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THE OPTIONAL SAMPLING THEOREM FOR PROCESSES INDEXED BY A PARTIALLY ORDERED SET

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The optional sampling theorem (OST) is not necessarily true for supermartingales indexed by a partially ordered set. However, if the index set satisfies a mild separability condition, without necessarily being directed or countable, we prove the OS inequality for a class of supermartingales that extends the concept of S-processes defined by Cairoli on the plane $\mathbb{R}^2_+$. Under a further restriction on these processes we obtain the OS equation, thus extending the corresponding result for martingales to the case of nondirected index sets. We then introduce strong martingales and strong supermartingales for separable partially ordered index sets, and show that these processes again satisfy the OST. By defining stopping domains as well as the value of a process for a stopping domain, we show that the strong (super)martingales are precisely those processes which satisfy the OST for all bounded stopping domains. This extends a result of Cairoli–Walsh and Wong–Zakai on $\mathbb{R}^2_+$.

0. Introduction. The optional sampling theorem for a positive supermartingale X with index set $[0, \infty]$ states that for any two stopping times $\tau_1 \le \tau_2$,

$$E\left[X_{\tau_2} | \mathscr{F}_{\tau_1}\right] \le X_{\tau_1} \tag{0.1}$$

with equality if X is a uniformly integrable martingale. On partially ordered sets other than the real line this theorem is not necessarily true anymore, as a counterexample by Chow [7] shows. However, in the case of a martingale on a directed set, Chow proved that (0.1) was indeed still correct for a restricted class of stopping times; Kurtz [19] removed this restriction on the stopping times and generalized the result to where the index set T is a topological lattice. As concerns (0.1) for supermartingales, Haggstrom [12] showed that the optional sampling theorem is true on certain graphs if the τ_i are *control variables*; Washburn and Willsky [26] extended this result to general countable index sets and to the larger class of *reachable stopping times*, a class that includes the *tactics* of Krengel and Sucheston [17], but in general does not contain all stopping times (see also [17], Section 2). Results for supermartingales with uncountable index sets other than $\mathbb{R}^n_+$ do not seem to be known.

The approach we take here is different: Rather than restricting the class of stopping times, we use a combinatorial method introduced in [13], [14] to characterize a special class of supermartingales. The basic idea is to interpret $E[X_s; F]$ as the value of a measure $P^{X, \ge}$ on the set $\{t \in T | t \ge s\} \times F$, where F belongs to an appropriate σ-algebra on Ω. X is then called an *S-process* if $P^{X, \ge}$

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is positive, and a *special M-process* if $P^{X,\geq}$ is concentrated on the maximal elements of T, a characterization that is possible both for countable and uncountable sets T. The concept of an S-process was first introduced on $\mathbb{R}^2_+$ by Cairoli in [4], and was then extended to more general partially ordered index sets in [13], [14]. In the classical case $T = \mathbb{R}$, the S-processes reduce to the positive supermartingales [8], [9], [11]. As to the special M-processes, they are precisely the L^1-bounded martingales if T is directed.

The precise definitions are given in Section 2. We show there that the S-processes satisfy the OST (0.1) for all stopping times, and that Kurtz's martingale version of (0.1) holds for the special M-processes, although it does not hold generally for martingales if T is not directed. Interpreting $E[X_s; F]$ as the value of a measure $P^{X,\leq}$ on the set $\{t \in T | t \leq s\} \times F$ then leads to the dual concept of a strong submartingale if $P^{X,\leq}$ is positive and a strong martingale if $P^{X,\leq}$ vanishes; on the plane these processes are well known [5], [23], [25], and again they satisfy the OST for stopping times.

In Section 3 we extend the OST to *stopping domains* δ, which on the plane correspond to the stopping lines of Merzbach [20], [21] and the stopping times of Wong and Zakai [27]. We define X_δ, and then show that X is a strong submartingale iff

$$E\left[X_{\delta_2} | \mathscr{F}_{\delta_1}\right] \geq X_{\delta_1} \tag{0.2}$$

for all bounded stopping domains $\delta_1 \leq \delta_2$, with equality iff X is a strong martingale. This generalizes results of Wong and Zakai [27], of Cairoli and Walsh [6], and of Walsh [25].

