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AN INVARIANCE PRINCIPLE FOR ϕ-MIXING SEQUENCES

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In this paper it is established that the normalized sample paths of a ϕ-mixing sequence converge to the Brownian motion, under the Lindeberg's condition and under some stationarity assumptions. No mixing rate is required.

1. Introduction and notations. Let $\{X_n\}_n$ be a sequence of random variables on a probability space (Ω, K, P). Let $F_n^m = \sigma(X_i; n \le i \le m)$, $1 \le n \le m < \infty$. We say that $\{X_n\}_n$ is ϕ-mixing if $\phi_n \to 0$, where ϕ_n is defined by

$$\phi(n) = \sup_{m \in N} \sup_{\{A \in F_1^m,\, P(A) \neq 0,\, B \in F_{n+m}^\infty\}} |P(B|A) - P(B)|.$$

Assume $EX_i^2 < \infty$ for every i and let $S_n = \sum_{i=1}^n X_i$ and $\sigma_n^2 = ES_n^2$. Define the random element W_n in $D[0,1]$ endowed with the Skorokhod topology (see [1], page 101) by:

$$W_n(t) = S_{[nt]}/\sigma_n, \qquad t \in [0,1],\ n \in N.$$

where $[x]$ denotes the greatest integer function, and $X_0 = 0$. The aim of this paper is to investigate the weak convergence of W_n to the standard Brownian motion process on $[0,1]$, for ϕ-mixing sequences of random variables having finite second moments. We shall denote the standard Brownian motion process on $[0,1]$ by W, the weak convergence by $\Rightarrow$, and L_p norm by $\|\cdot\|_p$. It is known that a strictly stationary centered ϕ-mixing sequence with $\sigma_n^2 \to \infty$ and $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$ satisfies the central limit theorem (C.L.T.) (Theorem 18.5.1, Ibragimov and Linnik, 1971) and invariance principle (Ibragimov, 1975). In Ibragimov and Linnik (1971), page 393, it is noted the following conjecture:

If a sequence $\{X_n\}_n$ is strictly stationary centered ϕ-mixing and satisfies $EX_1^2 < \infty$ and $\sigma_n^2 \to \infty$ does it satisfy C.L.T.?

Iosifescu (1977) noted the following conjecture: If $\{X_n\}_{n \in N}$ is a strictly stationary centered ϕ-mixing sequence with $EX_1^2 < \infty$ and $\sigma_n^2 \to \infty$, does the invariance principle hold?

Herrndorf (1983) showed in Remark 4.3 that, if there exists a strictly stationary ϕ-mixing sequence with $\sigma_n^2 \to \infty$ and $\liminf(\sigma_n^2/n) = 0$, this last conjecture is not true. A natural problem that arises is to study if the conjecture is true under the assumption $\liminf(\sigma_n^2/n) \neq 0$. This paper gives an affirmative answer to this problem. In fact we shall establish (Theorem 2.1), the weak convergence to W for centered ϕ-mixing sequences with $\sigma_n^2 \to \infty$, satisfying some stationarity

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restrictions, and the Lindeberg condition:

$$\text{(L)} \qquad \lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{i=1}^{n} EX_i^2 I_{\{X_i^2 > \varepsilon\sigma_n^2\}} = 0 \quad \text{for every } \varepsilon > 0.$$

If $\phi_1 < 1$, this condition also appears to be a necessary condition for the weak convergence to W (Proposition 2.2).

In the strictly stationary case the condition (L) becomes:

$$\text{(L}'\text{)} \qquad \lim_{n\to\infty} \frac{n}{\sigma_n^2} EX_1^2 I_{\{X_1^2 > \varepsilon\sigma_n^2\}} = 0 \quad \text{for every } \varepsilon > 0.$$

It is obvious that if $\liminf(\sigma_n^2/n) \neq 0$ and $EX_1^2 < \infty$ the condition (L′) is satisfied and, by Corollary 2.2, the invariance principle holds. A study of the condition (L′) and especially of the behavior of σ_n^2 will bring a definite light on this conjecture. We raise the following problem. Is it true that $\liminf(\sigma_n^2/n) > 0$ for every strictly stationary centered ϕ-mixing sequence with $EX_1^2 < \infty$ and $\sigma_n^2 \to \infty$? The positive (negative) answer to this problem will imply a positive (negative) answer to the Ibragimov–Iosifescu (Iosifescu) conjecture.

