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# LAWS OF THE ITERATED LOGARITHM FOR TIME CHANGED BROWNIAN MOTION WITH AN APPLICATION TO BRANCHING PROCESSES

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A functional law of the iterated logarithm for time changed Brownian motion is given for stopping times that increase at a geometric rate. This result is applied to various quantities associated with a Galton–Watson process.

**Introduction.** The work undertaken here is motivated by the desire to apply embedding techniques to obtain laws of the iterated logarithm for quantities associated with a supercritical Galton–Watson process. In the literature (for example, Heyde [4], [5], and [6], Heyde and Leslie [7], and Brown and Heyde [2]) asymptotic results for such processes have been proven via Berry–Esseen-type inequalities and in Scott [9] a functional central limit theorem for martingales that are not uniformly asymptotically negligible was applied to branching processes. This latter approach is rather long and complicated and seems to imply that standard martingale results with uniformly asymptotically negligible summands do not apply to branching processes. However in Asmussen and Keiding [1] it was shown that by considering the offspring of each individual a central limit theorem for a martingale difference array could be used to obtain a central limit theorem for quantities associated with branching processes. The major purpose of our work here is to similarly consider laws of the iterated logarithm for branching processes and as these results require new properties of Brownian motion and stopping times the bulk of this article is devoted to this area, the desired results being applications of these properties.

**1. Properties of Brownian motion and stopping times.** Let $\psi$ be the real valued function on $(e, \infty)$ defined by

$$\psi(t) = 2t \log\log t = 2t \log_2 t.$$

We define $\{\xi(t), t \geq 0\}$ to be a standard Brownian motion and $(C, \rho)$ to be the Banach space of all real valued continuous functions on $[0,1]$ with

$$\rho(x, y) = \sup_{0 \leq t \leq 1} |x(t) - y(t)|, \qquad x, y \in C.$$

Let $K$ be the set of absolutely continuous $x \in C$ such that $x(0) = 0$ and

$$\int_0^1 [\dot{x}(t)]^2 \, dt \leq 1,$$

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where $\dot{x}$ denotes the derivative of $x$ with respect to Lebesgue measure and is determined a.e.

The question of prime consideration concerning stopping times and Brownian motion is the following. For a sequence of random variables $\{\tau_n; n \geqq 1\}$ what is the fastest rate at which these variables can increase to infinity and the law of the iterated logarithm still hold for the sequence $(2\tau_n \log_2 \tau_n)^{-1/2} B(\cdot \tau_n)$. We show in Section 1 that this rate is at least that given by (1) and (2) below, i.e., the sequence may increase at a geometric rate.

Our main result is the following theorem which is an extension of Theorem 1 of Strassen [12].

THEOREM 1. *Let* $\{\xi(t), t \geq 0\}$ *be a standard Brownian motion and* $\{a_n, n \geq 1\}$ *a sequence of constants increasing to* $\infty$ *satisfying*

$$(1) \qquad a_n^{-1} a_{n+1} \to L, \qquad 1 \leq L < \infty.$$

*If for some nondecreasing sequence of positive almost surely finite random variables* $\{\tau_n, n \geq 1\}$, *where w.l.o.g.* $\tau_n > 3$,

$$(2) \qquad a_n^{-1} \tau_n \to W \quad a.s. \qquad 0 < W < \infty \quad a.s.,$$

*then the sequence*

$$(3) \qquad \left\{ (2\tau_n \log_2 \tau_n)^{-1/2} \xi(\tau_n t),\ t \in [0,1], n \geq 1 \right\}$$

*is with probability one relatively compact and the set of its limit points coincides with* $K$.

PROOF. Corollary 1 of Strassen [12] implies that the sequence in (3) is relatively compact and the set of its limit points is at most $K$ with probability one. To show that with probability one the set of its limit points is at least $K$ we proceed via several lemmas. The first is a corollary of Theorem 1 of Strassen [12].

LEMMA 1. *Define for* $t \in [0,1]$

$$(4) \qquad \xi_n(t) = \psi(a_n)^{-1/2} \xi(a_n t).$$

*Then, under condition* (1), *with probability one the sequence* $\{\xi_n, n \geq 1\}$ *is relatively compact and the set of its limit points coincides with* $K$.

