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RECURRENT SETS FOR TRANSIENT LÉVY PROCESSES WITH BOUNDED KERNELS

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In the study of recurrent sets for transient Lévy processes on the real line, we present two main results. As long as the process has a "well-behaved" (bounded in a particular way) kernel, a set is recurrent for the process if and only if the sum of the capacities of pieces of the set is infinite. In the second result, we show that a simple condition on the Lévy measure guarantees that the process has a "well-behaved" kernel. Finally, the results are applied to subordinators in order to construct examples of recurrent sets including a recurrent set with finite Lebesgue measure.

1. Introduction. Past investigations of recurrent sets for transient random walks have yielded some rather complete results. Bretagnolle and Dacuhna-Castelle (1967) showed that if a random walk on the real line is nonsingular (i.e., the distribution of X has a nonzero absolutely continuous part), and if $EX_1 > 0$ with $E|X_1| < \infty$, then a set B is recurrent (i.e., hit infinitely often) if and only if $\lambda(B \cap (0, \infty)) = \infty$ where λ is Lebesgue measure. This result does not directly generalize to continuous processes. For these processes, infinite Lebesgue measure is still sufficient for recurrence since the set will be recurrent for some imbedded random walk (i.e., $\{X_{nh}\}_{n=1}^{\infty}$). However, infinite Lebesgue measure is not necessary for recurrence (even if the process has no linear or Gaussian component—see the example in Section 6). One of the two main results in this paper gives a necessary and sufficient condition for recurrence under the assumption that the continuous process has a bounded kernel. In order to make this result useful, the second main result gives a criterion which guarantees a bounded kernel. This second result is of independent interest in investigating the behavior of kernels.

To be more precise, let $X = \{X_t, t \geq 0\}$ be a real-valued stochastic process with stationary independent increments (i.e., a Lévy process). Then the characteristic function of X_t, denoted $\phi_t(u)$, is $E^0 e^{iuX_t} = \exp\{-t\psi(u)\}$ where

$$\psi(u) = -iua + \left(\frac{\sigma^2}{2}\right)u^2 + \int\left(1 - e^{iux} + \frac{iux}{1 + x^2}\right)\nu(dx).$$

ψ is called the exponent of the process, and the measure ν is the Lévy measure which satisfies $\int |x|^2/(1 + |x|^2)\nu(dx) < \infty$. We assume that a version of X has been chosen so that the sample paths are right continuous with left limits. We also assume that $\nu(R - \{0\}) = \infty$, since otherwise the analysis of recurrent sets is best done using random walk methods. Without loss of generality, we may assume further that $\nu(0, \infty) = \infty$. It follows from $\nu(0, \infty) = \infty$ that

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$P[T_{(r,r+\varepsilon)} < \infty] > 0$ where $r > 0$, $\varepsilon > 0$, and $T_{(r,r+\varepsilon)} = \inf[t\colon X_t \in (r, r+\varepsilon)]$. The measure P^x is the distribution of the process when starting at x (i.e., $X(0) = x$) and E^x is the expectation with regard to the measure P^x. When $x = 0$, we drop the superscripts. Finally, let $U(0, A) = E^0 \int_0^\infty I_A(x_t)\,dt$ for all Borel sets A. The measure U is called the potential measure for the process, and we write $U(A)$ for $U(0, A)$.

A transient Lévy process is one such that $\liminf_{t\to\infty} |X_t| = \infty$ a.s., or, equivalently, for every compact B, $\lim_{t\to\infty} P^x[X_s \in B$, some $s > t] = 0$. For a Borel set B, let $T_B = \inf\{t > 0\colon X_t \in B\}$. Then we have the following definition.

Definition. A recurrent set is a Borel set such that $P^x[T_B < \infty] = 1$ a.e. x. If a set is not recurrent, it is transient.

Without loss of generality, we can restrict ourselves to considering the recurrence or transience of only those sets contained in $[0, \infty)$.

Definition. A partition is a collection of sets $\{S_n\}_1^\infty$ such that $S_n = \{x\colon a_n \le x_n < a_{n+1}\}$ where $a_n \ge 0$, $\{a_n\}_1^\infty$ is an increasing sequence with $a_n \to \infty$.

Definition. A partition $\{S_n\}_1^\infty$ is said to have nonzero mesh if $\inf_n(a_{n+1} - a_n) > \delta > 0$.

If we let $B_n = B \cap S_n$, it can be shown that B is recurrent if

$$P^x\left[\limsup_n \left[T_{B_n} < \infty\right]\right] = 1 \text{ a.e.}$$

The Borel–Cantelli lemma would supply a necessary and sufficient condition for recurrence if the sets $\{T_{B_n} < \infty]$ were independent. Unfortunately, they are generally not independent. A condition on the kernel of the process, however, does allow us to use an extension of the Borel–Cantelli lemma.

Recall that a process has a kernel u if and only if the potential measure U is absolutely continuous with respect to Lebesgue measure. In that event, $U(A) = \int_A u(x)\,dx$ where u can be chosen so $x \to u(-x)$ is an excessive function.

Definition. A kernel is well-behaved if it is bounded on $R \setminus (-a, a)$ and bounded away from zero on (a, ∞).

(Note that the definition of well-behaved is based on the fact that we are restricting attention to recurrent sets in $[0, \infty)$.)

From capacity theory for processes with stationary independent increments, we know that if $\bar{A}$ is a compact Borel set,

$$P^y[T_A < \infty] = \int_{-A} u(-y-x)\mu_{-A}(dx),$$

where u is the kernel of the process and μ_{-A} is the capacitary measure (finite) supported by $-\bar{A}$. Also, $C(B) = \mu_B(\bar{B})$ is the capacity of B. Now we can state the main results of this paper.