In Section 2 we give the results and in Section 3 the proofs.

2. Results.

Let us denote $\lim_{n\to\infty}\phi_n$ by ϕ^*.

PROPOSITION 2.1. *Let $\{X_n\}_{n\geq 1}$ be a centered sequence of random variables with $\phi^* < \frac{1}{4}$. Then, $\{\max_{1\leq i\leq n}(S_i^2/\sigma_n^2)\}_n$ is uniformly integrable if and only if $\{\max_{1\leq i\leq n}(X_i^2/\sigma_n^2)\}_n$ is uniformly integrable.*

This result appears to be an important step in proving $W_n \Rightarrow W$ for ϕ-mixing sequences, when using for instance, Theorem 19.2, Billingsley (1968). Philipp (1980) introduced the notion of L_p invariance principle. This proposition can also be useful to determine a class of ϕ-mixing sequences for which L_2 invariance principle holds. In the following, we shall make the following stationarity assumptions:

(A) $\sigma_n^2 = nh(n)$ where $h(n)$ is a slowly varying function defined on R.
(B) $\sup_{m\geq 0, n\geq 1}[(E(S_{m+n} - S_m)^2)/\sigma_n^2] < \infty$.

We shall establish:

THEOREM 2.1. *Let $\{X_n\}_n$ be a centered ϕ-mixing sequence of random variables having finite moments of second order, satisfying (A), (B), and (L). Then $W_n \Rightarrow W$.*

REMARK 2.1. It is easy to see that if
(i) $\liminf_n(\sigma_n^2/n) > 0$ and $\{X_n^2\}_n$ is uniformly integrable or
(ii) The sequence has finite moments of order $2+\delta$ for some $\delta > 0$ and

$$\max_{1\leq i\leq n} E|X_i|^{2+\delta} = o\left(\frac{\sigma_n^{2+\delta}}{n}\right)$$

then the condition (L) is satisfied. So this theorem includes the already known results of: Ibragimov (Theorem 3.2, 1975), McLeish (Theorem 3.8, 1975), Peligrad (Corollary 2.4, 1982, Theorem, 1983). It is known that a ϕ-mixing sequence, second order stationary, with $\sigma_n^2 \to \infty$, satisfies (A) (see Theorem 18.2.3, [5] and the remark that follows this theorem). So we have the following corollary:

COROLLARY 2.1. *Let $\{X_n\}_n$ be a centered, second-order stationary ϕ-mixing sequence with $EX_1^2 < \infty, \sigma_n^2 \to \infty$, and the condition (L) is satisfied. Then $W_n \Rightarrow W$.*

In a strictly stationary case we obtain:

COROLLARY 2.2. *If $\{X_n\}_n$ is a strictly stationary centered ϕ-mixing sequence with $EX_1^2 < \infty, \lim_{n\to\infty}\sigma_n^2 = \infty$, and the condition (L') is satisfied. Then $W_n \Rightarrow W$.*

In Remark 2.3, Herrndorf (1983) noticed that the condition (A) is a necessary condition for the invariance principle. In some cases the condition (L) also appears to be a necessary condition. We have:

PROPOSITION 2.2. *If $\{X_n\}_n$ is such that $W_n \Rightarrow W$, $\sigma_n^2 \to \infty$ and $\phi_1 < 1$, then the condition (L) is satisfied.*

REMARK 2.2. In Theorem (2.1) we can assume instead of the ϕ-mixing condition only that $\phi^* < \frac{1}{2}$ and the sequence is strong mixing (i.e., $\alpha_n \to 0$, where $\alpha_n = \sup_m \sup_{A \in F_0^m, B \in F_{m+n}^\infty} |P(A \cap B) - P(A)P(B)|$), Rosenblatt, 1956). However, this is not a major improvement of the ϕ-mixing assumption because, if the sequence is assumed to be strictly stationary and mixing, from Theorem 1 of Bradley (1980), it follows that $\phi^* < 1$ implies $\phi^* = 0$.