PROOF. Corollary 1 of Strassen [12] again implies that almost surely the sequence in (4) is relatively compact and the set of its limit points is at most $K$. To show that this set of limit points is at least $K$ set for any integer $m \geq 1$ and any $\delta > 0$

$$(5) \qquad n_j = \inf\left\{k: a_k \geq (L(1+\delta)m)^j\right\}.$$

Then for large enough $j$ and any $\varepsilon > 0$,

$$(6) \qquad (L^{-1} - \varepsilon) \leqq a_{n_j}^{-1}(L(1+\delta)m^j) \leqq 1$$

so that for large enough $j$,

(7) $$a_{n_j}^{-1} a_{n_{j+1}} > m.$$

For $x \in K$ and $m \geq 1$ an integer define

$$A_n^* = \left\{ \left| \xi_n\left(\frac{i}{m}\right) - \xi_n\left(\frac{i-1}{m}\right) - \left(x\left(\frac{i}{m}\right) - x\left(\frac{i-1}{m}\right)\right) \right| \right. \quad . \\ \left. < \delta \text{ for all } i \text{ with } 2 \leq i \leq m \right\}.$$

Now (7) ensures that for large enough $J$ the events $A_{n_j}^*$ are mutually independent (for all $j \geq J$). Following Strassen (p. 214) we have

$$P(A_n^*) \geq \frac{\text{const}}{\log a_n \sqrt{m \log_2 an}}.$$

Thus from (12) we have

$$P\left(A_{n_j}^*\right) \geq \frac{\text{const}}{\log(2Lm)^{j+1} \sqrt{m \log_2 (2Lm)^{j+1}}}$$

and as $\sum_j 1/(j\sqrt{\log j})$ diverges we see that

$$\sum_{j=J}^{\infty} P\left(A_{n_j}^*\right) = \infty.$$

The proof may be completed using the Borel–Cantelli lemma as in Strassen (p. 215).

The next step in our proof of the theorem is to strengthen this result to:

Lemma 2. *Let the sequence $\{a_n, n \geq 1\}$ be as in the theorem. Then for any random variable $W$, $0 < W < \infty$ a.s., if we define for $t \in [0,1]$*

(8) $$\zeta_n(t) = \psi(W a_n)^{-1/2} \xi(W a_n t),$$

*the sequence $\{\zeta_n, n \geq 1\}$ satisfies the conclusions of Theorem 1.*

Proof. We first consider the case when

(9) $$0 < a < W < b < \infty \quad \text{a.s.}$$

For each integer $m$ define for $1 \leq j \leq 2^m$

$$t_j^{(m)} = 2^{-m} j |b - a|$$

and random variables $p_m$ by

$$p_m = \max\left\{j \colon t_j^{(m)} \leq W\right\}.$$

By Lemma 1 for fixed $m$ and $j \leq 2^m$ the sequence $\{\psi(t_j^{(m)} a_n)^{-1/2} \xi(t_j^{(m)} a_n t), n \geq 1\}$ is with probability one relatively compact and the set of its limit points coincides with $K$. Therefore the same conclusion holds conditional on

$\{t_j^{(m)} \le W < t_{j+1}^{(m)}\}$ for the sequence

$$\left\{\psi\left(t_j^{(m)}a_n\right)^{-1/2}\xi\left(t_j^{(m)}a_n t\right)I\left(t_j^{(m)} \le W < t_{j+1}^{(m)}\right), n \ge 1\right\}$$

and hence for the sequence $\{\psi(t_{p_m}^{(m)}a_n)^{-1/2}\xi(t_{p_m}^{(m)}a_n t), n \geqq 1\}$.

Now define, for $t \in [0,1]$,

$$\zeta_n^{(m)}(t) = \psi\left(t_{p_m}^{(m)}a_n\right)^{-1/2}\xi\left(t_{p_m}^{(m)}a_n t\right).$$

Then for any $\varepsilon > 0$, $x \in K$ and for arbitrary but fixed $m$ we have almost surely

$$\sup_t |x(t) - S_n^{(m)}(t)| < \varepsilon \quad \text{i.o.} \tag{10}$$

To complete the proof of the lemma when (9) holds note that

$$\begin{aligned}&\psi\left(t_{p_m}^{(m)}a_n\right)^{-1/2}\left|\xi(Wa_n t) - \xi\left(t_{p_m}^{(m)}a_n t\right)\right| \\ &\qquad \le \sup\left\{\psi\left(t_{p_m}^{(m)}a_n\right)^{-1/2}\left|\xi(sa_n t) - \xi\left(t_{p_m}^{(m)}a_n t\right)\right|, t_{p_m}^{(m)} \le s < t_{p_m+1}^{(m)}\right\}\end{aligned}$$

and after some calculation we have that for any $\varepsilon > 0$, for large enough $m$, with probability one,

$$\sup_t \left|x(t) - \psi\left(t_{p_m}^{(m)}a_n\right)^{-1/2}\xi(Wa_n t)\right| < \varepsilon \quad \text{i.o.}, \tag{11}$$

which, as $t_{p_m}^{(m)} \to W$ a.s., completes the proof under (9). To now obtain the lemma it is sufficient to note that for $0 < W < \infty$ a.s. we may choose $a$ and $b$ so that (9) holds with arbitrarily large probability.