THEOREM 1. *Let X be a transient Lévy process with a well-behaved kernel. Let $\{S_n\}_1^\infty$ be a partition of nonzero mesh, and set $B_n = B \cap S_n$ where $B \subset (0, \infty)$ is a Borel set. Then the following are equivalent*:

(i) *B is recurrent for X.*

(ii) $\sum_1^\infty P^x[T_{B_n} < \infty] = \infty$.

(iii) $\sum_1^\infty C(B_n) = \infty$.

NOTE. Condition (iii) is interesting when compared to Lamperti's results (1963). It follows from Lamperti's arguments that for stable subordinators of index α, $0 < \alpha < 1$, if we set $B_n = \{2^n \le x < 2^{n+1}\} \cap B$, then B is recurrent if and only if $\sum_1^\infty C(B_n)/(2^n)^{1-\alpha} = \infty$.

In order to apply the theorem we need some way of determining if a process has a well-behaved kernel. Unfortunately, even the existence of bounded, continuous densities does not guarantee the behavior (see Section 4). Since the Lévy measure for a process is often known, the following result is useful and of interest when studying kernels by themselves. The most interesting case is the one where the process does not have a Gaussian component, so we make that assumption.

THEOREM 2. *Let X be a transient Lévy process with positive finite mean and Lévy measure ν. Suppose ν has a density g which is bounded on $R - \{(-\varepsilon, \varepsilon)\}$ and satisfies,*

$$|g(x) - C_1 x^{-1-\alpha}| < \delta \quad \text{on } (0, \varepsilon),$$

$$|g(x) - C_2 x^{-1-\alpha}| < \delta \quad \text{on } (-\varepsilon, 0),$$

where $C_1 > 0$, $C_2 > 0$, and $0 < \alpha < 1$. Then X has a well-behaved kernel.

NOTE. Theorem 2 covers many processes that do not hit points. Results of Kesten (1969) give criteria for a process to hit points, and Bretagnolle (1971) showed that these processes have bounded kernels. By showing that the kernel is bounded away from zero on (a, ∞), it can be established that the kernel is well-behaved.

We proceed with the proofs of Theorem 1 and Theorem 2 by first dispensing with some preliminary results in Section 2. Section 3 contains the proof of Theorem 1. Sections 4 and 5 present the proof of Theorem 2. Finally, in Section 6, we give some illustrative examples of recurrent sets for certain subordinators. With subordinators, the calculation of $C(B_n)$ is often tractable, so determining recurrence is often straightforward.

2. Preliminaries on recurrent sets. Port and Stone (1971) established the basic characterization leading to the definition of recurrent set.

THEOREM (PORT AND STONE). *Let B be a Borel set. Then, either $P^x[T_B < \infty] = 1$ a.e. x ($T_B = \inf\{t > 0\colon X_t \in B\}$) or $\lim_{t\to\infty} P^x[X_s \in B$ for some $s > t] = 0$ a.e. x.*

In the above theorem and in the first proposition below, "almost everywhere" can be replaced with "for all x" if the process is nonsingular (i.e., for some $t > 0$, the distribution of X_t has a nonzero absolutely continuous part).

PROPOSITION 1. *The set $M = \{\omega\colon [t\colon X_t \in B]$ is unbounded$\}$ is measurable and $P^x(M) = 0$ almost everywhere x or $P^x(M) = 1$ almost everywhere x. B is recurrent if and only if $P^x(M) = 1$* a.e. x.

PROOF. Let $T_n = \inf\{t > n\colon X_t \in B\}$. Since $\{[t\colon X_t \in B]$ unbounded$\} = \bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}[T_k < \infty]$, measurability is established. Assume B is recurrent. Port and Stone (1971) showed that $P^x[X_t \in B$ some $t > s] = 1$ for all s and a.e. x. Hence, $P^x[T_k < \infty] = 1$ for all k and a.e. x. Thus, $P^x[\omega\colon [t\colon X_t \in B]$ is bounded$] = 1$ a.e. x. If B is transient, let $\{t_n^x\}$ be an increasing sequence such that, for x outside an exceptional set of measure zero, $P^x[X_t \in B$ for some $t > t_n^x] < 2^{-n}$. Then

$$\sum_n P^x\left[X_t \in B \text{ for some } t > t_n^x\right] < \infty \text{ a.e. } x,$$

and by the Borel–Cantelli lemma,

$$P^x\left[\omega\colon [t\colon X_t \in B] \text{ is unbounded}\right] = P^x\left[\limsup_n \left[X_t \in B \text{ for some } t > t_n\right]\right] = 0 \text{ a.e. } x. \square$$

For the next proposition, let $\{S_n\}_1^\infty$ be a partition and set $B_n = B \cap S_n$. Let $T_{B_n} = \inf\{t > 0\colon X_t \in B_n\}$.

PROPOSITION 2. *For a transient process X,*

$$\{\omega\colon [t\colon X_t \in B] \textit{ unbounded}\} = \left\{\omega\colon \left[t_{B_n} < \infty \textit{ infinitely often}\right]\right\} = \limsup_n \left[T_{B_n} < \infty\right] \textit{ almost surely.}$$

PROOF. The result follows from the fact that almost all paths of a Lévy process are bounded on finite time intervals, and the fact that for transient processes, $\liminf_{t\to\infty}|X_t| = \infty$. $\square$

The above two propositions show that we can determine recurrence or transience by finding the probability of $\limsup_n[T_{B_n} < \infty]$.

3. Proof of Theorem 1. The first step in the proof is to show that $\sum_1^\infty P^x[T_{B_n} < \infty] = \infty$ a.e. x is a necessary and sufficient condition for recurrence. Then an argument from capacity theory will show that $\sum_1^\infty C(B_n) = \infty$ is an equivalent condition so the qualification "a.e. x" can be removed.