1. Basic definitions. Let T be a set with a partial order $\geq$; let T^d be a countable subset which is dense from above in T in the sense that for any $t \in T$ there exists a sequence $(t_n)_{n \in \mathbb{N}} \subseteq T^d$ with

$$t_n \downarrow t, \tag{1.1}$$

i.e., with $t_n \geq t_{n+1}$ and $\inf_n t_n = t$. We assume that there exists an increasing sequence $(K_n)_{n \in \mathbb{N}}$ of finite subsets of T^d such that

$$T^d = \bigcup_{n \in \mathbb{N}} K_n \tag{1.2}$$

and such that $\forall\ s, t \in K_n$ there exist integers $c_r^n(s,t), d_r^n(s,t)$, $r \in K_n$, satisfying

$$\mathbb{1}_{\{x \in T | x \leq s\} \cap \{x \in T | x \leq t\}} = \sum_{r \in K_n} c_r^n(s,t) \mathbb{1}_{\{x \in T | x \leq r\}}, \tag{1.3}$$

$$\mathbb{1}_{\{x \in T | x \geq s\} \cap \{x \in T | x \geq t\}} = \sum_{r \in K_n} d_r^n(s,t) \mathbb{1}_{\{x \in T | x \geq r\}}. \tag{1.4}$$

Examples of such posets are the countable trees, $\mathbb{R}^n_+$ furnished with the product order, the set of closed sets of a separable topological space with the partial order defined by inclusion, as well as all those lattices Kurtz [19] calls separable from above.

For $t \in T$ set

$$(1.5)\qquad t_n := \min\{s \in K_n | t \le s\} \in K_n, \quad \text{if } \{s \in K_n | t \le s\} \neq \varnothing, \qquad := 1, \quad \text{if } \{s \in K_n | t \le s\} = \varnothing.$$

Hypothesis (1.4) implies the existence of $\min\{s \in K_n \cup \{1\} | t \le s\} \in K_n \cup \{1\}$, where 1 is the artificially introduced largest element of $T \cup^d \{1\}$ ([14] Lemma 1.5 gives the dual statement). $(t_n)_{n \in \mathbb{N}}$ is then a decreasing sequence, and due to (1.1) $t = \inf_n t_n$ ([14] Lemma 1.9). If there is any ambiguity, we will denote $(t_n)_{n \in \mathbb{N}}$ by $\nu^{\le}(t)$.

Since K_n is finite, there exist two operators $D^{n,\ge}$ and $D^{n,\le}$ which map any function $f\colon K_n \to \mathbb{R}$ into new real valued functions $D^{n,\ge} f = (D_s^{n,\ge} f)_{s \in K_n}$ and $D^{n,\le} f = (D_s^{n,\le} f)_{s \in K_n}$ on K_n in such a way that

$$(1.6)\qquad f(s) = \sum_{t \in K_n,\, t \ge s} D_t^{n,\ge} f,$$

$$(1.7)\qquad f(s) = \sum_{t \in K_n,\, t \le s} D_t^{n,\le} f$$

for all $s \in K_n$. These two operators are uniquely defined by (1.6) and (1.7), and can be explicitly computed: Define the *Möbius function* $\mu^n\colon K_n \times K_n \to \mathbb{Z}$ recursively by

$$(1.8)\qquad \mu^n(s,t) := \begin{cases} 1, & s = t, \\ -\sum_{r \in K_n,\, s \le r < t} \mu^n(s,r), & s < t, \\ 0, & \text{otherwise;} \end{cases}$$

then

$$(1.9)\qquad D_s^{n,\ge} f = \sum_{r \in K_n} \mu^n(s,r) f(r) = \sum_{r \in K_n,\, r \ge s} \mu^n(s,r) f(r),$$

$$(1.10)\qquad D_s^{n,\le} f = \sum_{r \in K_n} f(r) \mu^n(r,s) = \sum_{r \in K_n,\, r \le s} f(r) \mu^n(r,s).$$

The functions $D^{n,\ge} f$ and $D^{n,\le} f$, called the *upper and lower Möbius inversions* of f, intuitively describe the "mass" allotted by f to the points $s \in K_n$, depending on whether $f(s)$ is regarded as describing the mass of $\{t \in T | t \ge s\}$ or $\{t \in T | t \le s\}$. Or, if the reader prefers, $D_s^{n,\ge} f$ can also be regarded as the mass of the atom $A_s^{n,\ge}$ of $\sigma\{\{t \in T | t \ge r\} | r \in K_n\}$ containing the element s of K_n (s is unique and minimal in $A_s^{n,\ge}$); similarly for $D_s^{n,\le}$ and $A_s^{n,\le}$. (For details see

[1], [13], [14], [24].) In the case where an artificial 1 had to be introduced, define

$$D_1^{n, \geq} f := 0,$$

$$(1.11) \qquad D_1^{n, \leq} f := f(1) - \sum_{t \in K_n} D_t^{n, \leq} f,$$

and note that (1.7) now holds for all $s \in K_n \cup \{1\}$, and also (1.6) if $f(1) = 0$.

