mathbf{F}$ with $F_2 \subset F_3$. One has $(F_1, G, F_3) \in \mathrm{CI}$ iff $(F_1, G, F_2) \in \mathrm{CI}$ and $\sigma(F_1 \mid F_3 \vee G) \subset (F_2 \vee G)$.*

PROOF. $\Rightarrow$ $(F_1, G, F_3) \in \mathrm{CI}$ implies by restriction that $(F_1, G, F_2) \in \mathrm{CI}$, and by 2.4. e

$$\sigma(F_1 \mid F_3 \vee G) \subset G \subset (F_2 \vee G).$$

$\Leftarrow$ Let $x_1 \in L^+(F_1)$. Then

$$\begin{aligned} &E[x_1 \mid F_3 \vee G] \\ &\quad = E[E[x_1 \mid F_3 \vee G] \mid F_2 \vee G] \quad \text{by} \quad \sigma(F_1 \mid F_3 \vee G) \subset (F_2 \vee G). \\ &\quad = E[x_1 \mid F_2 \vee G] \quad \text{by} \quad F_2 \subset F_3, \\ &\quad = E[x_1 \mid G], \end{aligned}$$

and the result follows from 2.4.c. □

Some consequences of 3.1 and related results are stated next.

3.2 PROPOSITION. *Let* $F_1, F_2, F_3, G \in \mathbf{F}$.

a. *One has that* $(F_1, G, F_2 \vee F_3) \in \mathrm{CI}$ *iff* $(F_1, G, F_2) \in \mathrm{CI}$ *and* $(F_1, G \vee F_2, F_3) \in \mathrm{CI}$.
b. $(F_1, G, F_2) \in \mathrm{CI}$ *and* $(F_1, G \vee F_2, F_3) \in \mathrm{CI}$ *iff* $(F_1, G, F_3) \in \mathrm{CI}$ *and* $(F_1, G \vee F_3, F_2) \in \mathrm{CI}$.
c. *Assume that* $G \subset F_2$. *Then* $(F_1, G, F_2) \in \mathrm{CI}$ *iff* $\sigma(F_1 \mid F_2) \subset G$.
d. *Assume that* $(F_1 \vee F_2 \vee G, F_3) \in I$. *Then* $(F_1, G, F_2 \vee F_3) \in \mathrm{CI}$ *iff* $(F_1, G, F_2) \in \mathrm{CI}$.
e. $(F_1 \vee F_3, G, F_2) \in \mathrm{CI}$ *and* $(F_1, G, F_3) \in \mathrm{CI}$ *iff* $(F_1, G, F_3 \vee F_2) \in \mathrm{CI}$ *and* $(F_3, G, F_2) \in \mathrm{CI}$.
f. *Assume that* $F_3 \subset (F_2 \vee G)$. *If* $(F_1, G, F_2) \in \mathrm{CI}$ *then* $(F_1, G, F_3) \in \mathrm{CI}$.

PROOF. a. By 2.4 $\sigma(F_1 \mid F_2 \vee F_3 \vee G) \subset (F_2 \vee G)$, and $(F_1, F_2 \vee G, F_3) \in \mathrm{CI}$ are equivalent. The result then follows from 3.1.

b. By a. both sides are equivalent with $(F_1, G, F_2 \vee F_3) \in \mathrm{CI}$.
c. By 2.4 $(F_1, G, F_2) \in \mathrm{CI}$ iff $\sigma(F_1 \mid F_2 \vee G) \subset G$. From $G \subset F_2$ then follows that $\sigma(F_1 \mid F_2) = \sigma(F_1 \mid F_2 \vee G) \subset G$.
d. $\Rightarrow$ This follows by restriction. $\Leftarrow$ $(F_1 \vee F_2 \vee G, F_3) \in I$ and 2.5.b imply that $(F_1, G \vee F_2, F_3) \in \mathrm{CI}$. The conclusion then follows from a.
e. $(F_1 \vee F_3, G, F_2) \in \mathrm{CI} \Leftrightarrow \{(F_3, G, F_2) \in \mathrm{CI}$ and $(F_1, G \vee F_3, F_2) \in \mathrm{CI}\}$, while $\{(F_1, G \vee F_3, F_2) \in \mathrm{CI}$ and $(F_1, G, F_3) \in \mathrm{CI}\} \Leftrightarrow (F_1, G, F_3 \vee F_2) \in \mathrm{CI}$, by applying a twice.
f. This follows directly from 2.4.d. □