According to Propositions 1 and 2, a set B is recurrent if $P^x[\limsup_n\{T_{B_n} < \infty\}] = 1$ a.e. x. By the Borel–Cantelli lemma, it follows that $\sum_1^\infty P^x[T_{B_n} < \infty] = \infty$ a.e. x. This establishes the necessity.

To show sufficiency, we will use a well-known extension of the Borel–Cantelli lemma. This result states that if $\Sigma_1^\infty P[A_n] = \infty$ and $P[A_n A_m] \le KP[A_n]P[A_m]$ for $n, m > M$, then $P[\limsup_n A_n] > 0$. In our case, we must show that $P^x[\limsup_n [T_{B_n} < \infty]] = 1$ a.e. x. From Propositions 1 and 2, we know this probability is either zero or one. Thus, we need only show it is greater than zero.

Since the kernel is well-behaved, there exist $z_0 > 0$ and $z_1 < 0$ such that if $z > z_0$, then $K_1 < u(z) < K_2$, and if $z < z_1$, then $u(z) < K_3$. The constants K_1, K_2, K_3 are all positive.

Let $z^* = \max\{z_0, |z_1|\}$. Since $\{S_n\}_{n=1}^\infty$ is a partition of nonzero mesh, $\inf_n(a_{n+1} - a_n) > \delta > 0$. Let k be an integer such that $k > z^*/\delta$. For the sufficiency, we assume $\Sigma_1^\infty P^x[T_{B_n} < \infty] = \infty$ a.e. x. We rewrite this sum in blocks of k summands to get, $\Sigma_{m=0}^\infty \Sigma_{p=1}^k P^x[T_{B_{mk+p}} < \infty]$. Since the sum is infinite, one of the sums $\Sigma_{m=0}^\infty P^x[T_{B_{mk+p}} < \infty]$, $p = 1, 2, \ldots, k$, is infinite. Given x, we choose the appropriate infinite sum, and consequently may suppose that we have sets $\{T_{B_r} < \infty\}$ where the distance between B_r and B_s, $r \ne s$, is greater than z^*, and $\Sigma_{r=1}^\infty P^x[T_{B_r} < \infty] = \infty$.

Consider the following inequality,

$$(\mathrm{A}) \qquad P\left[T_{B_r} < \infty, T_{B_s} < \infty\right] \le \int_{[T_{B_r} < \infty]} P^{X_{T_{B_r}}}\left[T_{B_s} < \infty\right] dP + \int_{[T_{B_s} < \infty]} P^{X_{T_{B_s}}}\left[T_{B_r} < \infty\right] dP.$$

We proceed to estimate the right-hand side of this inequality. Note first that if $r < s$, then $z \in B_r$, $y \in B_s$ implies $z < y$.

From capacity theory,

$$P^y[T_A < \infty] = \int_{-A} u(-y - z)\mu_{-A}(dz),$$

where μ_{-A} is the capacitary measure. If $-y - z > z_0$, then

$$K_1 C(-A) \le P^y[T_A < \infty] \le K_2 C(-A).$$

From the definition of sets B_r, $y \in B_r$ and $z \in -B_s$ $(r < s)$ implies $-y - z > z^* > z_0$. Further, if s is large enough $(s > r_0)$ then $P[T_{B_s} < \infty] \ge K_1 C(-B_s)$. Thus

$$P^y\left[T_{B_s} < \infty\right] \le K_2 K_1^{-1} P\left[T_{B_s} < \infty\right]$$

whenever $y \in B_r$, $r_0 < r < s$. Now, $X_{T_{B_r}} \in \overline{B}_r$, and therefore the first term on the right of (A) is bounded above by $K_2 K_1^{-1} P[T_{B_r} < \infty] P[T_{B_s} < \infty]$.

Return again to the capacity equality. If $-y - z < z_1 < 0$, then

$$P^y[T_A < \infty] \le K_3\, C(-A).$$

From the definition of sets B_r, if $r < s$, $y \in B_s$, and $z \in -B_r$, we have $-y - z < -z^* < z_1$. Thus, for $y \in B_s$ and $r > r_0$,

$$P^y\left[T_{B_r} < \infty\right] \le K_3 C(-B_r) \le K_3 K_1^{-1} P\left[T_{B_r} < \infty\right].$$

Since $X_{T_{B_s}} \in \overline{B}_s$, the second term on the right of (A) is bounded above by $K_3 K_1^{-1} P[T_{B_r} < \infty] P[T_{B_s} < \infty]$. Consequently, for all r, s sufficiently large,

$$P\left[T_{B_r} < \infty, T_{B_s} < \infty\right] \le (K_2 + K_3) K_1^{-1} P\left[T_{B_r} < \infty\right] P\left[T_{B_s} < \infty\right].$$

Finally, observe that given x, we can choose $r_0(x)$ sufficiently large so that $r > r_0(x)$, $s > r_0(x)$ implies

$$P^x\left[T_{B_r} < \infty, T_{B_s} < \infty\right] \le (K_2 + K_3) K_1^{-1} P^x\left[T_{B_r} < \infty\right] P^x\left[T_{B_s} < \infty\right].$$

This establishes the Borel–Cantelli extension, and hence

$$P^x\left[\limsup\left[T_{B_n} < \infty\right]\right] > 0$$

holds for every x for which $\Sigma_i^\infty P^x[T_{B_n} < \infty] = \infty$. Hence

$$P^x\left[\limsup\left[T_{B_n} < \infty\right]\right] = 1$$

a.e. x and B is recurrent.