3. Proofs.

REMARK 3.1. If $\{X_n\}_n$ is a sequence of random variables with $\phi^* < 1$, then:

(i) $P(\max_{1 \le i \le n}|X_i| > \varepsilon\sigma_n) \to 0$ for every $\varepsilon > 0$ is equivalent with

$$\sum_{i=1}^{n} P(|X_i| > \varepsilon\sigma_n) \to 0$$

for every $\varepsilon > 0$ and

(ii) (L) is equivalent with

$$(\mathrm{L}'') \qquad E \max_{1 \le i \le n} \frac{X_i^2}{\sigma_n^2} \to 0.$$

PROOF. One of the implications in (i) is trivial. For the other we shall use the same type of judgement that leads to the relation (3.28) in Lai (1977). Because

$P(\max_{1\le i\le n}|X_i| > \varepsilon\sigma_n) \to 0$, for every $\varepsilon > 0$ we can choose n_0 and p_0 such that:

$$P\Big(\max_{1\le i\le n} X_i^2 < \varepsilon\sigma_n^2\Big) - \phi_{p_0} \ge a > 0 \quad \text{for all } n \ge n_0. \tag{3.1}$$

Therefore, as in (3.28) [Lai (1977)] for every $x \ge \varepsilon > 0$ and $n \ge n_0$

$$a \sum_{i \le \left[\frac{(n-j)}{p_0}\right]} P\big(X_{ip_0+j}^2 > x\sigma_n^2\big) \le P\Big(\max_{1\le i\le n} X_i^2 > x\sigma_n^2\Big) \quad \text{for } j = 0,\dots,p_0-1.$$

So for every $x \ge \varepsilon > 0$ and $n \ge n_0$.

$$\sum_{j=1}^{n} P\big(X_j^2 > x\sigma_n^2\big) \le \frac{p_0}{a} P\Big(\max_{1\le j\le n} X_j^2 > x\sigma_n^2\Big). \tag{3.2}$$

From this relation, as a first consequence (i) follows. In order to prove (L″) implies (L), let us notice first that, under (L″), (3.1) holds and in the same time (3.2) holds. Now, we only have to apply the following well-known relation: For every positive integrable random variable X,

$$EXI_{\{X>a\}} = aP(X > a) + \int_a^\infty P(X > x)\,dx. \tag{3.3}$$

The fact (L) implies (L″) follows from:

$$E\max_{1\le i\le n}\frac{X_i^2}{\sigma_n^2} \le \varepsilon + E\max_{1\le i\le n}\frac{X_i^2}{\sigma_n^2} I_{\{X_i^2/\sigma_n^2>\varepsilon\}} \quad \text{for every } \varepsilon > 0.$$

The following lemma has a technical interest and is an extension of some results for independent random variables implicitly contained in Hoffman-Jørgensen (1974).

LEMMA 3.1. *Let $\{Y_n\}_n$ be a sequence of random variables. Let $T_m = \sum_{i=1}^m Y_i$ and suppose that for some $b > 0$, $p \in N$, and $a_0 > 0$.*

$$\phi_p + \max_{1\le i\le m} P\Big(|T_m - T_i| > \Big(\frac{b}{2}\Big)a_0\Big) \le \eta < 1. \tag{3.4}$$

