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ABSOLUTE CONTINUITY OF STABLE SEMINORMS

BY T. BYCZKOWSKI AND K. SAMOTIJ

Wrocław Technical University

Suppose that E is a complete separable real metric vector space. It is proved that if X is a symmetric E-valued p-stable random vector, $0 < p < 2$, and q is a lower semicontinuous, a.s. finite seminorm, then the distribution of $q(X)$ is absolutely continuous apart from a possible jump. If, additionally, q is strictly convex or $0 < p < 1$, then the distribution of $q(X)$ is either absolutely continuous or degenerate at 0. This result settles, in particular, the problem of absolute continuity of the supremum of stable sequences, extending thus Tsirel'son's theorem.

1. Introduction. Properties of the distribution of the supremum of separable Gaussian processes play an important role in theory as well as applications and have been extensively investigated for many years. For the Wiener process $W(t)$, the distribution of $\max|W(t)|$ is absolutely continuous and its density can be expressed explicitly. For stationary Gaussian processes the absolute continuity of the supremum was proved in [17]; for general separable Gaussian processes this problem was solved in [16].

The continuity of the distribution of the supremum (in fact, of an arbitrary measurable seminorm) for p-stable measures, $0 < p < 1$, was established in [18]. It was also shown there that for $p = 1$ this distribution can have at most one atom. When $p \geq 1$, [14] provides a simple example of a seminorm which is positive and constant a.s. In the same paper the problem of absolute continuity for separable or strictly convex seminorms is solved, by means of a representation of p-stable measures as mixtures of Gaussian measures and applying known results for Gaussian measures.

This paper proposes an approach based on a version of the Lévy–Khinchine formula. Our presentation is quite general, in the context of complete separable metric vector spaces. Readers interested only in locally convex or Banach space cases can use [4] or [3], instead of Theorem 3.1. Although our main motivation is the case of stable measures on R^∞ with the supremum seminorm, nonlocally convex applications are also of some interest. A fairly general example (including the previous one) is provided by stable measures on the space $L_0(m)$ of all measurable functions on a finite separable measure space with convergence in measure m and the essential supremum seminorm.

2. Preliminaries. In this section we introduce some notation and terminology and collect basic facts needed in the sequel.

For standard concepts of weak convergence, tightness, properties of convolution of probability measures, etc., the reader is referred to, e.g., [4], [11].

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Throughout the whole paper E will stand for a complete separable metric vector space over the real numbers. Let D be an additive subsemigroup of $R^+ = (0, \infty)$, dense in R^+. A family $(\mu_t)_{t\in D}$ of probability measures on E is called a convolution semigroup if

$$\mu_t * \mu_s = \mu_{t+s}$$

for all $t, s \in D$; it is called continuous if μ_t converges weakly to δ_0 (the point mass at 0) as $t \to 0+$, $t \in D$; it is called symmetric if all μ_ts are symmetric. The most typical situation is when $D = R^+$ and we then simply write $(\mu_t)_{t>0}$.

A Borel measurable function q: $E \to [0, \infty]$ is called a measurable pseudonorm if it is subadditive, i.e., $q(x+y) \le q(x) + q(y)$, for all $x, y \in E$, if $q(0) = 0$, and if it is monotonic, i.e., $q(ax) \le q(bx)$ whenever $|a| \le |b|$ for all $x \in E$; q is called a measurable seminorm if, additionally, $q(ax) = |a|q(x)$ for all $a \in R$ and $x \in E$.

Next, we state some inequalities needed in the sequel. The first two of them are taken from [1] and [2]; the remaining ones are versions of the Lévy Inequality (see [9]).

LEMMA 2.1. *Let q be a measurable pseudonorm on E and let X, Y be independent E-valued random vectors. Then for every $\varepsilon, \delta > 0$ we have*:

$$\text{(a)}\quad \begin{aligned} P\{q(X+Y) > \varepsilon\} \ge\ & P\{q(X) > (1+\delta)\varepsilon\} \cdot P\{q(Y) \le \delta\varepsilon\} \\ & + P\{q(Y) > (1+\delta)\varepsilon\} \cdot P\{q(X) \le \delta\varepsilon\}, \end{aligned}$$

$$\text{(b)}\quad \begin{aligned} P\{q(X+Y) > \varepsilon\} \le\ & P\{q(X) > (1-\delta)\varepsilon\} + P\{q(Y) > (1-\delta)\varepsilon\} \\ & + P\{q(X) > \delta\varepsilon\} \cdot P\{q(Y) > \delta\varepsilon\}. \end{aligned}$$

LEMMA 2.2. *Let $X_1, \dots, X_n$ be E-valued independent and symmetric random vectors and let q be a measurable pseudonorm. Then for every $\varepsilon > 0$ we have*

$$\text{(a)}\qquad P\left\{\max_{1\le j\le 2} q(X_j) > \varepsilon\right\} \le 2P\left\{q\left((1/2)\sum_{i=1}^{2} X_i\right) > \varepsilon/2\right\},$$

$$\text{(b)}\qquad P\left\{\max_{1\le j\le n} q\left(\sum_{i=1}^{j} X_i\right) > \varepsilon\right\} \le 2P\left\{q\left((1/2)\sum_{i=1}^{n} X_i\right) > \varepsilon/2\right\}.$$

A convolution semigroup $(\mu_t)_{t\in D}$ is called q-continuous if for every $\varepsilon > 0$ we have $\lim_{D\ni t\to 0+} \mu_t\{q > \varepsilon\} = 0$. Observe that if q generates the topology of E then the q-continuity of convolution semigroups reduces to the usual continuity. It follows easily by Lemma 2.1(b) that if $(\gamma_t)_{t\in D}$ and $(\kappa_t)_{t\in D}$ are q-continuous then so is $(\gamma_t * \kappa_t)_{t\in D}$. If we assume that $(\gamma_t)_{t\in D}$ and $(\kappa_t)_{t\in D}$ are symmetric then Lemma 2.2(a) implies that the converse is also true.

LEMMA 2.3. *Let q be a measurable pseudonorm on E and let $(\gamma_t)_{t\in D}$ and $(\kappa_t)_{t\in D}$ be two symmetric convolution semigroups on E. If $(\gamma_t * \kappa_t)_{t\in D}$ is q-continuous then so are $(\gamma_t)_{t\in D}$ and $(\kappa_t)_{t\in D}$.*

Now, we need one more result (for the proof, see [5]).