Finally, if n is sufficiently large, $-y - z > z_0$ for $z \in -B_n$ and y fixed. Hence,

$$K_1 C(-B_n) \le P^y\left[T_{B_n} < \infty\right] \le K_2 C(-B_n).$$

Since $C(B_n) = C(-B_n)$, $\Sigma_1^\infty P^y[T_{B_n} < \infty] = \infty$ if and only if $\Sigma_1^\infty C(B_n) = \infty$. This completes the proof. □

Corollary 1. *Under the hypotheses of Theorem* 1, *let* $B = \bigcup_1^\infty B_n$ *be a recurrent set where* $B_n = B \cap S_n$. *Set* $B' = \bigcup_1^\infty B_n'$ *where* $B_n' = B_n + x_n$ *and* $\{x_n\}_1^\infty$ *is a sequence such that* $B_n' \cap B_m' = \varnothing$ *if* $n \ne m$. *If there exists a partition of nonzero mesh*, $\{S_n'\}$, *such that* $B_{n_k}' = S_n' \cap B'$, *then* B' *is recurrent.*

Proof. $C(B_n') = C(B_n + x_n) = C(B_n)$. □

The corollary shows that suitable rearrangements of a recurrent set are still recurrent for processes with well-behaved kernels. In particular, the corollary holds for nondecreasing sequences, $\{x_n\}_1^\infty$, of positive numbers. In this case, the distances between successive B_n are made larger and larger—B is "spread out." Similarly, if $\{x_n\}_1^\infty$ is a sequence of negative numbers that insures the existence of the partition $\{S_n'\}$, then the resulting "compressed" set is still recurrent. □

4. Preliminaries on kernels. If u is a kernel for a process, then $x \to u(-x)$ is excessive and $Uf(x) = \int f(z) u(z - x)\, dx$ for all bounded Borel f (where by definition, $Uf(x) = \int e^{-t} E^x f(x_t)\, dt$). A kernel exists if and only if the potential measure is absolutely continuous with respect to Lebesgue measure. In that case, $U(A) = \int_A u(x)\, dx$. Finally, if X_t has a density f_t, then there exists a version $\tilde{f}_t$ of f_t such that $u(x) = \int_0^\infty \tilde{f}_t(x)\, dt$.

The next proposition shows that we can use the renewal theorem for random walks to investigate the potential measure for continuous processes. Recall that if

F_t is the distribution of X_t, and we write $U(x)$ for $U([0,x))$, then

$$U(x) = E^0 \int_0^\infty I_{[0,x)}(X_t)\,dt$$
$$= \int_0^\infty P[X_t \in [0,x)]\,dt = \int_0^\infty F_t(x)\,dt.$$

Proposition 3. *Let* $F(x) = c\int_0^\infty e^{-ct}F_t(x)\,dt$, $c > 0$. *Then* $U(x) = c^{-1}\sum_{n=1}^\infty F^{*n}(x)$ *where* F^{*n} *is the nth-fold convolution of* F.

Proof. Let $\{Y_n\}$ be independent, identically distributed random variables with distribution $F_Y(x) = 1 - e^{-cx}$. Then

$$P[X(Y_1) \le x] = c\int_0^\infty e^{-cy}P[X(y) \le x]\,dy = F(x).$$

Let $Z_n = Y_1 + Y_2 + \cdots + Y_n$. Then Z_n is a gamma random variable with density $e^{-cx}(cx)^{n-1}c/(n-1)!$. Now $X(Z_n) = \sum_1^n[X(Z_r) - X(Z_{r-1})]$ where $X(Z_r) - X(Z_{r-1})$ has the same distribution as Y_r, which is just F. Hence $X(Z_n)$ has distribution F^{*n}.

Also,

$$P[X(Z_n) \le x] = \int_0^\infty \frac{e^{-cy}(cy)^{n-1}c}{(n-1)!}F_y(x)\,dy.$$

Therefore,

$$F^{*n}(x) = \int_0^\infty \frac{e^{-cy}(cy)^{n-1}c}{(n-1)!}F_y(x)\,dy.$$

Finally,

$$\sum_{n=1}^\infty F^{*n}(x) = \int_0^\infty \sum_1^\infty \frac{e^{-cy}(cy)^{n-1}c}{(n-1)!}F_y(x)\,dy$$
$$= c\int_0^\infty F_y(x)\,dy. \square$$

Notice that if F_t has finite expectation $\mu > 0$, then F has finite positive expectation. Now we turn to some refinements of the renewal theorem established by Feller (1971).

Proposition 4. *Let the distribution* F *have finite expectation* $\mu > 0$, *and density* f. *If* f^{*n} *is bounded for some* n, *then* $U = \sum_{n=1}^\infty F^{*n}$ *has a density* u *such that* $u - f - f^{*2} - \cdots - f^{*n} \to \mu^{-1}$ *as* $x \to \infty$, *and* $u - \cdots - f^{*n} \to 0$ *as* $x \to -\infty$.

PROOF. $u - f - f^{*2} - \cdots - f^{*n}$ is a solution to Feller's renewal equation, so Feller's results prove the proposition. □

If the process X has a Gaussian component ($\sigma^2 > 0$), then the distribution F defined in Proposition 3 has a bounded, continuous density f. It is then easy to see from Proposition 4 that the kernel is well-behaved. Based on these cases, we might conjecture that any process with bounded, continuous density has a well-behaved kernel. However, the following example shows that the situation is more delicate than at first glance.