Result 3.2.a is also derived in [5; 7, Lemma 3, page 629; 14, Theorem 2.5]. Special cases of 3.2.f are given by [7; 9, 1.b; 14, Corollary 2.6].

3.3 THEOREM. *Let* $F_1, F_2, G_1, G_2 \in \mathbf{F}$ *with* $G_2 \subset G_1$. *One has* $(F_1, G_1, F_2) \in \mathrm{CI}$ *and* $\sigma(F_1 \mid G_1) \subset G_2$ *iff* $(F_1, G_2, F_2) \in \mathrm{CI}$ *and* $\sigma(F_1 \mid F_2 \vee G_1) \subset (F_2 \vee G_2)$.

PROOF. $\sigma(F_1 \mid G_1) = \sigma(F_1 \mid G_1 \vee G_2)$ by $G_2 \subset G_1$ and by 3.2.c $\sigma(F_1 \mid G_1) = \sigma(F_1 \mid G_1 \vee G_2) \subset G_2$ iff $(F_1, G_2, G_1) \in \mathrm{CI}$. Similarly $\sigma(F_1 \mid F_2 \vee G_1) \subset (F_2 \vee G_2)$ iff $(F_1, F_2 \vee G_2, F_2 \vee G_1) \in \mathrm{CI}$. By 3.2.a both sides of the theorem are equivalent with $(F_1, G_2, G_1 \vee F_2) \in \mathrm{CI}$. □

Next some consequences of 3.3 and related results are stated.

3.4. PROPOSITION. *Let* $F_1, G_2, F_3, G_1, G_2, \in \mathbf{F}$.

a. *One has* $(F_1, G_1, F_2) \in \mathrm{CI}$ *and* $(F_1, G_1 \vee F_2, F_3) \in \mathrm{CI}$ *iff* $(F_1, G_1 \vee G_2, F_2) \in \mathrm{CI}$ *and* $(F_1, G_1, G_2) \in \mathrm{CI}$.
b. *One has* $(F_1, G_1, F_2) \in \mathrm{CI}$, $(F_1, G_1 \vee F_2, G_2) \in \mathrm{CI}$, *and* $(F_1, G_2, G_1) \in \mathrm{CI}$ *iff* $(F_1, G_2, F_2) \in \mathrm{CI}$, $(F_1, G_2 \vee F_2, G_1) \in \mathrm{CI}$, *and* $(F_1, G_1, G_2) \in \mathrm{CI}$.
c. *If* $(F_1, G_1, F_2) \in \mathrm{CI}$ *and* $F_1 \subset F_3$, *then* $(F_1, \sigma(F_3 \mid G_1), F_2) \in \mathrm{CI}$.
d. $(F_1, \sigma(F_1 \mid F_2), F_2) \in \mathrm{CI}$ *and* $(F_1, \sigma(F_2 \mid F_1), F_2) \in \mathrm{CI}$.
e. *If* $(F_1, G_1, F_2) \in \mathrm{CI}$ *and* $\sigma(F_1 \mid G_1) = G_2 \subset (F_2 \vee G_1)$ *then* $(F_1, G_2, F_2) \in \mathrm{CI}$.
f. *If* $(F_1, G_1, F_2) \in \mathrm{CI}$ *then* $(F_1, \sigma(G_1 \mid F_1), F_2) \in \mathrm{CI}$. *Hence* $\sigma(F_2 \mid F_1) \subset \sigma(G_1 \mid F_1)$.
g. *Assume that* $(F_1 \vee F_2 \vee G_1, G_2) \in I$. *Then* $(F_1, G_1 \vee G_2, F_2) \in \mathrm{CI}$ *iff* $(F_1, G_1, F_2) \in \mathrm{CI}$.
h. $(F_1, \sigma(F_2 \mid F_1) \vee \sigma(F_1 \mid F_2), F_2) \in \mathrm{CI}$.