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathscr{F} = (\mathscr{F}_t)_{t \in T}$ an isotone family of σ-subalgebras of $\mathscr{F}$ satisfying $\mathscr{F}_t = \cap_n \mathscr{F}_{t_n}$ $\forall\ t \in T$. $X = (X_t)_{t \in T}$ denotes an adapted L^1-process, i.e., $X_t \in L^1(\Omega, \mathscr{F}_t, P)$; we assume X to be L^1 *right continuous* in the sense that $X_t = L^1 - \lim_n X_{t_n}$. Note that this does not imply that X is L^1 right continuous with respect to a general sequence $s_n \downarrow t$! X is a *martingale* if $E[X_s|\mathscr{F}_t] = X_t$ $\forall\ s \geq t \in T$, a *supermartingale* if $E[X_s|\mathscr{F}_t] \leq X_t$ and a *submartingale* if $E[X_s|\mathscr{F}_t] \geq X_t$. $(X_n)_{n \in \mathbb{N}}$ is a *quasimartingale* if $\sup_m(\sum_{n=1}^m \|E[X_n - X_{n+1}|\mathscr{F}_n]\|_1 + \|X_{m+1}\|_1) < \infty$. If $(\mathscr{F}_n)_{n \in \mathbb{N}}$ is decreasing and $E[X_n|\mathscr{F}_{n+1}] = (\leq) X_{n+1}$, we call $(X_n)_{n \in \mathbb{N}}$ a *reversed* (sub)martingale, and analogously for the other types of processes. We repeatedly use the fact that reversed quasimartingales converge a.s. and in $L^1(\Omega, \mathscr{F}, P)$ ([8]). A map τ: $\Omega \to T$ is a *stopping time* if $\{\omega \in \Omega | \tau(\omega) \leq t\} \in \mathscr{F}_t$ $\forall\ t \in T$; the associated σ-algebra is $\mathscr{F}_\tau := \{F \in \mathscr{F} | F \cap \{\tau \leq t\} \in \mathscr{F}_t, \forall\ t \in T\}$. For any stopping time τ we define τ^n by

$$(1.12) \qquad (\tau^n(\omega))_{n \in \mathbb{N}} = \nu^{\leq}(\tau(\omega)).$$

The τ^n are again stopping times, with values in $K_n \cup \{1\}$, and $\tau^n \downarrow \tau$ pointwise, $\mathscr{F}_\tau = \cap_n \mathscr{F}_{\tau^n}$. Here it becomes apparent why we only require T^d to be dense from above (1.1).

Furthermore we denote the maximal elements $\{s \in K_n | \nexists\ t \in K_n \text{ with } s < t\}$ in K_n by $K_n^{\max}$; $K_n^{\max}$ does not contain the artificial 1.

2. The optional sampling theorem for stopping times

DEFINITION 2.1. An adapted L^1 right continuous process $X = (X_t)_{t \in T}$ is an (*upper*) *S-process* iff

$$(2.1) \qquad E[D_s^{n, \geq} X | \mathscr{F}_s] \geq 0, \qquad \forall\ s \in K_n, \quad \forall\ n \in \mathbb{N},$$

and if the *variation* of X is bounded, i.e., if

$$(2.2) \qquad \operatorname{var} X := \sup_{n \in \mathbb{N}} E\left[\sum_{s \in K_n} \left|E[D_s^{n, \geq} X | \mathscr{F}_s]\right|\right] < \infty.$$

We will omit the word "upper" since we will not deal with any other kind of S-processes here. Due to

$$(2.3) \qquad E[X_s - X_t | \mathscr{F}_s] = \sum_{\substack{r \in K_n \\ r \geq s,\, r \ngeq t}} E[D_r^{n, \geq} X | \mathscr{F}_s]$$

$\forall\ s \le t \in K_n$, an S-process is a positive supermartingale; the converse however is not generally true, as can easily be seen on the plane $T = \mathbb{R}_+^2$. Note that if T has a smallest element, then (2.2) follows from (2.1).

LEMMA 2.2. *If X is an S-process, there exists an adapted modification $\overline{X}$ of X such that*

$$\overline{X}_t(\omega) = \begin{cases} \lim\limits_{n\to\infty} \overline{X}_{t_n}(\omega), & \text{if the limit exists,} \\ 0, & \text{otherwise,} \end{cases} \tag{2.4}$$

$\forall\ t \in T$, *where* $(t_n)_{n\in\mathbb{N}} = \nu^{\le}(t)$.