Lemma 2.4. *Let q be a measurable pseudonorm and let $(\mu_t)_{t>0}$ be a q-continuous convolution semigroup on E. Then for all $\varepsilon > 0$ the following holds:*

$$\limsup_{t\to 0+} (1/t)\mu_t\{q > \varepsilon\} < \infty. \tag{2.1}$$

Moreover, if $\alpha = 1 - 2\mu_1\{q > \varepsilon/5\} > 0$, then

$$\limsup_{t\to 0+} (1/t)\mu_t\{q > \varepsilon\} \le -\ln \alpha. \tag{2.2}$$

3. Lévy–Khinchine formula. In this section we state and prove an abstract version of the Lévy–Khinchine formula in a form suitable for our purposes. Instead of considering infinitely divisible distributions, as in [15], we work here with continuous convolution semigroups. The advantage of our technique is that we obtain more effective formulas, especially, the uniqueness of the Lévy measure.

We begin with a definition. Let ν be a finite Borel measure on E. For every $t > 0$ define

$$\exp(t\nu) = e^{-t\nu(E)} \sum_{k=0}^{\infty} (t^k/k!)\nu^{*k}.$$

It is clear that the above series is convergent in the total variation norm and defines a convolution semigroup. Observe that if $t \to 0+$, then $\exp(t\nu)$ converges to δ_0 in the total variation norm. In particular, $\exp(t\nu)$ is q-continuous, for every measurable pseudonorm q.

Now, we are able to formulate and prove our version of the Lévy–Khinchine formula.

Theorem 3.1. *Let $(\mu_t)_{t>0}$ be a symmetric continuous convolution semigroup on E. Then there exists a nonnegative measure ν such that for every open neighbourhood U of 0 $\nu|_{U^c}$ is finite and $(1/t)\mu_t|_{U^c}$ converges weakly to $\nu|_{U^c}$ as $t \to 0+$, whenever $\nu(\partial U) = 0$. Moreover, the following decomposition holds:*

$$\mu_t = \kappa_t * \gamma_t,$$

with κ_t, γ_t satisfying:

(i) *$(\kappa_t)_{t>0}$, $(\gamma_t)_{t>0}$ are symmetric and continuous convolution semigroups,*
(ii) *$\gamma_t = \lim \exp(t\nu|_{F_n})$, where F_n is any increasing sequence of Borel subsets such that $\nu|_{F_n}$ is finite and $\bigcap_n F_n^c = \{0\}$,*
(iii) *$\lim_{t\to 0+}(1/t)\gamma_t|_{U^c} = \nu|_{U^c}$ for every open neighborhood U of 0 such that $\nu(\partial U) = 0$,*
(iv) *$\lim_{t\to 0+}(1/t)\kappa_t(U^c) = 0$ for every open neighbourhood U of 0.*

Proof. Let q be a pseudonorm generating the topology of E. Let us choose a sequence η_n of positive numbers which decreases to 0. Using elementary facts from the abstract semigroup theory (see, e.g., [8] Chapter X) we obtain that $\exp((s/t)\mu_t)$ converges weakly to μ_s, as $t \to 0+$ for every $s > 0$. Since

$$\exp((s/t)\mu_t) = \exp\big((s/t)\mu_t|_{\{q>\eta\}}\big) * \exp\big((s/t)\mu_t|_{\{q\le\eta\}}\big),$$

standard compactness arguments [11] and the symmetry of all measures implies the existence of a sequence t' decreasing to 0 such that for a finite measure $\nu^{(n)}$ and probability measures $\kappa_s^{(n)}$ we have:

$$(1/t')\mu_{t'}|_{\{q>\eta_n\}} \Rightarrow \nu^{(n)} \quad \text{and} \quad \exp\Big((s/t')\mu_{t'}|_{\{q\le\eta_n\}}\Big) \Rightarrow \kappa_s^{(n)}$$

for all $s \in Q^+$ (positive rationals) and all positive integers n. Denote $\exp(s\nu^{(n)})$ by $\gamma_s^{(n)}$. Then we obtain $\mu_s = \kappa_s^{(n)} * \gamma_s^{(n)}$ for all $s \in Q^+$. Choosing again an appropriate subsequence n' we obtain that there exist probability measures κ_s and γ_s, $s \in Q^+$, such that $\kappa_s^{(n')} \Rightarrow \kappa_s$, $\gamma_s^{(n')} \Rightarrow \gamma_s$, and $\mu_s = \kappa_s * \gamma_s$ for all $s \in Q^+$. Moreover, $(\kappa_s)_{s\in Q^+}$ and $(\gamma_s)_{s\in Q^+}$ are symmetric convolution semigroups: hence, by Lemma 2.3, they are continuous on Q^+. Therefore, they can be uniquely extended, as continuous convolution semigroups, to R^+ [13]. Hence we obtain

$$\mu_s = \kappa_s * \gamma_s \quad \text{for all } s > 0. \tag{3.1}$$

By construction it follows that for all $\varepsilon \ge \eta_{n-1}$ with the property $\nu^{(n)}\{q = \varepsilon\} = \nu^{(n-1)}\{q = \varepsilon\} = 0$ we have $\nu^{(n)}|_{\{q>\varepsilon\}} = \nu^{(n-1)}|_{\{q>\varepsilon\}}$. Define

$$\nu = \lim_n \nu^{(n)}|_{\{q>\eta_n\}}. \tag{3.2}$$

It is not difficult to check that ν is σ-additive. Clearly, $\nu|_{\{q>\varepsilon\}} = \nu^{(n)}|_{\{q>\varepsilon\}}$ for $\varepsilon \ge \eta_n$.