Then for every $a \ge a_0$ and $m > p$ the following relations hold:

$$P\Big(\max_{1\le i\le m}|T_i| > (1+b)a\Big) \le \frac{1}{(1-\eta)}P(|T_m| > a) + \frac{1}{(1-\eta)}P\Big(\max_{1\le i\le m}|Y_i| > \frac{ba}{2(p-1)}\Big) \tag{3.5}$$

and

$$P(|T_m| > (1+2b)a) \le \frac{\eta}{(1-\eta)}P(|T_m| > a) + \frac{1}{(1-\eta)}P\Big(\max_{1\le i\le m}|Y_i| > \frac{ba}{2p}\Big). \tag{3.6}$$

PROOF. In the case $\phi_1 < 1$, the relation (3.5) was proved in Iosifescu and Theodorescu (1968), Lemma 1.1.6. The proof in the general case of the relation (3.5) can be deduced from Billingsley (1968, page 175) and is due to Peligrad (1981).

In order to prove (3.6), let us denote

$$E_k = \left\{ \max_{1 \le i < k} |T_i| \le (1 + b)a < |T_k| \right\}.$$

We have

$$P(|T_m| > (1 + 2b)a) \le P\left(|T_m| > (1 + 2b)a, \quad \max_{1 \le i \le m-p} |T_i| > (1 + b)a, \right.$$
$$\left. \max_{1 \le i \le m} |Y_i| \le \frac{ba}{2p} \right) + P\left(\max_{1 \le i \le m} |Y_i| > \frac{ba}{2p} \right)$$

Because $|T_m - T_{j+p-1}| \ge |T_m| - |T_{j-1}| - p\max_{1 \le i \le m}|Y_i|$ for all $1 \le j \le m - p$ it follows:

$$P(|T_m| > (1 + 2b)a) \le \sum_{j=1}^{m-p} P\left(E_j \cap \left\{ |T_m - T_{j+p-1}| > \frac{ba}{2} \right\} \right)$$
$$+ P\left(\max_{1 \le i \le m} |Y_i| > \frac{ba}{2p} \right).$$

Therefore, by the definition of the ϕ-mixing coefficients and by (3.4), for every $a \ge a_0$ we have

$$P(|T_m| > (1 + 2b)a) \le \eta P\left(\max_{1 \le i \le m} |T_i| > (1 + b)a \right)$$
$$+ P\left(\max_{1 \le j \le n} |Y_j| > \frac{ba}{2p} \right). \tag{3.7}$$

Now, (3.6) follows from (3.7) and (3.5). □

In the following $E_A X$ denotes $EXI_{\{X > A\}}$.

LEMMA 3.2. *If $\{Y_n\}_n$ is a sequence satisfying* (3.4), *then for every $A \ge a_0^2$ we have*

$$E_{(1+2b)^2 A} T_m^2 \le (1 + 2b)^2 \frac{\eta}{(1 - \eta)} E_A T_m^2$$
$$+ \left(\frac{2p(1 + 2b)}{b} \right)^2 \frac{1}{(1 - \eta)} E_{A(b/2p)^2} \max_{1 \le i \le m} Y_i^2.$$

PROOF. Let $A \ge a_0^2$. By (3.3) and a change of variables one obtains

$$E_{(1+2b)^2 A} T_m^2 \le (1 + 2b)^2 A P\left(T_m^2 > (1 + 2b)^2 A\right)$$
$$+ (1 + 2b)^2 \int_A^\infty P\left(T_m^2 > (1 + 2b)^2 y\right) dy.$$

Lemma follows by using (3.6) and then (3.3).

LEMMA 3.3. *Let* $\{X_n\}_n$ *be a centered sequence of random variables such that* $\phi^* < \frac{1}{4}$ *and* $\{\max_{1 \le i \le n}(EX_i^2/\sigma_n^2)\}_n$ *is bounded. Then*

$$\left(\max_{1 \le i \le n} \frac{E(S_n - S_i)^2}{\sigma_n^2} \right)_n$$

is bounded.