To prove the theorem note that for any $\varepsilon > 0$ and large enough $n$

$$\begin{aligned}&\psi(Wa_n)^{-1/2}|\xi(\tau_n t) - \xi(Wa_n t)| \\ &\qquad \leqq \sup\left\{\psi(Wa_n)^{-1/2}|\xi(st) - \xi(Wa_n t)|, Wa_n(1-\varepsilon) \leqq s < Wa_n(1+\varepsilon)\right\}\end{aligned}$$

since we have $|\tau_n - Wa_n| < Wa_n\varepsilon$.

For $Wa_n(1-\varepsilon) \leqq s < Wa_n(1+\varepsilon)$ put

$$t' = \frac{st}{Wa_n};$$

then for any $\varepsilon > 0$ for large enough $n$ with arbitrarily large probability

$$\psi(Wa_n)^{-1/2}|\xi(Wa_n t') - \xi(Wa_n t)| < \varepsilon,$$

which is sufficient to prove the theorem. □

REMARK. Let $\eta(t), t \in [0, \infty)$ be any process taking values in $C[0,\infty)$ (the space of continuous functions on $[0,\infty)$) such that

$$|\eta(t) - \xi(t)| = o\left((t \log_2 t)^{1/2}\right) \quad \text{a.s.} \tag{12}$$

Then under the conditions of Theorem 1 the sequence

$$\eta_n(t) = \psi(\tau_n)^{-1/2}\eta(\tau_n t), \qquad t \in [0,1],$$

satisfies the conclusions of Theorem 1.

COROLLARY 1. *For $n = 1, 2, \dots$ let $\{\tau_n(t), 0 \le t \le 1\}$ be a nondecreasing stochastic process such that $\tau_n \equiv \tau_n(1)$ satisfies the conditions of Theorem 1 and*

$$\sup_{0 \le t \le 1} \left| \frac{\tau_n(t)}{\tau_n(1)} - t \right| \to 0; \tag{13}$$

*then the sequence $\{\psi(\tau_n)^{-1/2}\xi(\tau_n(\cdot))\}$ is a.s. relatively compact and the set of its limit points is $K$.*

PROOF. Note that

$$\begin{aligned}
&\psi(\tau_n)^{-1/2}|\xi(\tau_n(t)) - \xi(\tau_n(1)t)| \\
&\quad \le \sup\left\{\psi(\tau_n)^{-1/2}|\xi(s) - \xi(\tau_n(1)t)|, \tau_n(1)(t-\varepsilon) < s < \tau_n(1)(t+\varepsilon)\right\} \\
&\quad \le \sup\left\{\psi(\tau_n)^{-1/2}|\xi(\tau_n(1)t') - \xi(\tau_n(1)t)|, (t-\varepsilon) < t' < (t+\varepsilon)\right\},
\end{aligned}$$

which is sufficient to give the corollary.

REMARK 1. Note that the above results do not contain Theorem A of Hall and Heyde [3] as their result does not require the existence of constants $\{a_n\}$ satisfying (1) and (2), however they do require $\tau_n^{-1}\tau_{n+1} \to 1$.

REMARK 2. Define for $t \in [0,1]$,

$$\tau_n(t) = \tau_j + p(t),$$

where

$$p(t) = \frac{a_n t - a_j}{a_{j+1} - a_j}(\tau_{j+1} - \tau_j)$$

and

$$j = \max\{k\colon a_k \le a_n t\}.$$

After some calculations (13) can be shown to hold. Thus if $\eta(\cdot)$ satisfies (12) and the conditions of Theorem 1 hold then the sequence $\{\eta_n^*, n \ge 1\}$, where $\eta_n^*(t) = \psi(\tau_n)^{-1/2}\eta(\tau_n(t))$, $t \in [0,1]$, satisfies the conclusions of Theorem 1.