Using the same arguments as to get (3.1) we obtain that for all n

$$\gamma_s = \lambda_s^{(n)} * \exp(s\nu^{(n)}) \quad \text{for all } s > 0, \tag{3.3}$$

for some symmetric continuous convolution semigroup $\lambda_s^{(n)}$. (3.1) and (3.3) together give

$$\mu_s = \kappa_s * \lambda_s^{(n)} * \exp(s\nu^{(n)}), \qquad s \in R^+. \tag{3.4}$$

By (3.4) and elementary inclusions of the type needed to prove Lemma 2.1(a) we obtain

$$\begin{aligned}&(1/s)\mu_s|_{\{q>\eta\}}(A)\\ &\quad \ge \int_{\{q\le\delta\eta\}} (1/s)\exp(s\nu^{(n)})\big((A-x)\cap\{q>(1+\delta)\eta\}\big)\big(\lambda_s^{(n)} * \kappa_s\big)(dx)\end{aligned} \tag{3.5}$$

for all $\varepsilon, \eta > 0$ and $0 < \delta < 1$, and all $s > 0$. Now, the right-hand side of (3.5) converges to $\nu^{(n)}(A \cap \{q > (1+\delta)\eta\})$, whenever A is open and such that $\nu(\partial A) = 0$. We thus have obtained

$$\liminf_{s\to 0+} (1/s)\mu_s|_{\{q>\eta\}}(A) \ge \nu^{(n)}(A \cap \{q > \eta\}) \tag{3.6}$$

for A as above and all $\eta > 0$ such that $\nu\{q = \eta\} = 0$.

By construction of ν we obtain, for ηs as above

$$\lim_{s\to 0+} (1/s)\mu_s|_{\{q>\eta\}} = \nu|_{\{q>\eta\}}. \tag{3.7}$$

Applying Lemma 2.1(a) to (3.4) and using (3.7) we obtain for $\eta > \eta_n$ that

$\lim_{s\to 0+}(1/s)\kappa_s * \lambda_s^{(n)}\{q > \eta\} = 0$. Using this lemma once again we get

$$\lim_{s\to 0+} (1/s)\kappa_s\{q > \eta\} = 0 \quad \text{for all } \eta > 0, \tag{3.8}$$

$$\lim_{s\to 0+} (1/s)\lambda_s^{(n)}\{q > \eta\} = 0 \quad \text{for all } \eta > \eta_n. \tag{3.9}$$

Furthermore, using (3.2) and arguing as we did to get (3.6), we get

$$\liminf_{s\to 0+} (1/s)\gamma_s|_{\{q>\eta\}}(A) \geq \nu^{(n)}(A \cap \{q > \eta\}), \tag{3.10}$$

for all $\eta > \eta_n$ such that $\nu\{q = \eta\} = 0$. Now, applying a version of Lemma 2.1(b) to (3.3), we obtain

$$\begin{aligned}(1/s)\gamma_s|_{\{q>\eta\}}(A) \leq \int &(1/s)\exp(s\nu^{(n)})((A - x) \cap \{q > (1-\delta)\eta\})\lambda_s^{(n)}(dx)\\ &+ (1/s)\lambda_s^{(n)}\{q > (1-\delta)\eta\}\\ &+ (1/s)\exp(s\nu^{(n)})\{q > \delta\eta\}\lambda_s^{(n)}\{q > \delta\eta\}\end{aligned}$$

for all $\eta > 0$ and $0 < \delta < 1$. By virtue of (3.9), if $(1-\delta)\eta > \eta_n$ is such that $\nu\{q = (1-\delta)\eta\} = 0$ and $\nu\{q = \eta\} = 0$, we obtain

$$\limsup_{s\to 0+} (1/s)\gamma_s|_{\{q>\eta\}}(A) \leq \nu^{(n)}(A \cap \{q > \eta\}) \tag{3.11}$$

for all open subsets A with the property $\nu(\partial A) = 0$. This, together with (3.10) gives the part (iii) of the conclusion.

To complete the proof of the theorem it suffices to show that if $\nu = \lim_n \nu^{(n)} = \lim_n \nu'^{(n)}$, where $\nu^{(n)}$, $\nu'^{(n)}$ are two increasing sequences of finite and symmetric measures, and if $\exp(t\nu^{(n)})$ and $\exp(t\nu'^{(n)})$ are conditionally compact, then the weak limits of $\exp(t\nu^{(n)})$ and $\exp(t\nu'^{(n)})$ exist, as $n \to \infty$, and are identical. This is, however, quite standard and can be found in [15].

The measure ν defined, by virtue of Theorem 3.1, as $\lim \nu^{(n)}$, will be called the Lévy measure of the convolution semigroup $(\mu_t)_{t>0}$. The semigroup $\gamma_t = \lim \exp(t\nu^{(n)})$ will be denoted in the sequel by $\exp(t\nu)$.

COROLLARY 3.2. *Suppose that q is a lower semicontinuous seminorm and that $(\mu_t)_{t>0}$ is a symmetric continuous and q-continuous convolution semigroup on E such that $\mu_t\{q < \infty\} = 1$ for all $t > 0$. Assume, further, that $\mu_t = \exp(t\nu)$, where the Lévy measure ν is infinite and such that $\nu\{q = 0\} = 0$. Let $\eta_0 = \infty$ and let η_n be a decreasing to 0 sequence of positive numbers such that $\nu(V_n) > 0$, where $V_n = \{x\colon \eta_n < q(x) \leq \eta_{n-1}\}$, $n = 1, \ldots$. Denote $V_0 = \varnothing$, $\lambda_n = \nu(V_n)$, $\beta_n = \sum_{i=0}^n \lambda_i$, and $F_n = \bigcup_{i=n+1}^\infty V_i$, $n = 0, 1, \ldots$. Then we have*

$$\mu_t = e^{-t\beta_n}\sum_{k=1}^{\infty}\left((1/k!)\left(\nu|_{F_n^c}\right)^{*k}\right) * \kappa_t^{(n)} + e^{-t\beta_n}\kappa_t^{(n)} \tag{3.12}$$

for $n = 1, \ldots,$ where $\kappa_t^{(n)} = \exp(t\nu|_{F_n})$, and the above series is convergent in the total variation norm.