PROOF. a. By 3.2.a both sides are equivalent with $(F_1, G_1, G_2 \vee F_2) \in \mathrm{CI}$.
b. By applying a twice one obtains

$$\left.\begin{array}{l}(F_1, G_1, F_2) \in \mathrm{CI} \\ (F_1, G_1 \vee F_2, G_2) \in \mathrm{CI}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}(F_1, G_1, G_2) \in \mathrm{CI} \\ \left.\begin{array}{l}(F_1, G_1 \vee G_2, F_2) \in \mathrm{CI} \\ (F_1, G_2, G_1) \in \mathrm{CI}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}(F_1, G_2, F_2) \in \mathrm{CI}, \\ (F_1, G_2 \vee F_2, G_1) \in \mathrm{CI}.\end{array}\right.\end{array}\right.$$

c. $F_1 \subset F_3$ implies that $\sigma(F_1 \mid G) \subset \sigma(F_3 \mid G) \subset G$. The result then follows from 3.3.
d. *For* $x_1 \in L^+(F_1)$ is $E[x_1 \mid F_2 \vee \sigma(F_1 \mid F_2)] = E[x_1 \mid F_2]$, which is $\sigma(F_1 \mid F_2)$ measurable, and the result follows from 2.4.d. By symmetry $(F_1, \sigma(F_2 \mid F_1), F_2) \in \mathrm{CI}$.
e. $F_2 \subset (F_2 \vee G_1)$ and 2.5.a imply that $(F_1, G_1 \vee F_2, F_2) \in \mathrm{CI}$. Furthermore, by 2.4.c $\sigma(F_1 \mid F_2 \vee G_1) = \sigma(F_1 \mid G_1) \subset G_2 \subset (F_2 \vee G_1)$. Now apply 3.3 with G_1 replaced by $F_2 \vee G_1$ to obtain $(F_1, G_2, F_2) \in \mathrm{CI}$.
f. Take in e $G_2 = \sigma(G_1 \mid F_1) \vee \sigma(F_2 \mid G_1)$. Then $(F_1, \sigma(G_1 \mid F_1) \vee \sigma(F_2 \mid G_1), F_2) \in \mathrm{CI}$. By 3.4.d $(F_1, \sigma(G_1 \mid F_1), G_1) \in \mathrm{CI}$, hence $(F_1, \sigma(G_1 \mid F_1), \sigma(F_2 \mid G_1)) \in \mathrm{CI}$. Combining these results with 3.2.a yields $(F_1, \sigma(G_1 \mid F_1), F_2 \vee \sigma(F_2 \mid G_1)) \in \mathrm{CI}$, hence $(F_1, \sigma(G_1 \mid F_1), F_2) \in \mathrm{CI}$. This and 3.2.c give $\sigma(F_2 \mid F_1) \subset \sigma(G_1 \mid F_1)$.
g. $(F_1 \vee F_2 \vee G_1, G_2) \in I$ and 2.5.b imply that $(F_1, G_1 \vee F_2, G_2) \in \mathrm{CI}$ and $(F_1, G_1, G_2) \in \mathrm{CI}$. The result then follows from a.
h. By d $(F_1, \sigma(F_2 \mid F_1), F_2) \in \mathrm{CI}$. Then $\sigma(F_1 \mid F_2 \vee \sigma(F_2 \mid F_1) \vee \sigma(F_1 \mid F_2)) = \sigma(F_1 \mid F_2 \vee \sigma(F_2 \mid F_1)) \subset \sigma(F_2 \mid F_1) \subset (\sigma(F_2 \mid F_1) \vee \sigma(F_1 \mid F_2))$, and the result follows from 2.4. □

3.5. PROPOSITION. *Let* $F_1, F_2, F_3, F_4, G_1, G_2 \in \mathbf{F}$. *Assume that* $(F_1 \vee F_2 \vee G_1, F_3 \vee F_4 \vee G_2) \in I$. *Then* $(F_1 \vee F_3, G_1 \vee G_2, F_2 \vee F_4) \in \mathrm{CI}$ *iff* $(F_1, G_1, F_2) \in \mathrm{CI}$ *and* $(F_3, G_2, F_4) \in \mathrm{CI}$.