PROOF. For every $t \in T$, $(X_{t_n})_{n\in\mathbb{N}}$ is a reversed supermartingale with $\sup_n \|X_{t_n}\|_1 \le \|X_t\|_1$, which therefore converges a.s. and in L^1. Define $\overline{X}_t(\omega) := \lim_n X_{t_n}(\omega)$ if the limit exists, and $\overline{X}_t(\omega) := 0$ otherwise. $\overline{X}$ satisfies (2.4) because $\overline{X}_t \equiv X_t\ \forall\ t \in T^d$, and because $\overline{X}_t = X_t$ a.s. $\forall\ t \in T$ due to the L^1 right continuity of X. □

THEOREM 2.3. *Let X be an S-process satisfying* (2.4), *and let $\tau_1 \le \tau_2$ be two stopping times. Then X_{τ_1} and X_{τ_2} are in L^1, and*

$$E\left[X_{\tau_2} | \mathscr{F}_{\tau_1}\right] \le X_{\tau_1}. \tag{2.5}$$

PROOF. If necessary introduce an artificial largest element 1 and define $X_1 \equiv 0$, $\mathscr{F}_1 \equiv \mathscr{F}$. Then $X_{\tau_i^n} = \sum_{s\in K_n} X_s \mathbb{1}_{\{\tau_i^n = s\}}$ is $\mathscr{F}_{\tau_i^n}$-measurable, and for $F \in \mathscr{F}_{\tau_1^n}$ we get

$$\begin{aligned} E\left[X_{\tau_1^n} - X_{\tau_2^n}; F\right] &= \sum_{t\in K_n} E\left[D_t^{n,\ge} X; F \cap \{\tau_1^n \le t\} \cap \{\tau_2^n \not\le t\}\right] \\ &\ge 0 \end{aligned} \tag{2.6}$$

since $F \cap \{\tau_1^n \le t\} \in \mathscr{F}_t$, $\{\tau_2^n \not\le t\} \in \mathscr{F}_t$. This proves (2.5) for stopping times assuming only values in K_m, $m \in \mathbb{N}$. Defining $\bar\tau_1 := \tau_1^n$, $\bar\tau_2 := \tau_1^{n-1}$ it follows in particular that $(X_{\tau_i^n})_{n\in\mathbb{N}}$ is a (one-dimensional) reversed supermartingale with $\sup_n \|X_{\tau_i^n}\|_1 \le \operatorname{var} X < \infty$. Therefore $X_{\tau_i^n}$ converges a.s. and in L^1, and due to our assumption (2.4) this limit must be X_{τ_i} a.s. The observation $\mathscr{F}_{\tau_i} = \bigcap_n \mathscr{F}_{\tau_i^n}$ completes the proof. □

The basic argument is thus to first apply the Möbius inversion to obtain the OST on finite subsets, and then to pass to the limit using one-dimensional reverse (quasi)martingale convergence. In the sequel this idea will be applied to a number of different situations, the details being left to the reader.

The following discrete example shows that the OST (2.5) does not necessarily hold for uniformly integrable martingales (unless T is directed, [19] Theorem 2.15): Let $T = \{1, 2, 2'\}$ be the poset with $1 \le 2$, $1 \le 2'$, and let $\Omega = \{a, b\}$ with $P\{a\} = P\{b\} = \frac{1}{2}$; define $\mathscr{F}_1 = \{\phi, \Omega\}$, $\mathscr{F}_2 = \mathscr{F}_{2'} = \{\phi, \{a\}, \{b\}, \Omega\}$, and $X_1 \equiv 1$, $X_2 = 2\mathbb{1}_{\{a\}}$, $X_{2'} = 2\mathbb{1}_{\{b\}}$; X is a martingale. If $\tau_1 \equiv 1$, $\tau_2 = 2'\mathbb{1}_{\{a\}} + 2\mathbb{1}_{\{b\}}$, $\tau_3 =$

$2\mathbb{1}_{\{a\}} + 2'\mathbb{1}_{\{b\}}$, then $\tau_1 \le \tau_2$, $\tau_1 \le \tau_3$ and yet $0 \equiv X_{\tau_2} < 1 \equiv X_{\tau_1} < X_{\tau_3} \equiv 2$. For such an index set the following concept is more appropriate:

DEFINITION 2.4. An adapted L^1 right continuous process X with $\operatorname{var} X < \infty$ is an (*upper*) *special M-process* if

$$E\left[D_s^{n, \ge} X \mid \mathscr{F}_s\right] = 0, \qquad \forall\, s \in K_n \setminus K_n^{\max}, \qquad \forall\, n \in \mathbb{N}. \tag{2.7}$$

The term *M process* on posets was introduced in [13], [14] as a generalization of the concept of a weak martingale on the plane [4], [5], [22], [27]. But since the stopping theorem does not even hold for weak martingales on the plane, we need this more restrictive class.

LEMMA 2.5. *If T is directed (not necessarily a lattice), then X is a special M-process iff X is a martingale with* $\operatorname{var} X = \sup_{t \in T} \|X_t\| < \infty$.