PROOF. We first show that for all $\eta > 0$ we have

$$\nu\{q > \eta\} < \infty. \tag{3.13}$$

By Theorem 3.1 we have $(1/t)\mu_t|_{\{\|\cdot\| > \varepsilon\}} \Rightarrow \nu|_{\{\|\cdot\| > \varepsilon\}}$, when $t \to 0+$, and where $\|\cdot\|$ is a pseudonorm generating the topology of E, for all continuity points ε of $\Theta(u) = \nu\{\|\cdot\| > u\}$. Thus, by Lemma 2.4 and by the fact that $\{q > \eta\}$ is open we obtain

$$\nu(\{q > \eta\} \cap \{\|\cdot\| > \varepsilon\}) \le \liminf_{t\to 0+} (1/t)(\mu_t|_{\{\|\cdot\| > \varepsilon\}})\{q > \eta\}$$
$$\le \limsup_{t\to 0+} (1/t)\mu_t\{q > \eta\} < \infty.$$

If ε tends to 0, we obtain

$$\nu\{q > \eta\} \le \limsup_{t\to 0+} (1/t)\mu_t\{q > \eta\}. \tag{3.14}$$

Now, (3.12) is an immediate consequence of the equality

$$\exp(t\nu) = \exp(t\nu|_{F_n^c}) * \exp(t\nu|_{F_n}).$$

Now, let $(\mu_t)_{t>0}$ be a symmetric continuous convolution semigroup on E. $(\mu_t)_{t>0}$ is called symmetric p-stable, $0 < p < 2$, if $\mu_t = \exp(t\nu)$, $t > 0$, and

$$\nu(sA) = (1/s^p)\nu(A) \tag{3.15}$$

for all Borel sets A and all $s > 0$. It is not difficult to see that (3.15) is equivalent to the property $\mu_t(A) = \mu_{ts}(s^{1/p}A)$ for all $t, s > 0$ and all Borel sets A. In particular, any such semigroup is q-continuous for every measurable seminorm q such that $\mu_1\{q < \infty\} = 1$.

4. Absolute continuity of seminorms in R^n. The following lemma is quite elementary and intuitive. Since we were unable to find a suitable reference, we present a simple proof for the sake of completeness.

LEMMA 4.1. *Let $n \ge 1$ and let f: $R^n \to [0, \infty)$ be a convex function such that $\lim_{\|\mathbf{r}\|\to\infty} f(\mathbf{r}) = +\infty$. Let η_1, η_2 be real numbers such that $m = \inf f(\mathbf{r}) < \eta_1 < \eta_2$. Then there exists a positive constant K such that*

$$L(\{\mathbf{r} \in R^n\colon t_1 < f(\mathbf{r}) < t_2\}) \le K(t_2 - t_1) \tag{4.1}$$

for each t_1, t_2 satisfying $\eta_1 \le t_1 < t_2 \le \eta_2$. L stands here for the n-dimensional Lebesgue measure. Hence if N is of linear Lebesgue measure 0 and $m \notin N$, then

$$L(f^{-1}(N)) = 0. \tag{4.2}$$

PROOF. Fix $\mathbf{r}_0 \in R^n$ such that $f(\mathbf{r}_0) = m$. Denote $S^{n-1} = \{\mathbf{y} \in R^n\colon \|\mathbf{y}\| = 1\}$, where $\|\cdot\|$ is the usual Euclidean norm in R^n. Observe that for each fixed $\mathbf{y} \in S^{n-1}$ we have that $f(\mathbf{r}_0 + \rho\mathbf{y})$ is a nondecreasing, convex, unbounded function of ρ where $\rho \ge 0$. Moreover, it is strictly increasing on $\{\rho > 0\colon f(\mathbf{r}_0 + \rho\mathbf{y}) > m\}$. Now, let $\rho_t(\mathbf{y})$ be the only positive number ρ for which $f(\mathbf{r}_0 + \rho\mathbf{y}) = t$ for each fixed $\mathbf{y} \in S^{n-1}$ and $t > m$. By the preceding observation we get

$$\frac{t_2 - t_1}{\rho_{t_2}(\mathbf{y}) - \rho_{t_1}(\mathbf{y})} \ge \frac{t_1 - m}{\rho_{t_1}(\mathbf{y})} \tag{4.3}$$

for each $\mathbf{y} \in S^{n-1}$ whenever $m < t_1 < t_2$. On the other hand,

$$M = \sup\{\rho_t(\mathbf{y}) \colon m < t \le \eta_2, \mathbf{y} \in S^{n-1}\} = \sup\{\rho_{\eta_2}(\mathbf{y}) \colon \mathbf{y} \in S^{n-1}\} < +\infty,$$

since ρ_{η_2} is a continuous function on the compact set S^{n-1}.

Integrating in polar coordinates we obtain for any t_1, t_2 such that $t_2 > t_1 > m$:

$$\begin{aligned} &L(\{\mathbf{r} \in R^n \colon t_1 < f(\mathbf{r}) < t_2\}) \\ &= \int_{S^{n-1}} s(d\mathbf{y}) \int_{\rho_{t_1}(\mathbf{y})}^{\rho_{t_2}(\mathbf{y})} \rho^{n-1} d\rho \\ &= (1/n) \int_{S^{n-1}} \left[(\rho_{t_2}(\mathbf{y}) - \rho_{t_1}(\mathbf{y})) \sum_{k=0}^{n-1} (\rho_{t_1}(\mathbf{y}))^k (\rho_{t_2}(\mathbf{y}))^{n-k-1} \right] s(dy), \end{aligned} \tag{4.4}$$

where s denotes the surface measure on S^{n-1}. Now, if $\eta_1 \le t_1 < t_2 \le \eta_2$ then (4.4) together with (4.2) and the fact that $M < \infty$, give (4.1) with

$$K = (M^n/\eta_1 - m) \int_{S^{n-1}} s(dy).$$

COROLLARY 4.2. *Let E be a separable metric vector space and let q be a finite measurable seminorm on E. Suppose that s_j, $j = 1, \ldots, n$, and y are some elements of E. Denote*

$$m = \inf_{\mathbf{r}} \left\{ q\left(\sum_{j=1}^{n} r_j s_j + y \right) \colon \mathbf{r} = (r_1, \ldots, r_n) \in R^n \right\}.$$

Then

$$\int_{R^n} \mathbb{1}_N\left(q\left(\sum_{j=1}^{n} r_j s_j + y \right) \right) dr_1 \cdots dr_n = 0$$

for each N of linear Lebesgue measure 0 that does not contain the point m.