PROOF. $\Rightarrow$ By restriction $(F_1, G_1 \vee G_2, F_2) \in \mathrm{CI}$, and by 3.4.g $(F_1, G_1, F_2) \in \mathrm{CI}$. By symmetry one obtains $(F_3, G_2, F_4) \in \mathrm{CI}$.

$\Leftarrow$. By 3.2.d $(F_1, G_1, G_2 \vee F_3 \vee F_2 \vee F_4) \in \mathrm{CI}$, and by 3.2.a $(F_1, G_1 \vee G_2, F_3 \vee F_2 \vee F_4) \in \mathrm{CI}$. Similarly one proves $(F_3, G_1 \vee G_2, F_1 \vee F_2 \vee F_4) \in \mathrm{CI}$, hence $(F_3, G_1 \vee G_2, F_2 \vee F_4) \in \mathrm{CI}$. The result then follows from 3.2.e. □

Next the invariance of the conditional independence relation with respect to a measure transformation is investigated. In the following there are two probability measures on $\{\Omega, F\}$, denoted by P_0, P_1. Expectation with respect to these measures is denoted by $E_0(\cdot)$, respectively $E_1(\cdot)$. If P_0, P_1 are equivalent probability measures on $\{\Omega, F_1 \vee F_2 \vee G\}$, then by the Radon-Nikodym theorem there exists a $F_1 \vee F_2 \vee G$ measurable random variable $\rho: \Omega \to R_+$ with $E_0[\rho] = 1$, such that for all $A \in F_1 \vee F_2 \vee G$

$$E_1[I_A] = E_0[I_A \rho].$$

The reader is reminded of the formula

$$E_1[X \mid G] = E_0[x\rho \mid G]/E_0[\rho \mid G]$$

valid for any random variable $x: \Omega \to R$ such that $E_0 \mid x\rho \mid < \infty$[11, 24.4]. The conditional independence relation with respect to the probability measure P_0, P_1 is denoted by $\mathrm{CI}(P_0)$ respectively $\mathrm{CI}(P_1)$.

3.6. THEOREM. *Let F_1, F_2, $G \in \mathbf{F}$, and P_0, P_1 be two equivalent probability measures on $\{\Omega, F\}$. Assume that $\{\Omega, F, P_0\}$ and $\{\Omega, F, P_1\}$ are both complete. Let $\rho: \Omega \to R_+$ be the Radon-Nikodym derivative dP_1/dP_0 with respect to $F_1 \vee F_2 \vee G$. Assume further that $(F_1, G, F_2) \in \mathrm{CI}(P_0)$. Then $(F_1, G, F_2) \in \mathrm{CI}(P_1)$ iff there exist $\rho_1 \in L^+(F_1 \vee G)$, $\rho_2 \in L^+(F_2 \vee G)$ such that $\rho = \rho_1 \cdot \rho_2$ a.s. The decomposition $\rho = \rho_1 \cdot \rho_2$ is nonunique in general.*

The result of 3.6 is related to one of the equivalent definitions of a sufficient statistic. The definition is that the statistic z is sufficient for the estimation of x given y if for the joint density p_1 of x and y there exist positive functions p_2 and p_3 such that

$$p_1(x, y) = p_2(x, z)p_3(y).$$

[Bahadur, 1; Rao, 16, page 131].

