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MULTIVARIATE DISTRIBUTIONS WITH UNIFORMLY DISTRIBUTED PROJECTIONS

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Is there a distribution in n space that has the same uniform distribution for its projection in all directions?

Section 1. There are numerous bivariate distributions with uniform marginal distributions (see e.g., [2]), but are there any for which all one-dimensional projections, not only in the directions of the coordinate axes, are uniform? Or, more generally, are there any probability measures on Euclidean n space, such that the projection on every one-dimensional subspace induces the uniform distribution on, say, $(-1, 1)$?

We show here that there is one such distribution in each of the dimensions 1, 2, and 3, and none for all higher dimensions.

Section 2. Clearly, any such distribution on n space induces one with the same property on every subspace. For $n = 3$, a solution is provided by P_3, the uniform distribution on the surface of the unit sphere. Archimedes' famous theorem [1] states that a sphere and a circumscribing cylinder are sliced into bands of the same area by any planes perpendicular to the axis of the cylinder. This translates easily into the required property of the uniform distribution on the surface of the unit sphere.

In the plane, a solution is obtained by projecting P_3 on an equatorial plane. A surface element of the unit sphere at distance r from the axis, forms an angle with cosine $(1 - r^2)^{1/2}$ with that plane; the density of P_3 on the surface is $(4\pi)^{-1}$, and, the projection is two-to-one. Hence, we obtain P_2 with density $(2\pi)^{-1}(1 - x^2 - y^2)^{-1/2}$ on the unit disc of the (x, y) plane. This distribution is characterized by its circular symmetry and the Beta$(1, \frac{1}{2})$ distribution it bestows on r^2.

Section 3. The uniqueness of P_3 and of P_2 follows from the fact that the distributions of all one-dimensional projections determine the n-dimensional characteristic functions, and hence, the distribution.

Section 4. In 4 space, assume there is a P_4 with the required property. We shall derive a contradiction. If (X, Y, Z, W) is distributed according to P_4, then, by the uniqueness of P_3, (X, Y, Z) is distributed according to P_3. Hence, $X^2 + Y^2 + Z^2 = 1$ with probability 1. Similarly, $X^2 + Y^2 + W^2 = 1$. Therefore, $W^2 = Z^2$

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with probability 1. But, by the uniqueness of P_2, (W, Z) has the distribution P_2, which assigns the event $W^2 = Z^2$ probability zero.

There is no such P_4, nor, clearly, is there a solution for any higher dimension.

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