PROOF. Since T is directed, $\{x \in T | x \ge s\} \cap \{x \in T | x \ge t\} \ne \varnothing\ \forall\, s \ne t \in K_n$; (1.4) therefore implies $|K_n^{\max}| = 1$, say $K_n^{\max} = \{1_n\}$. If X is a special M-process and $s \in K_n$, then

$$X_s = E\left[\sum_{t \in K_n,\, t \ge s} D_t^{n, \ge} X \mid \mathscr{F}_s\right] = E\left[X_{1_n} \mid \mathscr{F}_s\right]$$

implies that X is a martingale. Conversely, if X is a martingale, then X is L^1 right continuous, and

$$\begin{aligned} E\left[D_s^{n, \ge} X \mid \mathscr{F}_s\right] &= \sum_{t \in K_n,\, t \ge s} \mu^n(s,t) E\left[X_t \mid \mathscr{F}_s\right] \\ &= X_s\left(\mu^n(s, 1_n) + \sum_{t \in K_n,\, s \le t < 1_n} \mu^n(s,t)\right) \\ &= \begin{cases} X_s, & \text{if } s = 1_n, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where in the last step we have merely used (1.8). In particular, if X is a martingale or a special M-process, then

$$\begin{aligned} \operatorname{var} X &= \sup_{n \in \mathbb{N}} \sum_{s \in K_n^{\max}} \|X_s\|_1 \\ &= \sup_{n \in \mathbb{N}} \|X_{1_n}\|_1 = \sup_{t \in T} \|X_t\|_1. \ \square \end{aligned}$$

If T is not directed, the two concepts of special M-processes and L^1-bounded martingales are not equivalent. In particular, if X is a special M-process and $t \in T$, $(X_{t_n})_{n \in \mathbb{N}}$ does not necessarily have to be a reversed martingale. But since $(X_{t_n})_{n \in \mathbb{N}}$ is still a reversed quasimartingale, X has a modification satisfying (2.4).

PROPOSITION 2.6. *Let X be a special M-process satisfying* (2.4), *and let $\tau_1 \leq \tau_2$ be two stopping times. If*

$$\lim_{n\to\infty} E\left[\sum_{s\in K_n^{\max}} |X_s| \mathbb{1}_{\{\tau_1 \leq s,\, \tau_2 \nleq s\}}\right] = 0, \tag{2.8}$$

then

$$E\left[X_{\tau_2} | \mathscr{F}_{\tau_1}\right] = X_{\tau_1}. \tag{2.9}$$

PROOF. All that is needed is to show that $(X_{\tau_i^n})_{i\in\mathbb{N}}$ is a reversed quasi-martingale, which therefore converges to X_{τ_i} a.s. and in L^1. Taking the limit as $n \to \infty$ of (2.6), the proposition then follows from (2.7) and (2.8).

Since $\mathscr{F}_{\tau_i^n}$ restricted to $\{\tau_i^n \leq s\}$ is contained in $\mathscr{F}_s$, it follows that for $m \geq n$

$$\begin{aligned} &E\left[X_{\tau_i^{n+1}} - X_{\tau_i^n} | \mathscr{F}_{\tau_i^{n+1}}\right] \\ &\quad = \sum_{s\in K_{m+1}} E\left[E\left[D_s^{m+1,\geq} X | \mathscr{F}_s\right] \mathbb{1}_{\{\tau_i^{n+1}\leq s\}\cap\{\tau_i^n \nleq s\}} | \mathscr{F}_{\tau_i^{n+1}}\right], \end{aligned} \tag{2.10}$$

and therefore

$$\begin{aligned} &\sum_{n=1}^{m} E\left[\left|E\left[X_{\tau_i^{n+1}} - X_{\tau_i^n} | \mathscr{F}_{\tau_i^{n+1}}\right]\right|\right] \\ &\quad \leq \sum_{s\in K_{m+1}} E\left[\left|E\left[D_s^{m+1,\geq} X | \mathscr{F}_s\right]\right| \left(\sum_{n=1}^{m} \mathbb{1}_{\{\tau_i^{n+1}\leq s\}\cap\{\tau_i^n \nleq s\}}\right)\right]. \end{aligned} \tag{2.11}$$

For s fixed, $(\{\tau_i^{n+1} \leq s, \tau_i^n \nleq s\})_{n\in\mathbb{N}}$ are disjoint sets, so that

$$\begin{aligned} &\sup_{m\in\mathbb{N}} \sum_{n=1}^{m} \left\| E\left[X_{\tau_i^{n+1}} - X_{\tau_i^n} | \mathscr{F}_{\tau_i^{n+1}}\right] \right\|_1 \\ &\quad \leq \sup_{m\in\mathbb{N}} \sum_{s\in K_{m+1}} E\left[\left|E\left[D_s^{m+1,\geq} X | \mathscr{F}_s\right]\right|\right] \leq \operatorname{var} X < \infty \ \square \end{aligned} \tag{2.12}$$