PROOF. Define T: $R^n \to E$ by

$$T(r_1, \ldots, r_n) = \sum_{j=1}^{n} r_j s_j.$$

Assume that $q \circ T$ is nonconstant, since otherwise the corollary is trivial. Then there exists an integer k, $1 \le k \le n$, and linear operators T_1: $R^n \to R^k$ and T_2: $R^k \to E$ such that $\rho \circ T = \rho \circ T_2 \circ T_1$, $T_1(R^n) = R^k$ and that $q \circ T_2$ is a norm on R^k. Then the function f defined on R^k by $f(\mathbf{r}) = q(T_2(\mathbf{r}) + y)$ satisfies the assumptions of Lemma 4.1. Hence for each set N of linear Lebesgue measure 0 such that $m \notin N$ we have $L(f^{-1}(N)) = 0$. Since $\{\mathbf{r} \in R^n \colon q(\sum_{j=1}^{n} r_j s_j + y) \in N\} = T_1^{-1}(f^{-1}(N))$ and since the preimage by a nonzero surjective linear operator of a set of Lebesgue measure 0 is again a set of Lebesgue measure 0, the corollary follows.

REMARK 4.3. Lemma 4.1 can be also derived from the Brunn–Minkowski Theorem (see, e.g., [7]).

5. The main theorem. We begin with two lemmas, the second one may be of independent interest.

LEMMA 5.1. *Let* $(\mu_t)_{t>0} = (\exp(t\nu))_{t>0}$ *be a symmetric convolution semigroup on E and let q be a measurable seminorm such that* $\mu_1\{q < \infty\} = 1$. *Assume that* $\nu\{q = 0\} = 0$. *Let* $\kappa^{(\eta)} = \exp(\nu|_{\{q \le \eta\}})$. *Then there exists a constant* $c \ge 0$ *such that for every* $\varepsilon > 0$ *we have*

$$\lim_{\eta \to 0+} \kappa^{(\eta)}\{q > c + \varepsilon\} = 0, \tag{5.1}$$

$$\liminf_{\eta \to 0+} \kappa^{(\eta)}\{q > c - \varepsilon\} \ge 1/2. \tag{5.2}$$

PROOF. Let $\eta_0 = \infty$ and let η_n be a decreasing to 0 sequence of positive numbers. We first show that (5.1) and (5.2) hold for this sequence. Denote $V_n = \{x\colon \eta_n < q(x) \le \eta_{n-1}\}$ and $F_n = \bigcup_{i \ge n+1} V_i$, $n = 0, 1, \ldots$. Let $(X_i)_i$ be a sequence of independent E-valued random vectors with distributions $\exp(\nu|_{Vi})$, respectively. By Theorem 3.1(ii) it follows that the series $\sum_{i=n+1}^{\infty} X_i$ converges with probability 1 to a random vector with the distribution $\exp(\nu|_{F_n})$ for $n = 0, 1, \ldots$. Observe that for $n = 0$ we get μ_1. Denote $Z_k = q(\sum_{i=k+1}^{\infty} X_i)$. $(X_i)_i$ are independent, hence, by the Kolmogorov 0–1 Law we obtain that $\limsup Z_k = c \le \infty$ with probability 1. Next, denote $Z_m^{(k)} = \max_{0 \le i \le m} Z_{k+i}$. When $m \to \infty$, then $Z_m^{(k)}$ converges pointwise to $Z^{(k)} = \sup_{i \ge 0} Z_{k+i}$. Hence for every k and every $\varepsilon > 0$ there exists m_k such that for $m \ge m_k$ we have $P\{Z^{(k)} - Z_m^{(k)} > \varepsilon\} < 1/k$. By Lemma 2.2(b) we obtain $P\{Z_m^{(k)} > \alpha\} \le 2P\{Z_k > \alpha\}$. Thus, we get $P\{Z^{(k)} > \alpha + \varepsilon\} - 1/k \le 2P\{Z_k > \alpha\}$. Since $Z^{(k)}$ decreases to $\limsup Z_k$, as $k \to \infty$, we finally obtain $P\{\limsup Z_k > \alpha\} \le \liminf 2P\{Z_k > \alpha\}$ for all $\alpha > 0$.

Suppose now that $\alpha < c$. Then we have

$$\liminf P\{Z_k > \alpha\} \ge 1/2. \tag{5.3}$$

Next, dividing $\sum_{i=1}^{\infty} X_i$ into two parts and applying Lemma 2.2(a) we get:

$$P\left\{\max\left(q\left(\sum_{i=1}^{k} X_i\right), q\left(\sum_{i=k+1}^{\infty} X_i\right)\right) > \alpha\right\} \le 2P\left\{q\left(\sum_{i=1}^{\infty} X_i\right) > \alpha\right\}.$$

By this inequality we obtain that $1 - 2\mu_1\{q > \alpha\} \le P\{Z_k \le \alpha\}$, which, together with (5.3), gives that $2\mu_1\{q > \alpha\} \ge 1/2$ for every $\alpha < c$. Thus $c = \infty$ implies that $\mu_1\{q < \infty\} < 3/4$, which gives the contradiction. Hence $c < \infty$, what concludes the proof of the lemma if we show that c does not depend on the choice of η_n. Indeed, (5.3) then yields (5.2), while (5.1) is a consequence of the following obvious inequality: $\limsup P\{Z_k > \alpha\} \le P\{\limsup Z_k > \alpha\}$.

To prove this remaining part, suppose that η'_n is another sequence decreasing to 0 and the corresponding constant $c' < c$. Let k_n be such that $\eta'_{k_n+1} \le \eta_n < \eta'_{k_n}$. Applying Lemma 2.2(a) again we see that

$$1 - 2\kappa^{(\eta'_{k_n})}\{q > \alpha\} \le \kappa^{(\eta_n)}\{q \le \alpha\}.$$

If α is such that $c' < \alpha < c$, then the above inequality and the version of (5.1) written for η'_n show that $\kappa^{(n)}\{q > \alpha\} \to 0$, while, by (5.3) we get $\liminf \kappa^{(\eta_n)}\{q > \alpha\} \geq 1/2$. This contradiction completes the proof of this part and concludes the proof of the lemma.

LEMMA 5.2. *Let $(\mu_t)_{t>0} = (\exp(t\nu))_{t>0}$ be a symmetric p-stable convolution semigroup on E, $0 < p < 2$, and let q be a measurable seminorm. Assume that $\mu_1\{q < \infty\} = 1$. Then the following conditions are equivalent:*

(i) *q is bounded*, μ_1 a.s.,
(ii) $\nu\{q > 0\} = 0$,
(iii) *q = const.*, μ_1 a.s.