EXAMPLE. Let $X = \{X_t\}$ be a symmetric stable process of index $\alpha < 1$. Its characteristic function is $e^{-t|u|^\alpha}$, so if f_t is the density of X_t, $f_t(0) = \int_{-\infty}^{\infty} e^{-t|u|^\alpha}\,du = K_0 t^{(-1/\alpha)}$ by the Fourier inversion theorems. Let $Y = \{Y_t\}$ be a compound Poisson process with Lévy measure $\nu = \sum_1^\infty r_s \varepsilon_{x_s}$ where ε_{x_s} is the measure with unit mass at x_s and $\{x_s\}$ is a sequence such that $x_s \to \infty$. Now $Z_t = X_t + Y_t$ has a density h_t and $h_t(x) = \int_{-\infty}^{\infty} f_t(x-y) P[Y_t \in dy]$. Thus, $\sup_x h_t(x) \le \sup_x f_t(x) \le f_t(0) < \infty$. So for $\alpha < 1$, $Z = \{Z_t\}$ is transient and has bounded, continuous density. However,

$$\begin{aligned} h_t(x_k) &\ge P[Y_t = x_k]\, f_t(0) \\ &\ge P[Y \text{ has one jump of size } x_k \text{ and no other jumps}]\, f_t(0) \\ &\ge e^{-t\Sigma r_s} t r_k f_t(0). \end{aligned}$$

If $\sum_1^\infty r_s < \infty$, then $h_t(x_k) \ge K_1 t^{1-(1/\alpha)}$ near $t = 0$ and $\int_0^\delta h_t(x_k)\,dt = \infty$ for $\alpha < \frac{1}{2}$. This shows that the kernel is unbounded near every x_k. [This example was presented by Kanda (1975) in greater generality.]

The procedure for showing that a process has a well-behaved kernel will be the following. First we will show that under appropriate hypotheses, the distribution F of Proposition 3 has a density that is bounded on (R, ∞). This implies f^{*k} is bounded on (R_k, ∞). Next we show that f^{*n} is bounded everywhere for some n, and then Proposition 4 establishes that the density is well-behaved.

There is one technical detail that we will dispense with first. The density of the potential measure may not be a kernel.

PROPOSITION 5. *Suppose the potential measure U has a density g that is well-behaved. Then the process has a kernel that is well-behaved.*

PROOF. Since g is well-behaved, $g(x) < K_1$ for $|x| > x^* > 0$ and $g(x) > K_2$ for $x > x^*$. It is known that since U has a density it has a kernel u such that $u(-x)$ is excessive and $u = g$ a.e. Further, since U has a density, all excessive functions are lower semicontinuous. In particular, $u(-x)$ is lower semicontinuous, and therefore $u(x) < K_1$ for $|x| > x^* > 0$.

Take $x_0 > x^*$. Then $g(x_0) > K_2$. Choose a nonnegative continuous function f with compact support in $(-\infty, -x^*)$, with $f \le 1$, and with $f(-x_0) = 1$. Let $\hat{u}(x) = u(-x)$ and $\hat{g}(x) = g(-x)$. For excessive functions h, $h(x) =$

$\lim_{\lambda\to\infty}\lambda U^\lambda h(x)$, and therefore we have,

$$\begin{aligned} u(x_0) = \hat{u}(-x_0) &= \lim_{\lambda\to\infty}\lambda U^\lambda \hat{u}(-x_0) \\ &\geq \liminf_{\lambda\to\infty}\lambda U^\lambda(f\hat{u})(-x_0) \\ &= \liminf_{\lambda\to\infty}\lambda U^\lambda(f\hat{g})(-x_0) \\ &\geq K_2 \liminf_{\lambda\to\infty}\lambda U^\lambda f(-x_0) \\ &\geq K_2 f(-x_0) = K_2. \end{aligned}$$

Since x_0 is arbitrary, $u(x) \geq K_2$ for $x > x^*$ and u is well-behaved.

5. Proof of Theorem 2. We first need two lemmas.

LEMMA 1. *Let $Y = \{Y_t\}$ be a process with density g_t and kernel u_Y such that $u_Y(x) < K_1$ for $|x| > M$. Let $\tilde{Y} = \{\tilde{Y}_t\}$ be a process derived from Y_t by deleting all jumps bigger than J. Then, there exists $\lambda > 0$ such that*

$$u_{\tilde{Y}}^\lambda(x) = \int_0^\infty e^{-\lambda t}\tilde{g}_t(x)\,dt < K_1 \quad \textit{for } |x| > M.$$

PROOF.

$$\begin{aligned} P[Y_t \in A] &\geq P[\tilde{Y}_t \in A \text{ and no jumps bigger than } J \text{ occur up to time } t] \\ &= P[\tilde{Y}_t \in A]e^{-\nu(J,\infty)t} \end{aligned}$$

This shows $\tilde{Y}_t$ has a density $\tilde{g}_t$ and

$$g_t(x) \geq \tilde{g}_t(x)e^{-\lambda t} \quad (\lambda = \nu(J,\infty))$$

which implies $u_Y(x) \geq u_{\tilde{Y}}^\lambda(x)$. □

LEMMA 2. *Let $X = \{X_t\}$ have a density f_t^x and a λ-kernel u_x^λ such that $u_x^\lambda(x) < K_1$ for $|x| > M_1$. Let $Y = \{Y_t\}$ be a compound Poisson process with Lévy measure ν. Suppose that ν has a bounded density p. Then if $Z_t = X_t + Y_t$, $Z = \{Z_t\}$ is a process which has a λ-kernel u_z^λ such that $u_z^\lambda(x) < K_2$ for $|x| > M_2$.*

PROOF. Let f_t^z be the density of Z_t and denote the distribution of Y_t by F_t^Y.

$$u_z^\lambda(x) = \int_0^\infty e^{-\lambda t} f_t^z(x)\,dt = \int_0^\infty e^{-\lambda t}\int_{-\infty}^\infty f_t^X(x-y)F_t^Y(dy)\,dt.$$

Now, $F_t^Y = e^{-\nu(R)t}\varepsilon_0 + e^{-\nu(R)t}\sum_{k=1}^\infty((\nu(R)t)^k/k!)F^{*k}$ where $F(dx) = \nu(dx)/\nu(R)$, so F has a bounded density equal to $P(x)/\nu(R)$. Since each F^{*k} has a density bounded by the same bound, $F_t^Y - e^{-\nu(R)t}\varepsilon_0$ has a bounded density. Call this density f_t^Y.

$$\begin{aligned}
u_z^\lambda(x) &= \int_0^\infty e^{-\lambda t} \int_{-\infty}^\infty f_t^X(x-y)\left[e^{-\nu(R)t}\varepsilon_0(dy) + f_t^Y(y)\,dy\right] dt \\
&= \int_0^\infty e^{-\lambda t} e^{-\nu(R)t} f_t^X(x)\,dt + \int_{-\infty}^\infty e^{-\lambda t} \int_{-\infty}^\infty f_t^X(x-y) f_t^Y(y)\,dy\,dt \\
&\le u_x^\lambda(x) + \int_0^\infty e^{-\lambda t} M \int_{-\infty}^\infty f_t^X(x-y)\,dy\,dt \\
&\le u_x^\lambda(x) + M/\lambda.
\end{aligned}$$

This establishes the bound on u_z^λ. □

Before proving the full theorem, we prove the following restricted version.