REMARK 2.7. If T is a directed set, fix n_0 and note that $\{\tau_i \leq 1_{n_0}\} \equiv \{\tau_i^n \leq 1_{n_0}\} \in \mathscr{F}_{\tau_i^n}$ $\forall\, n \geq n_0$. (2.10) then simplifies to

$$\begin{aligned} &E\left[\left(X_{\tau_i^{n+1}} - X_{\tau_i^n}\right)\mathbb{1}_{\{\tau_i \leq 1_{n_0}\}} | \mathscr{F}_{\tau_i^{n+1}}\right] \\ &\quad = E\left[X_{1_{n+1}} \mathbb{1}_{\{\tau_i^{n+1}\leq 1_{n+1}\}\cap\{\tau_i^n \nleq 1_{n+1}\}} | \mathscr{F}_{\tau_i^{n+1}}\right] \mathbb{1}_{\{\tau_i \leq 1_{n_0}\}} \\ &\quad = 0 \end{aligned} \tag{2.10b}$$

since $\{\tau_i^n \nleq 1_{n+1}\} \cap \{\tau_i^n \leq 1_{n_0}\} = \varnothing$. Therefore $(X_{\tau_i^n}\mathbb{1}_{\{\tau_i \leq 1_{n_0}\}})_{n\geq n_0}$ is a reversed martingale $\forall\, n_0 \in \mathbb{N}$. The assumption $\operatorname{var} X < \infty$ of Proposition 2.6 can then be weakened to $X_{\tau_2} \in L^1$ to obtain Theorem 2.15 of Kurtz [19].

To obtain a dual version of the preceding results, involving $D^{n,\leq}$ rather than $D^{n,\geq}$, set

$$\mathscr{F}_s^{\,n} := \sigma\{\mathscr{F}_t | t \in K_n, t \ngeq s\} \tag{2.13}$$

for $s \in K_n$, $\mathscr{F}_s^{\,n} := \{\phi, \Omega\}$ if s is the smallest element in K_n; $\mathscr{F}_s^{\,n}$ can be regarded as the "wide past" of s with respect to K_n.

DEFINITION 2.8. An adapted L^1 right continuous process $X = (X_t)_{t \in T}$ is a *(lower) strong submartingale* if

$$E\left[D_s^{n, \le} X | \mathscr{F}_s^{\,n}\right] \ge 0, \qquad \forall\, s \in K_n, \quad \forall\, n \in \mathbb{N}, \tag{2.14}$$

and a *(lower) strong martingale* if

$$E\left[D_s^{n, \le} X | \mathscr{F}_s^{\,n}\right] = 0, \qquad \forall\, s \in K_n, \quad \forall\, n \in \mathbb{N}. \tag{2.15}$$

Applying the dual of (2.3) it is easily seen that a strong (sub)martingale is actually a (sub)martingale. In particular, a strong submartingale has a modification satisfying (2.4).

Examples for strong submartingales are the strong submartingales defined in [23] on the plane $T = \mathbb{R}_+^2$. Examples of strong martingales are the sums of i.i.d. random variables on a finite poset T, or the Gaussian processes with independent normally distributed increments defined in [15].

THEOREM 2.9. *Let X be a strong submartingale satisfying* (2.4) *and let $\tau_1 \le \tau_2$ be two stopping times. If τ_2 is bounded, i.e., if there exists an $n \in \mathbb{N}$ such that $\tau_2(\omega) \in \bigcup_{s \in K_n}\{t \in T | t \le s\}$ $\forall\, \omega \in \Omega$, then*

$$E\left[X_{\tau_2} | \mathscr{F}_{\tau_1}\right] \ge X_{\tau_1}. \tag{2.16}$$

If X is a strong martingale, then we have equality.

The proof is completely dual to the one of Theorem 2.3: Since for $s \in K_n$ and $F \in \mathscr{F}_s^{\,n}$ we have $F \cap \{\tau_1^n \ge s\} \in \mathscr{F}_s^{\,n}$ as well as $\{\tau_2^n \ge s\} \in \mathscr{F}_s^{\,n}$, (2.6) holds with $\le$ and $\ge$ interchanged.

REMARK 2.10. (i) If X is a strong submartingale, then $Y_n := \sum_{s \in K_n} D_s^{n, \le} X$ defines a one-dimensional submartingale with respect to $\mathscr{G}_n := \sigma\{\mathscr{F}_s | s \in K_n\}$. If $(Y_n)_{n \in \mathbb{N}}$ is uniformly integrable then (2.16) also holds for unbounded stopping times: Simply note that $(X_t)_{t \in T \cup \{1\}}$ with $X_1 := L^1 - \lim_n Y_n$ is again a strong submartingale.

(ii) A strong martingale satisfies (2.7): Applying (1.9) and then (1.7) it follows that $E[D_s^{n, \ge} X | \mathscr{F}_s] = \sum_{t \in K_n}(\sum_{r \in K_n, r \ge t, r \ge s} \mu^n(s, r)) E[D_t^{n, \le} X | \mathscr{F}_s]$. Now use $E[D_t^{n, \le} X | \mathscr{F}_s] = 0$ for $t \nleq s$, and $\sum_r \mu^n(s, r) = 0$ if $s \in K_n$ is not maximal, 1 otherwise.