PROOF. $\Leftarrow$. By 2.4 $(F_1 \vee G, G, G \vee F_2) \in \mathrm{CI}(P_0)$. Let $x_1 \in L^+(F_1)$. Then

$$\begin{aligned} E_1[x_1 \mid F_2 \vee G] &= E_0[x_1\rho_1\rho_2 \mid F_2 \vee G]/E_0[\rho_1\rho_2 \mid F_2 \vee G] \\ &= \rho_2 E_0[x_1\rho_1 \mid F_2 \vee G]/\rho_2 E_0[\rho_1 \mid F_2 \vee G] \\ &= E_0[x_1\rho_1 \mid G]/E_0[\rho_1 \mid G] \end{aligned}$$

because $P_1(\{\rho_2 = 0\}) \leq P_1(\{\rho = 0\}) = 0$, and by $(F_1 \vee G, G, G \vee F_2) \in \mathrm{CI}(P_0)$, hence $E_1[x_1 \mid F_2 \vee G]$ is G measurable and the result follows from 2.4.d.

$\Rightarrow$ Define

$$\rho_1 = E_0[\rho \mid F_1 \vee G],$$
$$\rho_2 = E_0[\rho \mid F_2 \vee G]/E_0[\rho \mid G].$$

Let $A_1 \in (F_1 \vee G)$, $A_2 \in (F \vee G)$. Then one has

$$\begin{aligned}
&E_0[I_{A_1} I_{A_2} \rho_1 \rho_2] \\
&= E_0[E_0[I_{A_1} \rho_1 I_{A_2} \rho_2 \mid G]] \\
&= E_0[E_0[I_{A_1} \rho_1 \mid G] E_0[I_{A_2} \rho_2 \mid G]] \quad \text{by} \quad (F_1 \vee G, G, G \vee F_2) \in \mathrm{CI}(P_0), \\
&= E_0[E_0[I_{A_1} \rho \mid G] E_0[I_{A_2} \rho \mid G]/E_0[\rho \mid G]] \quad \text{by the definition of} \quad \rho_1, \rho_2, \\
&= E_0[E_1[I_{A_1} \mid G] E_1[I_{A_2} \mid G] E_0[\rho \mid G]] \\
&= E_0[E_1[I_{A_1} I_{A_2} \mid G] E_0[\rho \mid G]] \quad \text{by} \quad (F_1, G, F_2) \in \mathrm{CI}(P_1) \\
&= E_0[E_1[I_{A_1} I_{A_2} \mid G] \rho] = E_1[I_{A_1} I_{A_2}].
\end{aligned}$$

An application of the monotone class theorem then yields that for all $A \in (F_1 \vee G) \vee (G \vee F_2) = F_1 \vee F_2 \vee G$

$$E_0[I_A \rho_1 \rho_2] = E_1[I_A],$$

hence $\rho_1 \rho_2$ is a version of ρ, or $\rho = \rho_1 \cdot \rho_2$ a.s. □

4. The projection operator. In this section some results for the projection operator are derived. These results have been used in [18, 19].

4.1. PROPOSITION. *Let* $F_1, F_2, F_3, G \in \mathbf{F}$.
a. *If* $F_1 \subset F_2$ *then* $\sigma(F_1 \mid F_2) = F_1$.
b. *If* $F_1 \supset F_2$ *then* $\sigma(F_1 \mid F_2) = F_2$.
c. *If* $(F_1, G, F_2) \in \mathrm{CI}$ *then* $\sigma(F_1 \mid F_2 \vee G) = \sigma(F_1 \mid G)$.
d. $\sigma(F_1 \mid \sigma(F_1 \mid F_2)) = \sigma(F_1 \mid F_2)$.
e. $\sigma(F_1 \mid \sigma(F_2 \mid F_1) \vee \sigma(F_1 \mid F_2)) = \sigma(F_2 \mid F_1)$.
f. *If* $(F_1, G, F_2) \in \mathrm{CI}$ *then* $\sigma(\sigma(G \mid F_1) \mid F_2) = \sigma(F_1 \mid F_2)$.
g. $\sigma(\sigma(F_2 \mid F_1) \mid F_2) = \sigma(F_1 \mid F_2)$.
h. $\sigma(\sigma(F_1 \mid F_2) \mid \sigma(F_2 \mid F_1)) = \sigma(F_2 \mid F_1)$.
i. *If* $F_1 \subset F_3$ *then* $F_1 \vee \sigma(F_2 \mid F_3) = \sigma(F_1 \vee F_2 \mid F_3)$.
j. $\sigma(\sigma(F_2 \mid F_1) \vee \sigma(F_1 \mid F_2) \mid F_1) = \sigma(F_2 \mid F_1)$.