PROOF. Suppose that q is almost surely bounded. Let U_n be a decreasing sequence of open neighbourhoods of 0 such that $U_1 = E$, $\bigcap_{n=1}^{\infty} U_n = \{0\}$. Let $V_n = U_{n-1} \setminus U_n$, $n = 1, \dots$, and let $(X_i)_i$ be a sequence of independent E-valued random vectors with distributions $\exp(\nu|_{V_i})$, respectively, $i = 1, \dots$. As in the proof of Lemma 5.1 we obtain that $\sum_{i=n+1}^{\infty} X_i$ is distributed like $\exp(\nu|_{U_n})$, $n = 1, \dots$, and that $\limsup q(\sum_{i=n+1}^{\infty} X_i) = c < \infty$, μ_1 a.s. Denote $\exp(\nu|_{U_n})$ by $\chi^{(n)}$. By Lemma 2.2(a) we have

$$1 - 2\mu_1\{q > \varepsilon\} \leq \exp\big(\nu|_{U_n^c}\big)\{q \leq \varepsilon\}\chi^{(n)}\{q \leq \varepsilon\}.$$

Therefore, if ε is greater than the essential upper bound of q then $\exp(\nu|_{U_n^c})\{q > \varepsilon\} = 0$. In particular, $(\nu|_{U_n^c})\{q > \varepsilon\} = 0$. When $n \to \infty$ we obtain that $\nu\{q > \varepsilon\} = 0$. By this fact and by (3.15) we obtain (ii).

Now, assume (ii) and let $U_n, V_n, \chi^{(n)}$, $n = 1, \dots$, the constant c and the sequence $(X_i)_i$ be as in the first part of the proof. Note that for fixed n and $\eta > 0$ we obtain, by Lemma 2.1(b)

$$\big(\nu|_{U_n^c}\big) * \big(\nu|_{U_n^c}\big)\{q > \eta\} \leq 2\big(\nu|_{U_n^c}\big)\{q > (1 - \varepsilon)\eta\} + \Big(\big(\nu|_{U_n^c}\big)\{q > \varepsilon\eta\}\Big)^2 = 0.$$

By the obvious induction we obtain that $\exp(\nu|_{U_n^c})\{q > \eta\} = 0$ for every $\eta > 0$ and each $n = 1, \dots$. From this fact and since for each $\varepsilon > 0$ there holds $\chi^{(n)}\{q \leq c + \varepsilon\} \to 1$ as $n \to \infty$, we obtain

$$\begin{aligned}\mu_1\{q \leq c + 2\varepsilon\} &= \exp\big(\nu|_{U_n^c}\big) \times \chi^{(n)}\{(x, y)\colon q(x + y) \leq c + 2\varepsilon\} \\ &\geq \exp\big(\nu|_{U_n^c}\big) \times \chi^{n}\{(x, y)\colon q(x) \leq \varepsilon, q(y) \leq c + \varepsilon\} \to 1.\end{aligned}$$

Hence $q \leq c$, μ_1 a.s. Next, observe that if $q(\sum_{i=n+1}^{\infty} X_i)$ is convergent in probability as $n \to \infty$, then choosing subsequences convergent almost surely and using (5.1) and (5.2) we obtain that $\lim q(\sum_{i=n+1}^{\infty} X_i) = c$, a.s. This, in turn, implies that

$$\begin{aligned}\mu_1\{q < c - \varepsilon\} &= \exp\big(\nu|_{U_n^c}\big) \times \chi^{(n)}\{(x, y)\colon q(x) < \varepsilon/2, q(x + y) < c - \varepsilon\} \\ &\leq \chi^{(n)}\{q < c - \varepsilon/2\} \to 0.\end{aligned}$$

Therefore, it is sufficient to show that $q(\sum_{i=n+1}^{\infty} X_i)$ converges in probability.

Suppose, to the contrary, that this is not true. Then there exists an $\varepsilon > 0$ such that

$$P\left\{\omega: \left| q\left(\sum_{k_{2\iota-1}}^{\infty} X_j\right) - q\left(\sum_{k_{2\iota}}^{\infty} X_j\right) \right| > \varepsilon\right\} > \varepsilon$$

for some increasing subsequence $k_i \to \infty$. By the triangle inequality we then get

$$\exp\left(\nu\left|\bigcup_{k_{2\iota-1}}^{k_{2\iota}-1} U_j\right|\right)\{q > \varepsilon\} = P\left\{\omega: q\left(\sum_{k_{2\iota-1}}^{k_{2\iota}-1} X_j\right) > \varepsilon\right\} \geq \varepsilon.$$

This, however, gives the contradiction, because the left-hand side of the above formula vanishes for all $\varepsilon > 0$. This proves that (ii) implies (iii) and, clearly, completes the proof.

The next lemma is a version of Lemma 2.1 in [10]. We first introduce some notation. Suppose that $\mu_t = \exp(t\nu)$ is a symmetric p-stable convolution semigroup on E, $0 < p < 2$, and that q is a lower semicontinuous seminorm such that $\mu_1\{q < \infty\} = 1$. Let $S_q = \{x: q(x) = 1\}$. Now, for any $x \in E$ such that $q(x) \neq 0$, $q(x) \neq \infty$, define $\varphi(x) = (q(x), x/q(x)) \in R^+ \times S_q$. Let B be a Borel subset of S_q. Denote

$$I_{r_1, r_2}(B) = \{x: r_1 < q(x) \leq r_2, x/q(x) \in B\}$$

for $r_1, r_2 \in R^+$ such that $r_1 < r_2$. If $\nu|_{\{q>0\}} \neq 0$ then (3.13) and (3.15) imply that there exists a finite Borel measure σ and S_q such that

$$\text{(5.4)} \quad \begin{aligned} \nu\left(I_{r_1, r_2}(B)\right) &= \int_{r_1}^{r_2} (1/r^{1+p})\, dr \int_{S_q} \mathbb{1}_B(s)\sigma(ds) \\ &= \int_{S_q}\int_{R^+} \mathbb{1}_{\varphi}\left(I_{r_1, r_2}(B)\right)(r, s)(1/r^{1+p})\, dr\, \sigma(ds). \end{aligned}$$