COROLLARY 2. *Suppose that $\eta(\cdot)$ satisfies (12) and the conditions of Theorem 1 hold with*

$$a_n^{-1}a_{n+1} \to m, \qquad m > 1. \tag{14}$$

*For $n = 1, 2, \dots$ let $\{b_{n,k}, 1 \le k \le n\}$ be constants that satisfy*

$$\limsup_n \sum_{k=1}^{n} b_{n,k} < \infty \tag{15}$$

*and*

$$\lim_{\substack{r<n\\ t,n\to\infty}} \max\left\{\sum_{k=1}^{r-1}\left[b_{n,k}x\left(\frac{a_{n-(k-1)}}{a_n}\right)-b_{r,k}x\left(\frac{a_{r-(k-1)}}{a_r}\right)\right],\right.$$
$$\left.\cdot\sum_{k=1}^{r-1}\left[b_{n,k}x\left(\frac{a_{n-k}}{a_n}\right)-b_{r,k}x\left(\frac{a_{r-k}}{a_r}\right)\right]\right\}=0. \tag{16}$$

*Define for* $x \in C$,

$$g_n(x)=\sum_{k=1}^{n-1} b_{n,k}\left[x\left(\frac{a_{n-(k-1)}}{a_n}\right)-x\left(\frac{a_{n-k}}{a_n}\right)\right],$$

*then*

$$\limsup_{n\to\infty} g_n(\eta_n^*)=\lim_{n\to\infty}\left[\sum_{k=1}^{n} b_{n,k}^2\left(m^{-(k-1)}-m^{-k}\right)\right]^{1/2} \quad a.s.$$

PROOF. Under the conditions of the corollary it is a straightforward matter to show that for $x \in K$, $\{g_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ and hence has a limit $g(x)$. Furthermore for any sequence $x_n \to x$ where $\{x_n\}$ and $x$ are contained in $C$ it is easy to show that $|g_n(x_n)-g_n(x)| \to 0$ so that if $x_n \to x$ where $\{x_n\}$ is contained in $C$ and $x \in K$ then $g_n(x_n) \to g(x)$. In view of the preceding remark the sequence $g_n(\eta_n^*)$ is a.s. relatively compact and the set of its limit points coincides with $K$. In particular $\limsup_{n\to\infty} g_n(\eta_n^*)$ is just $\sup_{x\in K} g(x)$ and as $g_n(x) \to g(x)$ uniformly on $K$ this is just $\lim_{n\to\infty}\sup_{x\in K} g_n(x)$. However (see p. 219 of Strassen [12])

$$\sup_{x\in K} g_n(x)=\left(\int_0^1 S_n(t)^2\,dt\right)^{1/2},$$

where

$$S_n(t)=\sum_{k=1}^{n} b_{n,k} I_{[m^{-k},\,m^{-(k-1)}]}(t)$$

so that

$$\sup_{x\in K} g_n(x)=\left[\sum_{k=1}^{n} b_{n,k}^2\left(m^{-(k-1)}-m^{-k}\right)\right]^{1/2} \quad a.s.$$

This gives the corollary as from (15) the limit of this last expression is finite.

**2. Application to branching processes.** Let $\{Z_0=1, Z_1, Z_2, \dots\}$ be a supercritical Galton–Watson process with $1 < EZ_1 = m$ and $0 < \operatorname{var} Z_1 = \sigma^2 < \infty$. It is well known for this process that $m^{-n}Z_n \to W < \infty$ a.s. and that $\{W>0\} = \{Z_n \neq 0 \text{ for all } n\}$. In this section as a notational convenience it is assumed that $P(W=0)=0$ to avoid making trivial exceptions on the set of extinction. We follow here the approach of Asmussen and Keiding [1].

Let $\tau_n = Z_0 + \cdots + Z_n$ and represent the $n$th generation as $\{k \in \mathbb{N}: \tau_{n-1} < k \le \tau_n\}$ ($\mathbb{N} = \{0,1,2,\dots\}$) where here $n = n(k)$ always represents the generation of $k$. The number of offspring produced by $k$ is denoted by $U_k$ and

$$G_k = \sigma\{U_l, l \le k\}, \qquad F_N = \sigma\{U_k, n \le N\} = G_{\tau_N}.$$

The approach here considers our branching process as a sequence of independently and identically distributed (i.i.d.) random variables $\{U_k, k \in \mathbb{N}\}$ and time change functions $\{\tau_N, N \ge 1\}$. It is easy to see that for each $l$ the random variable $\tau_l$ is a stopping time with respect to the $\sigma$ fields $\{G_k, k \ge 1\}$ and using the Toeplitz lemma we have

$$\frac{\tau_l}{\sum_{j=1}^{l} m^j} \to W \quad \text{a.s.,} \qquad 0 < W < \infty \quad \text{a.s.}$$

Thus we take $a_n = \sum_{j=1}^{n} m^j$.