It follows from 4.1.a that for any $F_1, F_2 \in \mathbf{F}$ $\sigma(\sigma(F_1 \mid F_2) \mid F_2) = \sigma(F_1 \mid F_2)$. Thus for any $F_2 \in F$, is $\sigma(\cdot \mid F_2)$ the projection operator onto F_2. The results 4.1.d, g, h, i have also been derived in [14, Corollary 4.9, Theorem 4.10], but are mentioned here for the sake of completeness.

PROOF. Let $F_{12} = \sigma(F_1 \mid F_2)$ and $F_{21} = \sigma(F_2 \mid F_1)$.
a.b. This is obvious from the definition of the projection of F_1 on F_2.

c. For $x_1 \in L^+(F_1)$, $(F_1, G, F_2) \in \mathrm{CI}$ implies that

$$E[x_1 \mid F_2 \vee G] = E[x_1 \mid G].$$

The result then follows from consideration of the generators of the two σ-algebras.

d. By 3.4.d $(F_1, F_{12}, F_2) \in \mathrm{CI}$, and the result follows from *c*.

e. Again $(F_1, F_{21}, F_2) \in \mathrm{CI}$, and by restriction $(F_1, F_{21}, F_{12}) \in \mathrm{CI}$. Then

$$\sigma(F_1 \mid F_{21} \vee F_{12}) = \sigma(F_1 \mid F_{21}) \quad \text{by} \quad (F_1, F_{21}, F_{12}) \in \mathrm{CI} \text{ and } c,$$

$$= F_{21} \quad \text{by} \quad F_{21} \subset F_1 \quad \text{and b.}$$

f. $\sigma(G \mid F_1) \subset F_1$ implies by a that $\sigma(\sigma(G \mid F_1) \mid F_2) \subset F_{12}$. $(F_1, G, F_2) \in \mathrm{CI}$ and 3.4.f imply $(F_1, \sigma(G \mid F_1), F_2) \in \mathrm{CI}$. Again by 3.4.f $(F_1, \sigma(\sigma(G \mid F_1) \mid F_2), F_2) \in \mathrm{CI}$. From this and 3.2.c follows that $F_{12} \subset \sigma(\sigma(G \mid F_1) \mid F_2)$.

g. By 2.5.a $(F_1, F_2, F_2) \in \mathrm{CI}$, and the result follows from f.

h. $F_{21} = \sigma(F_{12} \mid F_1)$ by g.

$$= \sigma(F_{12} \mid F_1 \vee F_{21}) = \sigma(F_{12} \mid F_{21}) \quad \text{by} \quad (F_1, F_{21}, F_{12}) \in \mathrm{CI}.$$

i. By assumption $F_1 \subset \sigma(F_1 \vee F_2 \mid F_3)$, and also $\sigma(F_2 \mid F_3) \subset \sigma(F_1 \vee F_2 \mid F_3)$, hence $F_1 \vee \sigma(F_2 \mid F_3) \subset \sigma(F_1 \vee F_2 \mid F_3)$. Let $x_1 \in L^+(F_1)$, $x_2 \in L^+(F_2)$. Then

$$E[x_1 x_2 \mid F_3] = x_1 E[x_2 \mid F_3]$$

is $F_1 \vee \sigma(F_2 \mid F_3)$ measurable. A monotone class argument shows that for all $y \in L^+(F_1 \vee F_2) E[y \mid F_3]$ is $F_1 \vee \sigma(F_2 \mid F_3)$ measurable, hence $\sigma(F_1 \vee F_2 \mid F_3) \subset F_1 \vee \sigma(F_2 \mid F_3)$.

j. By *i* $\sigma(F_{21} \vee F_{12} \mid F_1) = F_{21} \vee \sigma(F_{12} \mid F_1) = F_{21}$ by *g*. □

5. The σ-algebraic realization problem. A problem formulation and a brief discussion of the σ-algebraic realization problem follow.