3. The optional sampling theorem for stopping domains. The set of *(lower) domains*

$$\Delta^l := \left\{D \subseteq T | D = \lim_{n \to \infty} \bigcup_{s \in D} \{t \in T | t \le s_n\}\right\} \tag{3.1}$$

furnished with the inclusion order is a lattice, so that

$$K_n^l := \left\{\bigcup_{s \in I} \{t \in T | t \le s\} | I \subseteq K_n\right\}$$

satisfies (1.1) to (1.4). For $D \in \Delta^l$ define D_n by (1.5) applied to Δ^l and $(K_n^l)_{n \in \mathbb{N}}$; $\nu^l(D) := (D_n)_{n \in \mathbb{N}}$. $\mathscr{F}_{\Delta^l} = (\mathscr{F}_D)_{D \in \Delta^l}$ is defined by

$$\mathscr{F}_D := \sigma\Big(\bigcup_{s \in D} \mathscr{F}_s\Big), \qquad \forall\, D \in K_n^l \cup \{T\} \tag{3.2}$$

and by $\mathscr{F}_D = \bigcap_n \mathscr{F}_{D_n}$ otherwise.

DEFINITION 3.1. A map $\delta\colon \Omega \to \Delta^l$ is a *lower wide stopping domain* if $\{\omega | \delta(\omega) \subseteq D\} \in \mathscr{F}_D$ $\forall\, D \in \Delta^l$. The associated σ-algebra is $\mathscr{F}_\delta := \sigma\{F \in \mathscr{F} | F \cap \{\delta \subseteq D\} \in \mathscr{F}_D \;\forall\, D \in \Delta^l\}$.

REMARK 3.2. (i) A lower wide stopping domain is simply a stopping time with respect to the poset Δ^l and the σ-algebras $\mathscr{F}_{\Delta^l}$.

(ii) We use the term wide stopping domain because we do not require $\{\omega | t \in \delta(\omega)\} \in \mathscr{F}_t$ $\forall\, t \in T$.

(iii) If $\delta \equiv D$, then $\mathscr{F}_\delta = \mathscr{F}_D$.

If $T = \mathbb{N}^2$ and δ is a stopping domain in the sense of Walsh [25], then δ is also a lower wide stopping domain [the converse is not true, due to Remark 3.2(ii)]. On $T = \mathbb{R}_+^2$, if λ is a stopping line in the sense of Merzbach [2], [20], [21], [22], then $\delta(\omega) := \{t \in T | t \le s \text{ for an } s \in \lambda(\omega)\}$ is also a lower wide stopping domain.

For $D \in \Delta^l$ define X_D by

$$X_D := \sum_{s \in K_n,\, s \in D} D_s^{n,\le} X, \qquad \text{if } D \in K_n$$
$$X_D := \begin{cases} \lim_{n \to \infty} X_{D_n}, & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

THEOREM 3.3. *X is a strong (sub)martingale iff for any bounded lower wide stopping domains $\delta_1 \le \delta_2$,*

$$E\big[X_{\delta_2} | \mathscr{F}_{\delta_1}\big] = (\ge) X_{\delta_1}. \tag{3.4}$$

Here bounded means $\exists\; m \in \mathbb{N}$ such that $\delta_2(\omega) \subseteq \bigcup_{s \in K_m} \{t \in T | t \le s\}$ $\forall\, \omega \in \Omega$.

PROOF. Approximate δ_i by $(\delta_i^n)_{n \in \mathbb{N}}$ with $(\delta_i^n(\omega))_{n \in \mathbb{N}} = \nu^l(\delta_i(\omega))$; $X_{\delta_i^n} = \sum_{s \in K_n} D_s^{n,\le} X \mathbb{1}_{\{s \in \delta_i^n\}}$ is then $\mathscr{F}_{\delta_i^n}$-measurable, and (3.4) follows from $\{s \in \delta_2^n\} \in \mathscr{F}_s^{\,n}$, $F \cap \{s \notin \delta_1^n\} \in \mathscr{F}_s^{\,n}$ $\forall\; F \in \mathscr{F}_{\delta_1^n}$. The sufficiency of (3.4) is obtained by choosing δ_1 and δ_2 in such a way that $X_{\delta_2} - X_{\delta_1} = D_s^{n,\le} X \mathbb{1}_F$, $F \in \mathscr{F}_s^{\,n}$. □

REMARK 3.4. (i) If X is a strong (sub)martingale, then $(X_D)_{D \in \Delta^l}$ is a (sub)martingale with respect to $(\mathscr{F}_D)_{D \in \Delta^l}$. In particular, $\lim_n D_n$ exists a.e.