RESTRICTED THEOREM 2. *Let $X = \{X_t\}$ be a transient process with finite mean and Lévy measure ν. Suppose ν has a density g on $R - \{0\}$ that is bounded on $R - \{(-\varepsilon, \varepsilon)\}$. Suppose $g(x) = C_1 x^{-1-\alpha}$ on $(0, \varepsilon)$ and $g(x) = C_2|x|^{-1-\alpha}$ on $(-\varepsilon, 0)$ where $C_1 \ge 0$, $C_2 \ge 0$, $0 < \alpha < 1$. Then X has a well-behaved kernel.*

PROOF. X can be thought of as a stable process that is altered by first deleting jumps bigger than ε and then adding a compound Poisson process with a Lévy measure that has a bounded density. By Lemmas 1 and 2, X has a λ-kernel which is bounded for $|x| > M$.

Now let f_t be the density of X_t. [Since ν has a density, X_t has a density (Chung, 1970)]. So far we have,

$$u^\lambda(x) = \int_0^\infty e^{-\lambda t} f_t(x)\,dt < K \quad \text{for } |x| > M.$$

λu^λ is a density for $F(x) = \lambda \int_0^\infty e^{-\lambda t} F_t(x)\,dt$. By Propositions 3 and 4, we need to show that f^{*n} is bounded where $f = u^\lambda$.

The Fourier transform of f is $(\lambda + \psi(u))^{-1}$ so the transform of f^{*n} is $(\lambda + \psi(u))^{-n}$.

$$\begin{aligned}
\int_0^\infty |\lambda + \psi(u)|^{-n}\,du &\le 2\int_0^\infty \left[\lambda + \int (1 - \cos ux)\nu(dx)\right]^{-n} du \\
&\le 2\int_0^\infty \left[\lambda + \int_0^\varepsilon (1 - \cos ux) g(x)\,dx\right]^{-n} du \\
&\le K + 2\int_{\pi/2\varepsilon}^\infty \left[\lambda + g(\pi/2u)[(\pi - 2)/2u]\right]^{-n} du \\
&\le K + 2\int_{\pi/2\varepsilon}^\infty \left[K' u^{1+\alpha} u^{-1}\right]^{-n} du < \infty.
\end{aligned}$$

By Fourier inversion, f^{*n} is bounded. Therefore X has a well-behaved kernel. □

Now we are in a position to complete the proof of Theorem 2. Recall that if a process has a Gaussian component, then the kernel is well-behaved. Therefore, in both Restricted Theorem 2 and Theorem 2 we assume the process X has no Gaussian component.

PROOF OF THEOREM 2. Define the following functions:

$$g_0(x) = \begin{cases} C_1 x^{-1-\alpha} & \text{on } (0,\varepsilon) \\ C_2 x^{-1-\alpha} & \text{on } (-\varepsilon,0) \\ g(x) & \text{on } R - \{(-\varepsilon,\varepsilon)\} \end{cases}$$

$$g_1(x) = \begin{cases} \min\left(g(x), C_1 x^{-1-\alpha}\right) & \text{on } (0,\varepsilon) \\ \min\left(g(x), C_2 x^{-1-\alpha}\right) & \text{on } (-\varepsilon,0) \\ g(x) & \text{on } (R - \{(-\varepsilon,\varepsilon)\}) \end{cases}$$

$$g_2(x) = \begin{cases} C_1 x^{-1-\alpha} - g(x) & \text{on } \left\{x \in (0,\varepsilon)\colon C_1 x^{-1-\alpha} - g(x) > 0\right\} \\ C_2 x^{-1-\alpha} - g(x) & \text{on } \left\{x \in (-\varepsilon,0)\colon C_2 x^{-1-\alpha} - g(x) > 0\right\} \\ 0 & \text{elsewhere} \end{cases}$$

$$g_3(x) = \begin{cases} g(x) - C_1 x^{-1-\alpha} & \text{on } \left\{x \in (0,\varepsilon)\colon g(x) - C_1 x^{-1-\alpha} > 0\right\} \\ g(x) - C_2 x^{-1-\alpha} & \text{on } \left\{x \in (-\varepsilon,0)\colon g(x) - C_2 x^{-1-\alpha} > 0\right\} \\ 0 & \text{elsewhere.} \end{cases}$$

If ν_2 and ν_3 are Lévy measures with densities g_2 and g_3, respectively, then the corresponding processes are compound Poisson. By Restricted Theorem 2, if Z_t is a process with Lévy measure having density g_0, Z_t has a well-behaved kernel.