LEMMA 5.3. *Let $(\mu_t)_{t>0}$ be a symmetric p-stable convolution semigroup, $0 < p < 2$, and let q be a lower semicontinuous seminorm such that $\mu_1\{q < \infty\} = 1$. Assume that $\nu|_{\{q>0\}} \neq 0$. Then for all $y \in E$ and all positive numbers η_1, η_2, t, such that $\eta_1 < \eta_2$ the following holds*:

$$\text{(5.5)} \quad \left(\nu|_{\{\eta_1 < q \leq \eta_2\}}\right)\{x: q(x + y) < t\} = \int_{S_q}\int_{\eta_1}^{\eta_2} \mathbb{1}_{(0, t)} q(rs + y) \frac{dr}{r^{1+p}} \sigma(ds).$$

PROOF. Observe that by (2.2) and (3.14) we obtain

$$\text{(5.6)} \quad \nu\{q = \infty\} = 0.$$

Hence, we may assume that $q(x) < \infty$ for all $x \in E$. Then the formula (5.5) is a simple consequence of (5.4). All that has to be proved is that the sets $\{x: q(x + y) < t, \eta_1 < q(x) \leq \eta_2\}$ belong to the σ-algebra generated by the sets of the type $I_{r_1,r_2}(B)$. This is a consequence of the following easy to check formula:

$$\{x: q(x+y) < t, \eta_1 < q(x) < \eta\} = \bigcup_{\eta_1 < r_1 < r_2 < \eta;\ r_1, r_2 \in Q^+} I_{r_1,r_2}\big(B_{r_1,r_2}\big),$$

where

$$B_{r_1,r_2} = \{s \in S_q: q(rs+y) < t, r_1 < r < r_2\}$$
$$= \bigcap_{r \in (r_1,r_2) \cap Q^+} \{s \in S_q: q(rs+y) < t\}.$$

THEOREM 5.4. *Let* $(\mu_t)_{t>0}$ *be a symmetric p-stable convolution semigroup on* E, $0 < p < 2$, *and let* q *be a lower semicontinuous seminorm. Assume that* $\mu_1\{q < \infty\} = 1$ *and let* $F(u) = \mu_1\{q < u\}$. *Then* $F(u)$ *is absolutely continuous except possibly at* $u_0 = \inf\{u > 0: F(u) > 0\}$. *If, additionally,* $0 < p < 1$ *or* q *is strictly convex then* F *is either absolutely continuous or degenerate at* 0.

PROOF. We first prove the theorem under the additional assumption that the Lévy measure ν of $(\mu_t)_{t>0}$ has the property

(5.7) $$\nu\{q = 0\} = 0.$$

We divide the proof into several steps.

Step 1. In this step we show that the following holds:

(5.8) $$\lim_{\eta \to 0+} \exp\big(\nu|_{\{q \le \eta\}}\big)\{q \ge \varepsilon\} = 0 \quad \text{whenever } \mu_1\{q < \varepsilon\} > 0.$$

To prove this, let $(X_i)_i$ be a sequence of independent random vectors defined as in Lemma 5.1. Note that the series $\sum_{n=1}^{\infty} X_n$ converges a.s.; hence $\sum_{n=k+1}^{\infty} X_n$ converges a.s. to 0 as $k \to \infty$. Therefore, $\sum_{n=1}^{k} X_n - \sum_{n=k+1}^{\infty} X_n$ converges a.s. to $\sum_{n=1}^{\infty} X_n$ as $k \to \infty$. Since q is lower semicontinuous, we obtain

$$q\left(\sum_{n=1}^{\infty} X_n\right) \le \liminf q\left(\sum_{n=1}^{k} X_n - \sum_{n=k+1}^{\infty} X_n\right) \quad \text{a.s.}$$

Denote $Y = q(\sum_{n=1}^{\infty} X_n)$ and $Y_k = q(\sum_{n=1}^{k} X_n - \sum_{n=k+1}^{\infty} X_n)$. Then, for every $\delta > 0$ we have

(5.9) $$\lim P\{Y_k - Y > -\delta\} = 1.$$

It is evident that

(5.10)
$$\begin{aligned}\{Y_k \le \varepsilon - \delta, Y_k - Y > -\delta\} &\subseteq \{Y \le \varepsilon, Y_k \le \varepsilon, Y_k - Y > -\delta\}\\ &\subseteq \{Y \le \varepsilon, Y_k \le \varepsilon\}\\ &\subseteq \left\{q\left(\sum_{n=k+1}^{\infty} X_n\right) \le \varepsilon, q\left(\sum_{n=1}^{k} X_n\right) \le \varepsilon\right\}.\end{aligned}$$

Comparing (5.9), (5.10) and using the fact that Y_k, Y have the same distributions as μ_1, we obtain:

$$\mu_1\{q \le \varepsilon - \delta\} = \lim_k P\{Y_k \le \varepsilon - \delta, Y_k - Y > -\delta\}$$

$$\le \liminf P\left\{q\left(\sum_{n=k+1}^{\infty} X_n\right) \le \varepsilon, q\left(\sum_{n=1}^{k} X_n\right) \le \varepsilon\right\} \tag{5.11}$$

$$\le \limsup \gamma^{(k)}\{q \le \varepsilon\} \cdot \liminf \kappa^{(k)}\{q \le \varepsilon\},$$

where $\gamma^{(k)}$, $\kappa^{(k)}$ are the distributions of $\sum_{n=1}^{k} X_n$ and $\sum_{n=k+1}^{\infty} X_n$, respectively. Next, since $\{q \le \varepsilon\}$ is closed, we get $\limsup \gamma^{(k)}\{q \le \varepsilon\} \le \mu_1\{q \le \varepsilon\}$. Therefore, (5.11) implies

$$\mu_1\{q < \varepsilon\} \le \mu_1\{q \le \varepsilon\} \cdot \liminf \kappa^{(k)}\{q \le \varepsilon\}. \tag{5.12}$$

It is easy to see that (5.12) completes the proof of this step.