PROOF. Let p be such that $\phi_p < \frac{1}{4}$. We have

$$\max_{1 \le i \le n} E(S_n - S_i)^2 \le \max_{1 \le i < n-p} E(S_n - S_i)^2 + \max_{1 \le i \le n} p^2 EX_i^2. \tag{3.8}$$

For every $i < n - p$ we also have

$$\big| \|S_n\|_2 - \|S_i + (S_n - S_{i+p})\|_2 \big| \le p \max_{1 \le i \le n} \|X_i\|_2.$$

By Lemma 17.2.3, Ibragimov and Linnik (1971) we have

$$\|S_i + (S_n - S_{i+p})\|_2 \ge \left(1 - 2\phi_p^{1/2}\right)^{1/2} \left(\sigma_i^2 + E\left(S_n - S_{i+p}\right)^2\right)^{1/2}.$$

From the preceding two inequalities it follows that

$$\sigma_i < \frac{1}{\left(1 - 2\phi_p^{1/2}\right)^{1/2}} \left(\sigma_n + p \max_{1 \le i \le n} \|X_i\|_2\right)$$

for every $i < n - p$. Whence, from (3.8),

$$\max_{1 \le i \le n} \frac{E(S_n - S_i)^2}{\sigma_n^2} \le 2 + \frac{4}{\left(1 - 2\phi_p^{1/2}\right)} + p^2 \left(1 + \frac{4}{\left(1 - 2\phi_p^{1/2}\right)}\right) \max_{1 \le i \le n} \frac{EX_i^2}{\sigma_n^2}.$$

REMARK 3.2. In this lemma $\phi^* < \frac{1}{4}$ can be replaced by $\lim \rho_n < 1$ where $\{\rho_n\}_n$ are the coefficients based on the maximal correlations (see Ibragimov, 1975).

PROOF OF PROPOSITION 2.1. First, because

$$P\left(\max_{1 \le i \le n} \frac{|X_i|}{\sigma_n} > 2x \right) \le P\left(\max_{1 \le i \le n} \frac{|S_i|}{\sigma_n} > x \right) \tag{3.9}$$

for every $x > 0$, one of the implications follows by the relation (3.3).

Let us assume now that $\{\max_{1 \le i \le n}(X_i^2/\sigma_n^2)\}_n$ is uniformly integrable. By Tchebycheff's inequality and by Lemma 3.3 for every $b > 0$

$$\lim_{t \to \infty} \sup_n \max_{1 \le i \le n} P\left(\frac{(S_n - S_i)^2}{\sigma_n^2} > \left(\frac{b}{2}\right)^2 t \right) = 0. \tag{3.10}$$

By the fact that $\phi^* < \frac{1}{2}$ and by (3.10), we can find some constants $b > 0$, $\eta < \frac{1}{2}$, $p \in N$, and $a_0 \in R$, that do not depend on n and such that

$$\frac{(1 + 2b)^2 \eta}{(1 - \eta)} < 1 \tag{3.11a}$$

and

$$\text{(3.11b)}\quad \phi_p + \max_{1 \le i \le n} P\left(\frac{(S_n - S_i)^2}{\sigma_n^2} > \left(\frac{b}{2}\right)^2 a_0^2 \right) \le \eta \quad \text{for every } n \ge 1.$$

From Lemma 3.2 and (3.11) it follows

$$E_{(1+2b)^2 A} \frac{S_n^2}{\sigma_n^2} \le (1 + 2b)^2 \frac{\eta}{(1-\eta)} E_A \frac{S_n^2}{\sigma_n^2} + \left(\frac{2p(1+2b)}{b} \right)^2 \frac{1}{(1-\eta)} \cdot E_{A(b/2p)^2} \max_{1 \le i \le n} \frac{X_i^2}{\sigma_n^2}$$

for every $A > a_0^2$ and every $n \ge 1$. Taking in this relation the supremum on n and taking into account that $\sup_n E_A(S_n^2/\sigma_n^2)$ is decreasing in A, and that $\{\max_{1 \le i \le n}(X_i^2/\sigma_n^2)\}_n$ is uniformly integrable it follows that

$$\lim_{A \to \infty} \sup_n E_A \frac{S_n^2}{\sigma_n^2} \le (1+2b)^2 \frac{\eta}{(1-\eta)} \lim_{A \to \infty} \sup_n E_A \frac{S_n^2}{\sigma_n^2}$$

whence, by (3.11), it follows $\{S_n^2/\sigma_n^2\}_n$ is uniformly integrable. This implies by (3.5) and (3.3) that $\{\max_{1 \le i \le n}(S_i^2/\sigma_n^2)\}_n$ is uniformly integrable.