Let Y_t^i be a process with Lévy measure ν_i (density of ν_i is g_i), $i = 1,2,3$. We have $Z_t = Y_t^1 + Y_t^2$ where Y_t^1, Y_t^2 are taken to be independent. Hence,

$$\begin{aligned} P[Z_t \in A] &= P\left[Y_t^1 + Y_t^2 \in A\right] \\ &\geq P\left[Y_t^1 \in A, Y_t^2 = 0\right] \\ &= P\left[Y_t^1 \in A\right] e^{-ct} \quad \text{where } c = \nu_2(R). \end{aligned}$$

Since Z_t has a kernel bounded on $|x| > M$, Y_t^1 has a λ-kernel bounded on $|x| > M$. Now, $X_t = Y_t^1 + Y_t^3$, and by Lemma 2, X_t has a λ-kernel bounded for $|x| > M$. The proof now proceeds as for Restricted Theorem 2. Notice that when estimating the integral, $|g(x) - x^{-1-\alpha}| < \delta$ implies $g(x) > x^{-1-\beta}$ for some $\beta < \alpha$. The proof is complete. □

NOTE. In the hypotheses of Theorem 2, the density of the Lévy measure is required to be close to the corresponding density for a stable process. In the proof, the stable process is needed because it has a kernel bounded on $R - (-a, a)$. The same method of proof can be used starting with other processes that have bounded kernels. For example, Hawkes (1975) showed that for subordinators if $\nu(x,\infty)$ is log convex, the kernel is monotone. This result can be used to weaken the hypotheses of Theorem 2 when considering subordinators.

6. Examples of recurrent sets. A Lévy process with increasing paths is called a subordinator. In this section we prove some results about subordinators that allow us to use Theorem 1 and Theorem 2 to give examples of interesting recurrent sets.

Recall that for subordinators the support of ν is contained in $(0,\infty)$ and $\int_0^1 x\nu(dx) < \infty$. Subordinators can be characterized by the Laplace transform $E\exp(-\lambda X_t) = \exp(-tg(\lambda))$ where g is called the subordinator exponent and $g(\lambda) = a\lambda + \int_0^\infty (1 - e^{-\lambda x})\nu(dx)$ where $a > 0$. The constant "a" is called the drift, and if $a = 0$, the subordinator has no drift.

If we set $\sigma(x) = \nu(x,\infty)$ for $x > 0$, then the following are well known:

1. $x\sigma(x) \to 0$ as $x \to 0$.
2. If $\int_0^\infty x\nu(dx) < \infty$, then $x\sigma(x) \to 0$ as $x \to \infty$.
3. $\int_0^\infty x\nu(dx) = \int_0^\infty \sigma(x)\,dx$.

If $\int_0^\infty x\nu(dx) < \infty$, we say the subordinator has finite mean.

In order to use Theorem 1 to determine recurrence, we can either estimate the probability of hitting a set, or estimate the capacity of a set. Both approaches are feasible with subordinators, but here we concentrate on capacities since the results are more easily obtained.

For subordinators with finite mean and nonzero drift, the following proposition gives rise to many examples.

PROPOSITION 6. *Every unbounded set is a recurrent set for a subordinator with finite mean and nonzero drift.*

PROOF. It is known (Kesten, 1969) that subordinators with drift and finite mean have kernels which are bounded on (a,∞) and are in fact continuous. Hence, the kernels are well-behaved and Theorem 1 applies. Kesten also showed that these processes hit points and therefore $C(\{r\}) > 0$ for $r > 0$. Since $C(A + x) = C(A)$, $C(\{r\}) = C(\{s\})$ for $r < 0$, $s < 0$. Let $B_n = B \cap [n, n+1)$ and pick $r_n \in B_n$ if $B_n \neq \varnothing$. Then $\sum_1^\infty C(B_n) \geq \sum_1^\infty C(\{r_n\}) = \infty$, and hence B is recurrent. □

Turning now to subordinators with finite mean and zero drift, we have the following definition.

DEFINITION. A "set of intervals" is a set $B = \bigcup_1^\infty B_n$ such that the B_n are intervals (closed, open, or half-open), B is unbounded, and the B_n are pairwise disjoint. Let $|B_n|$ denote the length of B_n.

Notice that if for infinitely many n, $|B_n| > \delta > 0$, then for these n, $C(B_n) > \delta^*$ and hence, $\sum_1^\infty C(B_n) = \infty$ which implies B is recurrent. We might as well focus, therefore, on sets of intervals where $|B_n| \to 0$ as $n \to \infty$.

PROPOSITION 7. *If a subordinator has a kernel and no drift* ($a = 0$), *then* $M_1 C(B_n) \leq \int_0^{|B_n|}\sigma(x)\,dx \leq M_2 C(B_n)$ *where* B_n *is an interval and* M_1, M_2 *are positive constants independent of* B_n.

PROOF. We first utilize an estimate obtained by Kanda (1981) for transient Lévy processes (not just subordinators). Following Kanda's development, let

$J = [-r, r]$ and $I = [-r/2, r/2]$, then

$$4r \geq \int_{-2r}^{2r} P^x[T_J < \infty]\,dx = \int_{-2r}^{2r}\int u(y-x)\mu_J(dy)\,dx$$
$$\geq C(J) \inf_{y \in J} \int_{-2r}^{2r} u(y-x)\,dx$$
$$\geq C(J) \int_{-r}^{r} u(x)\,dx.$$

Hence, $C(J) \leq 4r[\int_{-r}^{r} u(x)\,dx]^{-1}$. Further,

$$r = \int_{-r/2}^{r/2} P^x[T_I < \infty]\,dx \leq C(I) \sup_{y \in 1} \int_{-r/2}^{r/2} u(y-x)\,dx$$
$$\leq C(I) \int_{-r}^{r} u(x)\,dx.$$

Since $C(I) \leq C(J)$, we have

$$r\left[\int_{-r}^{r} u(x)\,dx\right]^{-1} \leq C(I) \leq 4r\left[\int_{-r}^{r} u(x)\,dx\right]^{-1}.$$

In the present context, if we let $r = |B_n|$ and recall $C(B_n) = C(B_n + x)$, we can replace $C(I)$ with $C(B_n)$ in the above inequality.