5.1. DEFINITION. The *minimal conditional independence relation* $\mathrm{CI}_{\min}$ for a triple of σ-algebras $F_1, F_2, G \in \mathbf{F}$ is defined by the conditions

1. $(F_1, G, F_2) \in \mathrm{CI}$;
2. if $H \in \mathbf{F}$, $H \subset G$, and $(F_1, H, F_2) \in \mathrm{CI}$, then $H = G$.

Notation: $(F_1, G, F_2) \in \mathrm{CI}_{\min}$. Then one says that F_1, F_2 are *minimal conditional independent* given G, or that G *splits* F_1, F_2 *minimally.*

5.2. *Problem.* The *σ-algebraic realization problem* is given $\{\Omega, F, P\}$ and $F_1, F_2 \in \mathbf{F}$ to solve the following subproblems.

a. To show existence of a $G \in \mathbf{F}$ such that $(F_1, G, F_2) \in \mathrm{CI}_{\min}$.
b. To classify all $G \in \mathbf{F}$ such that $(F_1, G, F_2) \in \mathrm{CI}_{\min}$ and $G \subset (F_1 \vee F_2)$; and to provide an algorithm that constructs all those σ-algebras G.

The existence subproblem of 5.2 is trivial. It is known that $(F_1, \sigma(F_1 \mid F_2), F_2) \in \mathrm{CI}_{\min}$ and that $(F_1, \sigma(F_2 \mid F_1), F_2) \in \mathrm{CI}_{\min}$ [McKean, 13, page 343, property e; Mouchart, Rolin, 14, Theorem 4.3]. Moreover, if $G \subset F_1$, then $(F_1, G, F_2) \in \mathrm{CI}_{\min}$ iff $G = \sigma(F_2 \mid F_1)$.

There remains thus the classification subproblem of 5.2. In this subproblem one can distinguish three major questions:

1. what are necessary and sufficient conditions for a σ-algebra G such that $(F_1, G, F_2) \in \mathrm{CI}_{\min}$?;
2. what is the classification of such σ-algebras G;
3. how to construct an algorithm that produces all such G's?

As to the first question, assume that $(F_1, G, F_2) \in \mathrm{CI}$. A necessary condition for F_1, F_2 to be minimal conditional independent given G is that

$$\sigma(F_1 \mid G) = G = \sigma(F_2 \mid G).$$

This follows directly from 3.4.c. However this condition is not sufficient, see [19, Example 4.4]. This question is still open.

The questions of classification and algorithm construction have not been solved. A step in the construction of minimal G's is given by 3.4.c, if $(F_1, G, F_2) \in \mathrm{CI}$ then $(F_1, \sigma(F_1 \mid G), F_2) \in \mathrm{CI}$. Based on the analogy with the Hilbert space framework a partial result is given by [19, Theorem 4.11].

The structure of all σ-algebras G such that $(F_1, G, F_2) \in \mathrm{CI}_{\min}$ is rather puzzling. For $G = \sigma(F_1 \mid F_2)$ or $G = \sigma(F_2 \mid F_1)$ one has $(F_1, G, F_2) \in \mathrm{CI}_{\min}$. Under a condition $(F_1, G, F_2) \in \mathrm{CI}_{\min}$ and $G \subset (F_1 \vee F_2)$ imply that $G \subset (\sigma(F_2 \mid F_1) \vee \sigma(F_1 \mid F_2))$. However this is not true in general. Also $\sigma(\sigma(F_2 \mid F_1) \mid \sigma(F_1 \mid F_2)) = \sigma(F_1 \mid F_2)$ by 4.1.h, but his property does not hold for all minimal G's. Additional information and results are given in [18, 19].

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