REMARK. We have assumed here that the index $k$ of the $U_k$ refers to the order of appearance of individuals with a suitable convention to cover ties, i.e., $U_k$ is the offspring produced by the $k$th individual.

Now define for $t \in [0, \infty)$

$$\eta(t) = ([t] + 1 - t)S_{[t]} + (t - [t])S_{[t]+1},$$

where

$$S_k = \sum_{j=1}^{k} X_j, \qquad X_j = \frac{U_{j-m}}{\sigma},$$

and $[t]$ as usual denotes the largest integer smaller than $t$. Theorem 2 of Strassen [11] now implies

$$|\eta(t) - \xi(t)| = O\left((2t \log_2 t)^{1/2}\right) \quad \text{a.s.}$$

and if we define for $t \in [0, 1]$,

$$\eta_N(t) = \psi(\tau_N)^{-1/2} \eta(\tau_N t),$$

the remark following Theorem 1 implies that $\{\eta_N, N \ge 1\}$ is relatively compact and the set of its limit points coincides with $K$. An application of this is given by observing that for a Galton–Watson process the maximum likelihood estimator of $m$ based on observing $Z_0, \dots, Z_n$ is $\hat{m} = (Z_1 + \cdots + Z_N)/(Z_0 + \cdots + Z_{N-1})$. See for example Asmussen and Keiding ([1], p. 117). Thus

$$\hat{m} - m = \sum_{j=1}^{N} (Z_j - mZ_{j-1}) \bigg/ \left( \sum_{j=0}^{N-1} Z_j \right) = \tau_{N-1}^{-1} \sum_{k=1}^{\tau_{N-1}} \{U_k - m\}.$$

As was shown by Strassen ([11] p. 218) the above invariance principle yields the ordinary law of the iterated logarithm and hence

$$\limsup_{N \to \infty} \tau_{N-1}^{1/2} (2\sigma^2 \log N)^{-1/2} (\hat{m} - m) = \pm 1 \quad \text{a.s.}$$

Thus our invariance principle is sufficient to obtain the laws of the iterated logarithm for the maximum likelihood estimator of the offspring mean of a supercritical Galton–Watson process.

In fact using Corollary 2 more than this is given, e.g., if for $k < \infty$ $b_1, \dots, b_k$ are constants then we have

$$\limsup_{N\to\infty} \sigma^{1}\psi(\tau_N)^{-1/2} \sum_{l=1}^{k} b_l\left[Z_{N-(l-1)} - mZ_{N-l}\right] = \left[\sum_{l=1}^{k} b_l^2\left(m^{-(l-1)} - m^{-l}\right)\right]^{1/2} \quad \text{a.s.}$$

and letting $b_l = m^{l-1}$, $1 \le l \le k$, we have

$$\limsup_{N\to\infty} \sigma^{-1}\psi(\tau_N)^{-1/2}\left(Z_N - m^k Z_{N-k}\right) = m^{-1}\left(m^k - 1\right)^{1/2} \quad \text{a.s.},$$

which is equivalent to Theorem 2 of Heyde and Leslie [7].

Now let $b_{N,k} = \delta^k$ for $\delta < 1$ and $b_{N,k} = (\sum_{k=1}^{N}\delta^k)^{-1}\delta^k$ for $1 < \delta < m^2$. Then from Corollary 2 we have for $\delta^2 < m$,

$$\limsup_{N\to\infty} \sigma^{-1}\psi(\tau_N)^{-1/2} \sum_{k=1}^{N} \delta^k\left[Z_{N-(k-1)} - mZ_{N-k}\right] = \left(\frac{(m-1)\delta^2}{m-\delta^2}\right)^{1/2}.$$

Next let $b_{N,k} = (\sum_{k=1}^{N} m^{k/2})^{-1} m^{k/2}$. Then again from Corollary 2 we have,

$$\limsup_{N\to\infty} \sigma^{-1}\psi(\tau_N)^{-1/2} N^{-1/2} \sum_{k=1}^{N} m^{k/2}\left[Z_{N-(k-1)} - mZ_{N-k}\right] = (m-1)^{1/2}.$$

Finally for $\delta^2 > m$ let $b_{N,k} = (\sum_{k=1}^{N}\delta^k)^{-1}\delta^k$. Then

$$\limsup_{N\to\infty} \sigma^{-1}\psi(\tau_N)^{-1/2}\left(\frac{\delta^2}{m}\right)^{-N/2} \sum_{k=1}^{N} \delta^k\left[Z_{N-(k-1)} - mZ_{N-k}\right] = \left(\frac{(m-1)\delta^2}{\delta^2 - m}\right)^{1/2}.$$

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