From an inequality in Hawkes (1975) for subordinators, we have,

$$N_1\left[\int_0^{|B_n|} u(x)\,dx\right]^{-1} \leq g(1/|B_n|) \leq N_2\left[\int_0^{|B_n|} u(x)\,dx\right]^{-1},$$

where g is the subordinator exponent and N_1, N_2 are constants. It now follows from the previous inequality that

$$N_3 C(B_n) \leq |B_n| g(1/|B_n|) \leq N_4 C(B_n).$$

From Horowitz (1968), we have if there is no drift,

$$N_6 \int_0^{|B_n|} \sigma(x)\,dx \leq |B_n| g(1/|B_n|) \leq N_7 \int_0^{|B_n|} \sigma(x)\,dx.$$

Putting the inequalities together gives the result. □

NOTE. Kanda's inequality used in the above proof establishes a criteria which can be applied to processes other than subordinators to show $\sum_1^\infty C(B_n) = \infty$.

PROPOSITION 8. *Let B be a set of intervals such that the distance between any two intervals is greater than δ. Let X be a subordinator with no drift satisfying the hypotheses of Theorem 2. $\sum_1^\infty \int_0^{|B_n|} \sigma(x)\,dx = \infty$ if and only if B is recurrent.*

PROOF. The hypotheses of Theorem 1 are satisfied and by Proposition 7, $\sum_1^\infty C(B_n) = \infty$. □

Using Proposition 8, we can construct an example of a recurrent set with finite Lebesgue measure.

EXAMPLE. Let X be a subordinator with no drift, finite mean and $EX_t > 0$. Let the Lévy measure have a density g such that g is bounded on $[\varepsilon, \infty)$ and $g(x) = x^{-1-\alpha}$ on $(0, \epsilon)$ where $0 < \alpha < 1$.

Now

$$\sigma(z) = \int_z^{\infty} g(y)\,dy = \int_z^{\varepsilon} y^{-1-\alpha}\,dy + \nu(\varepsilon, \infty)$$

$$= C_1 z^{-\alpha} + K \quad \text{where } z < \varepsilon.$$

Hence, $\int_0^x \sigma(z)\,dz = C_2 x^{1-\alpha} + Kx$. Let $|B_n| = n^{-(1+\alpha)}$ and let $B = \bigcup_1^{\infty} B_n$ be a set of intervals with the length of B_n equal to $|B_n|$, and with the distance between any two intervals greater than $\delta > 0$. We have,

$$\sum_1^{\infty} \int_0^{|B_n|} \sigma(y)\,dy = \sum_1^{\infty} C_2 n^{-(1+\alpha)(1-\alpha)} + K n^{-(1+\alpha)}$$

$$= C_2 \sum_1^{\infty} n^{-(1-\alpha^2)} + K' = \infty$$

Thus, B is recurrent, but $\sum_1^{\infty} |B_n| = \sum_1^{\infty} n^{-(1+\alpha)} < \infty$. B has finite Lebesgue measure. Notice that in this example, X is a process that does not hit points.

In Theorem 1 and throughout this section, we have broken a set B into sets $B_n = B \cap S_n$ where $\{S_n\}_1^{\infty}$ is a partition of nonzero mesh. The requirement of nonzero mesh is in some sense essential since we can now show that $\sum_1^{\infty} C(B_n)$ is finite or infinite depending on whether the partition has nonzero mesh or not.

EXAMPLE. Let X be the subordinator designated in the previous example. Then $\sigma(z) = C_1 z^{-\alpha} - C_1 \varepsilon^{-\alpha} + K$. Hence, $\sigma(z) = C_1 z^{-\alpha} - K_1$ if ε is sufficiently small and $K_1 > 0$. It follows that $\int_0^x \sigma(z)\,dz = C_2 x^{1-\alpha} - K_1 x$. Let $x_n = 2^{-(n\alpha/(1-\alpha))}$ and let $B_n = [r_n, r_n + x_n]$ where we choose the r_n sufficiently large that the B_n are mutually disjoint and at least a distance δ from each other. Hence $B_n = B \cap S_n$ where $\{S_n\}_{n=1}^{\infty}$ is a partition of nonzero mesh. By Proposition 7,

$$C(B_n) \le M_2 \int_0^{x_n} \sigma(y)\,dy.$$

Then

$$\sum_1^{\infty} C(B_n) \le M_2 \sum_{n=1}^{\infty} \int_0^{x_n} \sigma(y)\,dy$$

$$= M_2 \sum_{n=1}^{\infty} C_2 2^{-n\alpha} - K_1 2^{-(n\alpha/(1-\alpha))} < \infty.$$

Now divide B_n into 2^n intervals of equal length. Call the smaller intervals B_{n_i}, $i = 1, 2, 3, \ldots, 2^n$. $B_n = \bigcup_{i=1}^{2^n} B_{n_i}$. Again by Proposition 7, $C(B_{n_i}) \ge$

$M_1 \int_0^{|B_{n_i}|} \sigma(y)\, dy$. Then we have

$$\sum_{i=1}^{2^n} C(B_{n_i}) \ge M_1 \sum_{i=1}^{2^n} \int_0^{|B_{n_i}|} \sigma(y)\, dy$$
$$\ge M_1' \sum_{i=1}^{2^n} \left(2^{-n} \cdot 2^{-(n\alpha/(1-\alpha))}\right)^{1-\alpha} = M_1'.$$

Therefore, $\sum_{n=1}^{\infty} \sum_{i=1}^{2^n} C(B_{n_i}) = \infty$. Thus if we let $B_{n_i} = \hat{B}_m = B \cap S_m$ where $\{\hat{S}_m\}$ is a partition of zero mesh, then $\sum_1^{\infty} C(\hat{B}_m) = \infty$ whereas $\sum_1^{\infty} C(B_n) < \infty$ for the partition $\{S_n\}$ of nonzero mesh.

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