REMARK 3.3. If the condition (A) is assumed, by the properties of a slowly varying function on R (Appendix 1 of [5]) it follows that

$$\left(\max_{1 \le i \le n} \frac{(S_n - S_i)^2}{\sigma_n^2} \right)$$

is bounded and so we can replace in Proposition 2.1 the condition $\phi^* < \frac{1}{4}$ by the couple of conditions $\phi^* < \frac{1}{2}$ and (A).

PROOF OF THEOREM 2.1. In order to prove Theorem 2.1 we shall apply Theorem 19.2 of Billingsley (1968). It was proved in Billingsley, page 174, that the strong mixing property implies that W_n has asymptotically independent increments. By Remark 3.1 it follows that under (L), $\{\max_{1 \le i \le n}(X_i^2/\sigma_n^2)\}_n$ is uniformly integrable, whence by (A) and Proposition 2.1, it follows $\{W_n^2(t)\}_n$ is uniformly integrable for each t. Once again by (A), $EW_n^2(t) \to t$ for each t and because $\{X_n\}_n$ is centered, $EW_n(t) = 0$ for each t. It remains only to verify the tightness condition. In the strictly stationary case, the uniform integrability of $\{\max_{1 \le i \le n}(S_i^2/\sigma_n^2)\}_n$ implies that the tightness criterion contained in Theorem 8.4 [Billingsley (1968)] is verified. In our settling the tightness will result from the following:

PROPOSITION 3.1. *Let $\{X_n\}_n$ be a centered sequence, satisfying (A), (B), and (L), and such that $\phi^* < \frac{1}{2}$. Then W_n is tight in D.*

Proof. From the proof of Theorem 8.3 of Billingsley (1968) it is enough to establish

$$(3.12)\qquad \lim_{\delta\to 0,\,1/\delta\in N}\ \limsup_{n}\ \sum_{i=0}^{1/\delta-1} P\Big(\max_{i\delta\le s\le(i+1)\delta}|W_n(s)-W_n(i\delta)|>\varepsilon\Big)=0.$$

For every $0\le i\le 1/\delta-1$ denote

$$f_i=f_i(n,\delta,a)=\max_{in\delta\le j\le(i+1)n\delta} P\left(\frac{\left(\sum_{k=j}^{(i+1)n\delta}X_k\right)^2}{\sigma_n^2}>\left(\frac{b}{2}\right)^2 a\right).$$

By Tchebycheff's inequality we have

$$f_i<\left(\frac{2}{b}\right)^2\frac{1}{a}\cdot\max_{in\delta\le j\le(i+1)n\delta} E\frac{\left(\sum_{k=j}^{(i+1)n\delta}X_k\right)^2}{\sigma_n^2}.$$

By (A) and (B) and by the properties of slowly varying functions that follow from Karamata representation (Appendix 1, Ibragimov-Linnik, 1971) it follows that

$$(3.13)\qquad \lim_{\delta\to 0}\ \limsup_{n}\ \max_{0\le i\le 1/\delta-1} f_i=0.$$

Choose p and b such that

$$\frac{\phi_p}{(1-\phi_p)}(1+2b)^2<1$$

and δ_0 and n_0 such that for every $\delta<\delta_0$ and $n>n_0$.