(ii) As in Theorem 2.9, δ_2 bounded can be replaced by $(Y_n)_{n \in \mathbb{N}}$ uniformly integrable.

(iii) If T is countable and $|\{t \in T | t \leq s\}| < \infty \ \forall \ s \in T$, Theorem 3.3 shows that X is a strong martingale iff $(X_D)_{D \in \Delta^l}$ is an additive martingale with respect to $\mathscr{F}_{\Delta^l}$ in the sense of Edgar [10].

The analogous results involving S-processes are only possible in a weaker form. Let the *upper domains*

$$\Delta^u := \left\{ D \subseteq T | D = \lim_{n \to \infty} \bigcup_{s \in K_n \cap D} \{t \in T | t \geq s\} \right\} \tag{3.5}$$

be furnished with the partial order $D_1 \leq D_2$ iff $D_1 \supseteq D_2$.

$$K_n^u := \left\{ \bigcup_{s \in I} \{t \in T | t \geq s\} | I \subseteq K_n \right\}$$

again satisfies (1.1) to (1.4), so let $\nu^u(D) = (D_n)_{n \in \mathbb{N}}$ denote the sequence associated with $D \in \Delta^u$. $\mathscr{F}_{\Delta^u} = (\mathscr{F}_D)_{D \in \Delta^u}$ is defined by $\mathscr{F}_D = \bigcap_{s \in D} \mathscr{F}_s$.

An *upper stopping domain* is a map $\delta \colon \Omega \to \Delta^u$ satisfying $\{\omega | t \in \delta(\omega)\} \in \mathscr{F}_t$ $\forall \ t \in T$; the associated σ-algebra is $\mathscr{F}_\delta := \{F \in \mathscr{F} | \mathscr{F} \cap \{\omega | t \in \delta(\omega)\} \in \mathscr{F}_t \ \forall \ t \in T\}$. If τ is a T valued stopping time, then $\delta_\tau(\omega) := \{t \in T | t \geq \tau(\omega)\}$ defines an upper stopping domain with $\mathscr{F}_{\delta_\tau} = \mathscr{F}_\tau$ (a result that has no analog for lower wide stopping domains).

For $D \in K_n^u$ define $X_D := \sum_{s \in K_n \cap D} D_s^{n, \geq} X$. Since X_D is not necessarily $\mathscr{F}_D$-measurable, X_δ is not necessarily $\mathscr{F}_\delta$-measurable either. Therefore the analog to Proposition 3.3 becomes:

PROPOSITION 3.5. *Let X be an adapted L^1 right continuous process with $\operatorname{var} X < \infty$. Then X is an S-process iff*

$$F\left[X_{\delta_1} - X_{\delta_2} | \mathscr{F}_{\delta_2}\right] \geq 0 \tag{3.6}$$

for all upper stopping domains $\delta_1 \leq \delta_2$ assuming only values in K_m^u for some $m \in \mathbb{N}$, and X is a special M-process iff equality holds in (3.6) *for all such upper stopping domains satisfying $K_m^{\max} \in \delta_2(\omega) \ \forall \ \omega \in \Omega$.*

So if X is an S-process and $D \in \Delta^u$, we can only define the projection

$$X_D^p := \begin{cases} \lim_{n \to \infty} E\left[X_{D_n} | \mathscr{F}_{D_n}\right], & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

For any two stopping domains $\delta_1 \leq \delta_2$ the usual arguments then show that $E[X_{\delta_1}^p - X_{\delta_2}^p | \mathscr{F}_{\delta_1}] \geq 0$, with equality if X is a special M-process and δ_2 is bounded.

REMARK 3.6. In [13], [14] we constructed a positive finite measure P^X associated with the S-process X by means of

$$P^X\left[\{t \in S | t \geq s\} \times F\right] = E\left[X_s; F\right], \qquad \forall \, F \in \mathscr{F}_s, \quad \forall \, s \in T,$$

where S is a certain completion of T, $T \subseteq S$. If τ is a stopping time and δ an upper stopping domain, denote

$$(\tau, 1] \times F := \bigcup_{n \in \mathbb{N}} \{(x, \omega) | \omega \in F, x \geq \tau^n(\omega)\},$$

$$(\delta, 1] \times F := \bigcup_{n \in \mathbb{N}} \{(x, \omega) | \omega \in F, x \in \delta^n(\omega)\}.$$

It is then possible to prove that

$$P^X[(\tau, 1] \times F] = E[X_\tau; F], \qquad \forall F \in \mathscr{F}_\tau,$$

$$P^X[(\delta, 1] \times F] = E[X_\delta; F], \qquad \forall F \in \mathscr{F}_\delta,$$

from which the various OST can also be deduced. Similarly for strong submartingales.

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