Step 2. We prove that if $F(\varepsilon) > 0$ then F is absolutely continuous on (ε, ∞). To do this, we apply Corollary 3.2. By virtue of (5.6) we may assume that $q(x) < \infty$ for all $x \in E$. Let $\eta_0 = \infty$ and let η_n be a decreasing to 0 sequence of positive numbers. Denote, as in this corollary, $\beta_n = \nu\{q > \eta_n\}$ and $\kappa^{(n)} = \exp(\nu|_{\{q \le \eta_n\}})$. Formula (3.12) then gives

$$\begin{aligned}\mu_1\{q \in N\} &= e^{-\beta_n} \sum_{k=1}^{\infty} (1/k!) \int \left(\nu|_{\{q > \eta_n\}}\right)^{*k} \{x\colon q(x+y) \in N\} \kappa^{(n)}(dy) \\ &\quad + e^{-\beta_n} \kappa^{(n)}\{q \in N\}\end{aligned} \tag{5.13}$$

for $n = 1, \ldots,$ and all Borel subsets N of R^+. Now, by Step 1 we have $\kappa^{(n)}\{q > \varepsilon\} \to 0$ as $n \to \infty$. Since also $e^{-\beta_n} \to 0$, it is enough to show that for all Borel subsets $N \subseteq (\varepsilon, \infty)$ such that $|N| = 0$ we have

$$\int_{\{q \le \varepsilon\}} \left(\nu|_{\{q > \eta_n\}}\right)^{*k} \{x\colon q(x+y) \in N\} \kappa^{(n)}(dy) = 0, \tag{5.14}$$

for all positive integers n and k. By Lemma 5.3 the expression under the integral sign is equal to

$$\int_{S_q^k} \int_{\eta_n}^{\infty} \cdots \int_{\eta_n}^{\infty} \mathbb{1}_N\left(q\left(\sum_{j=1}^{k} r_j s_j + y\right)\right) \frac{dr_1 \cdots dr_k}{(r_1 \cdots r_k)^{1+p}} \sigma(ds_1) \cdots \sigma(ds_k). \tag{5.15}$$

By Corollary 4.2, (5.15) is 0 whenever

$$m = \inf_{\mathbf{r}} q\left(\sum_{j=1}^{k} r_j s_j + y\right) \le \varepsilon. \tag{5.16}$$

However, since we integrate (5.15) over the set $\{y\colon q(y) \le \varepsilon\}$ and it is easily seen that $m \le q(y)$, (5.14) follows.

Step 3. We prove that if q is, additionally, strictly convex then F is absolutely continuous on R^+.

Recall that q is said to be strictly convex if the equality $q(x) = q(y) = q((x+y)/2)$ implies that $x = y$.

Let N be a set of linear Lebesgue measure 0. Using Corollary 4.2 and the fact that the infimum in (5.16) is attained exactly at one point for each y, we deduce that (5.15) vanishes. This implies that each term in the series (5.13) is 0. Moreover, the last summand in (5.13) can be made arbitrarily small. Therefore, $\mu_1\{q \in N\} = 0$.

Step 4. W prove here that if $0 < p < 1$ then for all $t > 0$ we have $\mu_1\{q < t\} > 0$.

To show this we apply the idea of Proposition 3.1 in [12]. Namely, for every $t > 0$ we have the following simple inequality:

$$\mu_1 \times \mu_1\{q(x) < t, q(y) < t\} \leq \mu_1 \times \mu_1\{q(x + y) < 2t\}.$$

Since μ_1 is p-stable, we obtain $(\mu_1\{q < t\})^2 \leq \mu_1\{q < \alpha t\}$, where $\alpha = 2^{1-1/p}$. This clearly yields the conclusion of Step 4.

Step 5. In this step we eliminate the restriction (5.6). For this purpose, let us write $\nu = \nu^{(1)} + \nu^{(2)}$, where $\nu^{(1)} = \nu|_{\{q>0\}}$ and $\nu^{(2)} = \nu|_{\{q=0\}}$. It is easy to see that $\nu^{(1)}, \nu^{(2)}$ are Lévy measures of some p-stable semigroups. Denote the corresponding semigroups by $(\mu_t^{(1)})_{t>0}$, $(\mu_t^{(2)})_{t>0}$, respectively. By Steps 1 and 2 the function $F^{(1)}(u) = \mu_1^{(1)}\{q < u\}$ is absolutely continuous on (u_0^1, ∞), where $u_0^1 = \inf\{u > 0: F^{(1)}(u) > 0\}$. On the other hand, by Lemma 5.2 we have that $q = \text{const.}$, $\mu_1^{(2)}$ a.s. We show that $q = 0$, $\mu_1^{(2)}$ a.s. By the proof of Lemma 5.2 we obtain

$$\left(\exp\left(\nu^{(2)}|_{U_n^c}\right)\right)\{q > \varepsilon\} = 0$$

for all $\varepsilon > 0$ and all open neighbourhoods U_n of 0. If now U_n is decreasing with $\bigcap_n U_n = \{0\}$, then $\exp(\nu^{(2)}|_{U_n^c}) \Rightarrow \mu_1^{(2)}$ as $n \to \infty$. Since $\{q > \varepsilon\}$ is open, we obtain

$$\mu_1^{(2)}\{q > \varepsilon\} \leq \liminf \exp\left(\nu^{(2)}|_{U_n^c}\right)\{q > \varepsilon\} = 0.$$

Hence $q = 0$, $\mu_1^{(2)}$ a.s. By the triangle inequality we get $q(x + y) = q(x)$ for $\mu_1^{(2)}$ almost all $y \in E$ and thus

$$\begin{aligned} F(u) &= \mu_1\{q > u\} \\ &= \mu_1^{(1)} \times \mu_1^{(2)}\{(x, y): q(x + y) < u\} \\ &= \mu_1^{(1)}\{q < u\} = F^{(1)}(u) \end{aligned}$$

This completes the proof of this step and of our theorem.

EXAMPLE 5.5. Let $E = R^\infty$ be the countable product of real lines with the product topology. Let $(\mu_t)_{t>0}$ be a symmetric p-stable convolution semigroup on E, $0 < p < 2$, and let $q(x) = \sup|x_i|$. Suppose that $\mu_1\{q < \infty\} = 1$. By Theorem 5.4 $F(u) = \mu_1\{q < u\}$ is absolutely continuous on (u_0, ∞), where $u_0 = \inf\{u > 0: F(u) > 0\}$. If $0 < p < 1$ then $F(u) > 0$ for all $u > 0$ and F is absolutely continuous on R^+.

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INSTITUTE OF MATHEMATICS
WROCŁAW TECHNICAL UNIVERSITY
WYB. WYSPIANSKIEGO 27
50-370 WROCŁAW
POLAND