$$(3.14)\qquad \phi_p+\max_{1\le i\le 1/\delta} f_i=\eta'(n,\delta,a)=\eta'<1.$$

From (3.14) Lemma 3.2 we obtain for every $0\le i\le 1/\delta-1$

$$(3.15)\qquad \begin{aligned} E_{(1+2b)^2a}\frac{\left(\sum_{j=ni\delta}^{n(i+1)\delta}X_j\right)^2}{\sigma_n^2} &\le (1+2b)^2\frac{\eta'}{(1-\eta')}E_a\frac{\left(\sum_{j=ni\delta}^{n(i+1)\delta}X_j\right)^2}{\sigma_n^2}\\ &\quad+\left(\frac{2p(1+2b)}{b}\right)^2\frac{1}{(1-\eta')}\sum_{j=ni\delta}^{n(i+1)\delta}E_{a(b/2p)^2}X_j^2/\sigma_n^2. \end{aligned}$$

Let us note at this point that by (A) and (B), for every $\delta>0$ we have

$$\limsup_{n\to\infty}\sum_{i=0}^{1/\delta-1}\frac{E\left(\sum_{j=ni\delta}^{n(i+1)\delta}X_j\right)^2}{\sigma_n^2}=O\Big(\limsup_{n\to\infty}\sum_{i=1/\delta}\sigma_{[n\delta]}^2/\sigma_n^2\Big)=O(1).$$

Denote

$$l(a)=\limsup_{\delta\to 0}\ \limsup_{n}\ \sum_{i=0}^{1/\delta-1}E_a\frac{\left(\sum_{j=ni\delta}^{n(i+1)\delta}X_j\right)^2}{\sigma_n^2}.$$

From (3.15) we obtain by (3.13) and by condition (L):

$$l\big((1+2b)^2a\big) \le \frac{(1+2b)^2\phi_p}{(1-\phi_p)} l(a) \quad \text{for every } a > 0.$$

Because $l(a)$ is a decreasing function in a, and $[(1+2b)^2\phi_p]/(1-\phi_p) < 1$ by the preceding inequality we obtain $\lim_{a\to 0} l(a) = 0$ and so $l(a) = 0$ for every $a > 0$. Therefore for every $\varepsilon > 0$

$$\lim_{\delta\to 0} \limsup_n \sum_{i=0}^{1/\delta - 1} P\left(\left|\sum_{j=ni\delta}^{n(i+1)\delta} X_j\right| > \varepsilon\sigma_n\right) = 0.$$

The relation (3.12) follows now by (3.5), (L) and (3.13).

PROOF OF PROPOSITION 2.2. Remark (2.3) in Herrndorf (1983) implies $\sigma_i^2 = ih(i)$ where h is a slowly varying function on R^+, whence,

$$\left(\max_{1\le i\le n} \frac{E(S_n - S_i)^2}{\sigma_n^2}\right)_n$$

is bounded. So we can find $t_0 > 0$ such that for every n:

$$\phi_1 + \max_{1\le i\le n} P(|S_n - S_i| > t_0\sigma_n) \le c < 1.$$

Then, by Lemma 1.1.6 in [8], for every $x > t_0^2$ and for each $n \in N$, we obtain:

$$P\left(\max_{1\le i\le n} \frac{S_i^2}{\sigma_n^2} > 4x\right) \le \frac{1}{(1-c)} P\left(\frac{S_n^2}{\sigma_n^2} > x\right). \tag{3.16}$$

On the other hand, the weak convergence to W implies, by Theorem 5.4 in Billingsley (1968), the uniform integrability of $\{S_n^2/\sigma_n^2\}_n$. This fact together with (3.16) and (3.3) implies $\{\max_{1\le i\le n}(S_i^2/\sigma_n^2)\}_n$ is uniformly integrable. By (3.9) it follows that $\{\max_{1\le i\le n}(X_i^2/\sigma_n^2)\}_n$ is uniformly integrable. By Remark (2.3) in Herrndorf (1983) it follows that $P(\max_{1\le i\le n}|X_i| > \varepsilon\sigma_n) \to 0$ for every $\varepsilon > 0$, and (L″) follows, because $\{\max_{1\le i\le n}(X_i^2/\sigma_n^2)\}_n$ is uniformly integrable. By the Remark (3.1) condition (L) is a necessary condition for the weak convergence to W.

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